Toying with TAD

BILL HIGGINSON
Doctoral Student, University of Alberta
Edmonton, Alberta

Toying with TAD

Psychologists doing research in the area of human problem-solving have discovered a phenomenom called "functional fixedness", which refers to the tendency humans have to identify objects with some specific role or function. In many cases, this identification is so strong that it impedes productive problem-solving. This occurs when problem-solvers cannot see some given object as being a tool which might help them because they have fixated on the standard function which the object serves and this use differs from the one required to solve the problem.

Some mathematics teachers sometimes exhibit a form of functional fixation with respect to teaching aids. In this variation of functional fixedness, one particular use of a teaching aid is so strongly identified with the aid that it tends to block out its other potential uses. Hence, for example, Cuisenaire rods are employed almost exclusively to teach basic number operations to young children and rectangular grids are seldom used except for purposes of "graphing". To say that some mathematics teachers tend to functionally fixate with regard to some of their teaching aids is, in some sense, to say that they are not getting as much "mileage" out of these aids as they might.

Of the many types of mathematics teaching aids, perhaps the worst "mileage-getters" of all are mathematical games. Many teachers use mathematical games as a form of reward: "If you answer all your questions correctly, then you can go to the back of the room and play 'Cube-Fusion' or 'Hi-Q'." and so on. While this is certainly one valid use of these aids, it is likely that fixating on the reward function may well lead a teacher to overlook some of the other functions this type of aid might serve. This would be most unfortunate since some of these mathematical games offer extremely rich frameworks for significant mathematical activity on the part of students.

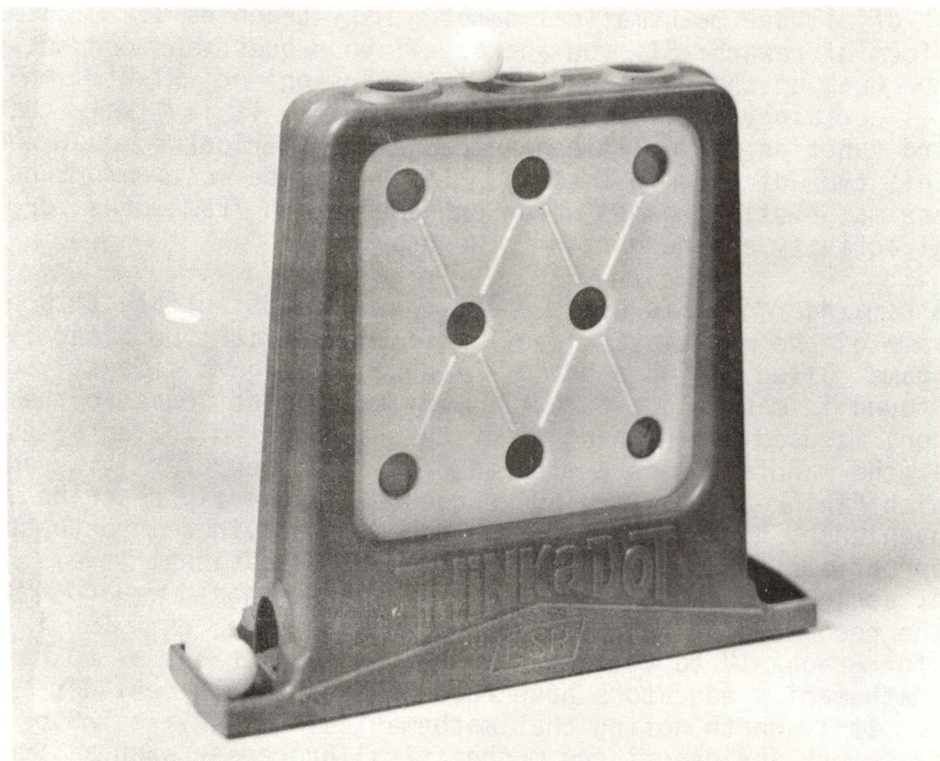
The remainder of this article attempts to substantiate this position by outlining some of the mathematical activities which might be generated by the structural game called "Think-a-Dot". Although this game is easily obtainable[1] and can be found in many classrooms, it very seldom (at least in the experience of the author) is used as anything other than a toy. Following a description of the game (the manufacturers prefer to call it a "computer"), a range of activities which "Think-a-Dot" (or TAD) suggests are briefly described. For the sake of convenience, these activities have been subdivided into those which might be appropriate for learners at 4 different educational levels: elementary, junior high, senior high and university. (Teachers should regard with some suspicion the recommended age levels for mathematical games. TAD is usually prescribed for Grades IV to VIII with almost no suggestions as to how it might be used.) Mathematics educators have recently started to consider "process objectives". It is worth noting that mathematical games like TAD provide an excellent framework for practicing mathematical processes such as "generalizing", "proving", "symbolizing" and "clarifying" (Morley, 1973).

---

[1]"Think-a-Dot" is available from Western Educational Activities Ltd., 10577 - 97 Street, Edmonton, for $3.75.

TAD is a small plastic box with dimensions approximately 6" x 5" x 1 1/2" (see photograph). On one face, there are 8 "windows" arranged in a 3-2-3 pattern, behind each of which the color yellow or blue appears. At the top of the box there are three "holes". When a marble is dropped in any of these holes, some of the windows "change color" and the marble emerges on either the right or the left side of the box. Inside the box there is a "flip-flop" inclination changing the color of its window at the same time. Certain positions or "states" can be set on the face by either tilting the box or by changing the color of each window individually.

The following activities/games/problems are certainly not the only ones that are suggested by TAD nor are they necessarily either the most obvious or the best. They serve only to indicate some of the possible areas open to investigation. For any learner, the most interesting questions are the ones he himself poses. Students should be encouraged to generate, and to work on, their own problems. The job of the teacher in this case is to help students learn how to attack these problems. Conjectures should be formulated, tested and modified. In some cases, it may be possible to construct proofs. Some problems, such as those relating to symbolization and notation, will be found at all age and grade levels. However, it is to be expected, for example, that older students will have more sophisticated systems of representation and terminology.



"Think-a-Dot"

ELEMENTARY LEVEL ACTIVITIES

With elementary school students, one may wish to play various sorts of *prediction games*.

Exit I

One prediction game which any number of children could play is Exit I. In this game, each player chooses an "exit side" and drops marbles until a marble exits on the "wrong" side. The winner is the player having the highest number of correct exits after some fixed number of turns. A variation on this basic theme can be introduced by limiting the number of drops that any player can make in a row in any one hole.

Exit II

In Exit II, each player predicts the side that a marble, dropped in a particular hole, will exit on. The winner is the player having the most correct predictions after, say, 10 drops. A more difficult version of Exit II is one in which the blank face of TAD, rather than the window face, is facing the players.

One-Change

A type of two-player game suitable for this level is the One-Change game. In this game, a player chooses one of the 8 windows and challenges his opponent to have it changed in color after, say, 3 drops. After perhaps 5 turns, the player who has most frequently been able to meet his opponent's challenge is the winner. Variations can be introduced by limiting the position of the challenge holes or by increasing the number of holes to be changed.

JUNIOR HIGH LEVEL ACTIVITIES

At the junior high level, students should be able to work on problems such as those relating to a binary representation of the states of TAD. How many different states are there in TAD? How can these states most conveniently be represented? According to the representation(s), what characterizes all states which have a blue window in the upper left-hand corner?

Given one state, how many drops are required to change TAD to another given state? If it is possible to move from one state to another by a series of drops (Note that this isn't always possible!), is this number of drops unique? (It isn't, but what can you say about it?)

Competitive games appropriate for this age group include: the *Ratio game* (from a given state, one player challenges another to make the ratio of blue windows to yellow windows say 3:5, in as few drops as possible); the *Maximum-change game* (from a given state make, say, 3 drops and change the colors of as many windows as possible); the *Symmetry game* (in as few moves as possible produce, say, a top row which is color-symmetric, or anti-symmetric, to the bottom row); and the *Lines game* (from a given state, produce after, say, 3 drops as many blue or yellow "lines" - three windows in a row - as possible).

SENIOR HIGH LEVEL ACTIVITIES

Senior high students might like to address themselves to TAD problems such as "accessibility", "operator analysis", "proof" and "duality".

Accessibility

From a given state, only certain other states are *accessible*. These are the states that can be reached after a series of drops. How many of these accessible states are there for any given state? Characterize them. Does your representation method give you any insights into the problem of accessibility? What does "parity" have to do with accessibility? Put an upper limit on the number of drops required to transform a given state into some other accessible state.

Operator Analysis

Consider any finite sequence of drops to be an *operator*. (Use some method of distinguishing the 3 different types of drops. We will use "l", "m" and "r" for drops in the left, middle and right holes respectively.) How many "essentially different" operators are there? (There are 128.) Given any operator (say lmlrmmrllmrllrll), can you find its "canonical" representation? To what extent are operators independent of states?

Proof

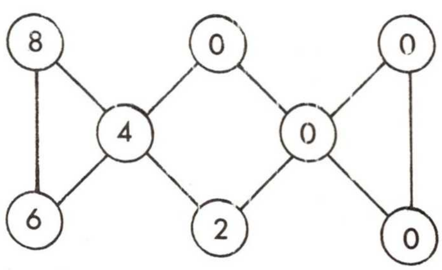
Prove the following "theorems" about operators:

1. $l^8=m^8=r^8=1$ (the identity operator).

2. $l^2m^2r^2= 1$.

3. lm=ml, lr=rl, mr=rm.

For example, to prove that $l^8=1$, consider the following diagram which shows the number of "window changes" brought about by 8 drops in the left hand hole.



Generalize Theorem 2.

Prove one part of Theorem 3 in 2 different ways.

Duality

Call states which differ in color in every window *dual states*. Call an operator which transforms a given state into its dual, a *dualizing operator*. Is the dual of a given state always accessible from that state? (Yes. Proof?) What is the minimum number of drops in a dualizing operator? Give an example of a state which dualizes in this number of drops. How many of these states are there? Characterize them. How many states can be dualized in 4 drops? Generalize. Prove. Define "inverse-operator". Which operators are self-inverse? What is the relation between self-inverse operators and dualizing operators?

UNIVERSITY LEVEL ACTIVITIES

The following activities involve concepts not usually encountered at the secondary school level. They might, however, provide an introduction to such concepts for the capable high school student. In fact the number of quite sophisticated mathematical concepts embodied by this so-called toy is surprisingly high.

TAD as computer

From a mathematical viewpoint all digital computers are finite state machines or automata. A *finite state machine* is "a five-tuple [A, S, Z, u, v] where A is a finite list of *input signals*, $A = a_0, a_1, \ldots, a_n$; Z is a list of *output signals*, $Z = z_0, z_1, \ldots, z_m$; S is a set of *internal states*, $S = s_0, s_1, \ldots, s_r$; u is a *next-state function* from SZA into S and v is an *output function* from SZA into Z (Birkhoff & Bartee, 1970, p.68)".

Can TAD be considered a finite state machine? If so, what are the values of n, m and r? Is it possible to construct a state diagram and a state table here? (See Berkhoff and Bartee, 1970, especially Chapter 3, for an elaboration of this topic.)

Group-theoretic aspects of TAD

Does the set of operators, G, form a group? If so, what is the order of the group and what properties does it have? Consider the set of self-inverse operators, H. Is H a group? Can you find a subgroup of H? Is this a normal subgroup? Why? What special set of operators is a subgroup of H?

Consider the Abelian group, M, which has a presentation a,b,c: $a^8=b^8=c^8=a^2b^2c^2$. Is M isomorphic to G? (See Macdonald, 1970, especially Chapter 8 for an elaboration of this topic.)

Computer-simulation of TAD

TAD presents many opportunities for computing science students to practice their programming skills. One good project, particularly for students who have access to some form of visual display apparatus, is the programming of a computer simulation of TAD. It may also be interesting to consider TAD-like systems which differ in only a few ways from TAD.

What *Boolean* algebra aspects does TAD have? How is the question of accessibility related to the concept of *equivalence classes*?

CONCLUSION

In the preceding sections, an attempt has been made to substantiate the claim that there is more mathematical potential in some common teaching aids than is usually recognized. Although TAD may be a particularly rich situation, similar activities can be created centering on other aids. The mutual formulation and investigation of such activities is, in the opinion of the author, a most worthwhile pursuit for both mathematics teachers and mathematics students.

REFERENCES

Birkhoff, G. and T.C. Bartee, *Modern applied algebra*. New York: McGraw-Hill, 1970.

Macdonald, I., *The theory of groups*. Oxford: Oxford University Press, 1968.

Morley, A., "Mathematics as process", *The Mathematics Teacher*, 1973, 66, pp.39-45.