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Manipulative materials for teaching and learning mathematics

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Editorial

ALPHA AND OMEGA

This publication marks the end of one era and the beginning of another with respect to Mathematics Council, ATA (MCATA) publications. Beginning in 1964, MCATA adopted a policy of producing an *Annual* each year, whose purpose was to help keep members informed on major ideas in mathematics education and to give members practical ideas to help them do a first-class job in the classroom.

At a recent executive meeting, it was decided to abandon the idea of an *Annual* but not the philosophy of a major annual publication. The pressure to produce an *Annual* in the year for which it was scheduled and the feeling that the prominent dating of a publication might "outdate" it long before the material was out of date were the two major reasons for this decision.

Beginning with this publication, the Mathematics Council has embarked on a series of monographs to appear at the rate of more or less one per year. Monograph No. 2 is already in preparation. It will consist of teacher-developed activities keyed to the Alberta Curriculum Guides. Some plans have been made for Monograph No. 3 and an editor has been appointed.

The executive believes that this change will better reflect the purpose of the major MCATA publication without decreasing the service of the Council to its members.

PURPOSE OF THE MONOGRAPH

The general purpose of this first monograph is to demonstrate that many mathematical concepts at all levels can be taught with very simple manipulative aids. Mathematics teachers have always been ingenious at devising their own aids. This skill is as important today with budget restrictions on every side as it was years ago when school administrators thought that all you needed to teach mathematics was a textbook and a piece of chalk. The need for children to manipulate has been well-documented in professional literature.

The specific purpose of this monograph is to show the versatility of relatively simple and inexpensive manipulative aids. Some of the articles demonstrate horizontal versatility; that is, the author concentrates on a narrow range of grade levels and shows how an aid may be used to teach a number of different mathematical concepts at that level. Other articles demonstrate the vertical versatility of an aid; that is, they show how an aid can be used to teach mathematical concepts over a wide range of grade levels. Still other articles demonstrate both horizontal and vertical versatility.

OVERVIEW

It may be wise to begin with a word of caution. We should not use manipulative aids just for the sake of using an aid. The aid must fit into our objectives for our lesson and program. In the first article, *Robert E. Reys* outlines some very useful considerations for anyone using manipulative aids.

The horizontal versatility of the geoboard is well illustrated by *Werner W. Liedtke*. He outlines a number of geoboard activities covering a wide range of elementary school topics including number, operations, patterns, geometric figures, measurement, graphing, fractions and games of various kinds. *T.P. Atkinson* shows how popsicle sticks can be used to teach number, numeration, and geometric concepts.

Separate articles by *Sr. Marie Benoit* and *J.E. Kirkpatrick* outline aids for use in skill development. Sr. Benoit outlines some activities with wooden cubes while Joan Kirkpatrick suggests a number of activities and easily-constructed devices to provide computation practice.

The use of thumbacks in developing concepts of weight and probability is suggested by *W. George Cathcart*. *Mary A. Beaton* outlines 10 experiences for developing concepts of 3-D space with materials such as straws, pipe cleaners, paper, cardboard containers, cubes, and tin cans.

Beth Blackall gives a number of suggestions for open-ended activities with shapes cut from a 12" by 12" floor tile. The activities suggested demonstrate both the vertical and horizontal versatility of a very simple manipulative aid.

Logical thinking can be developed through the use of attribute blocks. An article by *James H. Vance* contains some valuable games and activities using attribute blocks.

Separate articles by *Thomas E. Kieren*, *Bill Higginson*, and *Alton T. Olson* clearly demonstrate that much mathematics can be derived from simple aids. Kieren develops a number of mathematical concepts from upper elementary school through high school using a simple rectangle. The toy, Think-a-Dot, is used by Higginson as a source of mathematical activities from elementary through university level. Olson outlines five methods, all involving simple aids, for representing a reflection in the plane.

This monograph ends with a bibliography compiled by *Bill Higginson*. Let us not forget that books are invaluable aids in teaching mathematics.

THANKS

The editor is very thankful for the excellent cooperation received from all contributors. In particular, thank you for taking the time to write quality papers. A word of thanks is also due Hilda Lindae and the staff at Barnett House for their fine work in the final production.

Hopefully, mathematics teachers will find in this monograph a number of worthwhile aids to assist in the teaching and learning process, which is what education is all about. May the ideas presented here trigger your imagination with respect to the mathematics which can be found in many common concrete objects not mentioned in this monograph.

The Editor

Considerations for teachers using manipulative materials

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Considerations for teachers using manipulative materials

Classroom teachers of mathematics are witnessing an unprecedented period of proliferation in manipulative materials. Commercial catalogs list a great variety of available materials; professional journals carry many advertisements claiming that this device or that aid will provide a panacea for learning a certain mathematics topic; and professional meetings are frequently inundated with exhibits displaying new manipulative materials. This influx of newly available materials has precipitated many problems. The wide range of quality found among various materials has made the problem of selection much more difficult. It has made it impossible to list all available materials and discuss the merit - or lack of merit - of each. It has created doubts in some teachers' minds about the educational value of the materials. It has raised additional teacher-oriented questions such as, "What are some guidelines for selecting manipulative materials?", "What materials should be used?", "What are some dos and don'ts of using them?"

During the decade of the '60s, several fine articles appeared discussing considerations in the selection of learning materials (Berger and Johnson, 1959; Bernstein, 1963; Davidson, 1968; Hamilton, 1966; Spross, 1964). The present article is limited to a discussion of manipulative materials as opposed to other teaching aids. Furthermore, it is addressed specifically to classroom teachers in an effort to provide some current rationale, as well as guidelines, for the selection and use of manipulative materials.

WHAT ARE MANIPULATIVE MATERIALS?

The use of the term *manipulative materials* raises one fundamental question, namely, "Just what are manipulative materials?" In this context, manipulative materials are objects or things that the pupil is "able to feel, touch, handle, and move. They may be real objects which have social application in our everyday affairs, or they may be objects which are used to represent an idea" (Grossnickle, Junge, and Metzner, 1951, p.162). Hence, not all teaching aids or instructional materials are manipulative materials. Suffice it to say here that manipulative materials appeal to several senses and are characterized by a physical involvement of pupils in an active learning situation.

RATIONALE

In teaching mathematics, we are primarily concerned with concept formation as opposed to the memorization of facts. The mental processes involved in concept formation are much more complex than those associated with the memorization of a mass of isolated details. There is little disagreement among contemporary psychologists regarding the role of concept formation in the learning of mathematics. However, there are several existing theories about how to best foster proper concept formation. The results of recent psychological investigations into the ways children learn mathematics by men such as Jerome Bruner, Zolton Dienes, Robert Gagné, Jean Piaget, and Richard Skemp are beginning to have an influence on mathematical pedagogy. In short, more is known today about the way children learn mathematics, and the general nature of the mathematics they are capable of learning at various stages, than has previously been known. Ironically, it is still not known precisely how children learn, but the efforts of researchers are continually providing new evidence to suggest (and oftentimes refute) various learning theories. Since learning is an individual matter and invariably dependent on numerous factors, some of which are quite elusive, it is highly unlikely that a comprehensive learning theory that is completely satisfactory to all people will evolve in the foreseeable future.

A comparison of prominent learning theories will not be made here, but it seems appropriate to identify the following statements, subscribed to by most learning psychologists:

1. Concept formation is the essence of learning mathematics.
2. Learning is based on experience.
3. Sensory learning is the foundation of all experience and thus the heart of learning.
4. Learning is a growth process and is developmental in nature.
5. Learning is characterized by distinct, developmental stages.
6. Learning is enhanced by motivation.
7. Learning proceeds from the concrete to the abstract.
8. Learning requires active participation by the learner.
9. Formulation of a mathematical abstraction is a long process.

This list is not exhaustive, nor are the statements independent. In fact, they are closely interrelated. Suffice it to say that the above statements the basic foundation underlying the rationale for using manipulative materials in learning mathematics.

Many prominent mathematics educators have strongly urged greater use of manipulative materials in teaching mathematics. The rationale for this emphasis seems educationally sound. Unfortunately, research studies in this area have "not been conclusive in either supporting or refuting the value of manipulative aids" (Beougher, 1967, p.31). Most of the questions cited by Brown and Abell (1965, p.548), such as, "Are there certain manipulative devices that lend themselves better to different methods of instruction?" and "Will a device help one child and hinder another?" are yet to be answered. One can only hope that quality research focused on manipulative materials and mathematics learning will provide some objective evidence relevant to the issues. In the meantime, classroom teachers are still faced with the problem of selecting and using manipulative materials in their classroom.

SELECTION CRITERIA

The rapid increase in available commercial materials has made the job of selection not only difficult but also more crucial as the market is flooded with products. There are many criteria to consider in developing and procuring manipulative materials. In order to simplify this discussion, only important criteria in two basic categories, namely, pedagogical and physical, will be considered. The proposed criteria are not exhaustive, nor is any hierarchy of importance suggested by the order in which they are discussed. Although some considerations are more significant than others, the relative importance attached to each criterion should be determined by the teacher. Any final evaluation of manipulative materials should weigh strengths and weaknesses against the educational potential.

Pedagogically there are many criteria to consider in selecting manipulative materials. One of the most important considerations is whether or not the materials serve the purpose for which they are intended. Furthermore, do these materials do something that could not be done as well or better without them? Since mathematics is mental, do the materials develop the desired mental imagery?

The following criteria should be included in any list purporting to identify pedagogical considerations in the selection of manipulative materials:

1. *The materials should provide a true embodiment of the mathematical concept or ideas being explored.* The materials are intended to provide concrete representations of mathematical principles. Therefore it is important that, above all else, the materials be mathematically appropriate.

2. *The materials should clearly represent the mathematical concept.* Concepts are embedded so deeply in some materials that few, if any, pupils extract relevant ideas from their experience with the materials. This problem is further compounded by materials that have extraneous distractors, such as bright colors, which actually serve as a hindrance to concept formation. These experiences result in an "I can't see the forest for the trees" complex. This is, of course, not all bad, as it requires pupils to cull out extraneous data, yet in many cases such materials serve more as a deterrent to correct concept formation than as an aid.

3. *The materials should be motivating.* There are many factors that ultimately contribute to motivation. Several of these, such as attractiveness and simplicity, will be discussed later. Materials with favorable physical characteristics will frequently stimulate the pupil's imagination and interest.

4. *The materials should be multipurpose if possible.* That is, they should be appropriate for use in several grade levels as well as for different levels of concept formation. Ideally, the materials should also be useful in developing more than a single concept. Such wide applicability is frequently achieved by using a portion or subset of materials. For example, logic or attribute blocks have much multiapplicability through the careful selection and use of pieces.

   This requirement should not preclude the procurement and use of materials designed exclusively for embodying one concept. In fact, if use of certain materials results in concept formation that is otherwise impossible, then such items should be considered. In other disciplines, such as science and physical education, considerable funds are spent on devices that teach a single concept. Shouldn't mathematics teachers have a similar opportunity? Besides, using materials (even those designed for one specific function) often suggests additional topics or concepts that might be explored.

5. *The materials should provide a basis for abstraction.* This underscores the importance of the requirement that materials correctly embody the concept. In addition, caution should be exercised to ensure that the concept being developed is commensurate with the level of abstraction needed to form the mental image. Care must also be taken to ensure that the level of abstraction is commensurate with the ability of the student to abstract.

6. *The materials should provide for individual manipulation.* That is, each pupil should have ample opportunity to physically handle the materials. This may be done individually or within a group, as circumstances dictate. Such manipulation utilizes several senses, including visual, aural, tactile, and kinesthetic. In general, the materials should exploit as many senses as possible. Compliance with this generalization is particularly important with younger children.

Physical criteria are important, since many sources of information available to teachers, such as commercial catalogs and brochures, describe physical features of the materials. A careful scrutiny of physical criteria would be helpful in initially screening manipulative materials. Among the physical characteristics to consider in selecting manipulative materials are the following:

1. *Durability* - The device must be strong enough to withstand normal use and handling by children. If and when maintenance is needed, it should be readily available at a reasonable cost.

2. *Attractiveness* - The materials should appeal to the child's natural curiosity and his desire for action. Materials in themselves should not divert attention away from the central concepts being developed. Nevertheless there are certain qualities - such as aesthetically pleasing design; precision of construction; durable, smooth, and perhaps colorful finish - that are desirable. Nothing can be more distracting than pieces of a tangram puzzle that do not fit properly or a balance beam that doesn't quite balance.

3. *Simplicity* - The degree of complexity is of course a function of the concept being developed and of the children involved, but generally the materials should be simple to operate and manipulate. Although the materials may lend themselves to a host of complex and challenging ideas, for example, the attribute or logic blocks, they should be simple to use. In an effort to construct and use simple devices, there is the inherent danger of oversimplifying or misrepresenting the concept. In all cases, care must be taken to ensure that the device properly embodies the mathematical concept. In addition, the design of materials should not require time-consuming, mundane chores such as distributing, collecting, and keeping an extensive inventory record of a large number of items.

4. *Size* - The materials should be designed to accommodate children's physical competencies and thus be easily manipulated. Storage is an important consideration directly related to size - no device should take up more than a reasonable amount of storage space. Suitability of size is also important in preventing misconceptions or distorted mental images within the child's mind.

5. *Cost* - The index used to assess the worth of materials must ultimately weigh their use against cost. In this context, cost is used in a broad sense. Thus cost must include the initial expenditure and maintenance and replacement charges based on the life expectancy of materials under normal classroom use. The teacher-related cost, a function of the time required to learn how to use the materials effectively, is an item of utmost importance. It is not uncommon for someone other than a classroom teacher to order mathematics materials; but without proper planning, orientation, and preparation, it is ludicrous to expect teachers to use new materials effectively with their pupils. Therefore, *any purchase of new materials should be accompanied by a planned program designed to familiarize the teacher with these materials.* As a result, any cost estimate for manipulative materials should reflect the teacher-education phase as well as the expenditure for materials.

Teachers are often confronted with the dilemma of whether to use commercial or homemade manipulative materials. Many manipulative materials are relatively easy to make and can often be produced by the teacher and/or pupils. There are many priceless, intangible by-products, such as additional mathematical insight and increased motivation, that result directly from classroom projects. Nevertheless, one should weigh production costs for homemade materials, including labor, materials, and so on, against the cost of similar commercial products. Quality, of course, is another consideration. Frequently there is a marked difference in quality between homemade and commercially produced materials.

It would be ideal if manipulative materials could meet all the aforementioned criteria. Finding such materials would be tantamount to finding a "fish that runs fast and flies high". Consequently the search continues. It is hoped, however, that these criteria will provide teachers with some guidelines for both the selection and the use of manipulative materials.

USING MANIPULATIVE MATERIALS

There have been several fine lists summarizing uses and functions of teaching aids. Many such lists apply specifically to manipulative materials. Among the most common uses of manipulative materials are the following:

1. To vary instructional activities,
2. To provide experiences in actual problem-solving situations,
3. To provide concrete representations of abstract ideas,
4. To provide a basis for analyzing sensory data, so necessary in concept formation,
5. To provide an opportunity for students to discover relationships and formulate generalizations,
6. To provide active participation by pupils,
7. To provide for individual differences,
8. To increase motivation related, not to a single mathematics topic, but to learning in general.

From this list, it should be evident that manipulative materials may be used in a variety of ways. It should also be noted that the mere use of manipulative materials does not ensure that they are being used properly. Manipulative materials must be used at the right time and in the right way if they are to be effective. Failure to select appropriate manipulative materials and failure to use them properly can destroy their effectiveness. Some specific dos and don'ts for teachers who plan to use manipulative materials follow:

1. *Do consider pedagogical and physical criteria in selecting manipulative materials.* A prerequisite for effective use of manipulative materials is their appropriateness. The physical criteria for manipulative materials as well as the pedagogical considerations should not be taken lightly.

2. *Do construct activities that provide multiple embodiment of the concept.* It is difficult, and often foolhardy, to abstract or generalize from a single experience. Thus the pupil should be presented with different situations manifesting the concept or structure to be learned. For example, in developing the concept of three, children might examine sets with three elements for one activity. The number line, balance beam, and Minnebars might also be used to provide different embodiments for the same concept. The case for multiple embodiment has been ably presented by Dienes. Although the idea is pedagogically sound, it has yet to receive widespread use by classroom teachers.

3. *Do prepare in advance for the activity.* Be sure you, as the teacher, use the manipulative materials in the complete activity before they are used by pupils. As you make this trial run, you should consider questions such as: What prerequisite skills are needed before these manipulative materials are introduced? Are the directions clear, and can they be easily followed? Are there an adequate number of leading questions? Are the manipulative materials commensurate with the level of the pupils and appropriate for the mathematical concept? What are some potential problem areas, and how might they be alleviated?

4. *Do prepare the pupils.* The type of preparation depends on both the manipulative materials being used and the age of the pupils. Above all else, be sure the pupils are ready to profit from experience with the materials. Care should be taken to provide the necessary directions for beginning the activity. One must guard against telling pupils precisely what to do with the materials, as this might sterilize the learning experience. On the other hand, sufficient direction should be provided to prevent mass confusion, which may quickly lead to discipline problems.

5. *Do prepare the classroom.* Check to ensure that all required materials are on hand. Also be sure they are operative, accessible, and available in sufficient quantity. The arrangement of the classroom furniture should be examined to ensure that it is suitable for the planned activities.

6. *Do encourage pupils to think for themselves.* The use of manipulative materials in an informal situation provides an ideal climate for creativity, imagination, and individual exploration. This atmosphere encourages pupils to think for themselves. However, in order to get students to begin and then continue to think for themselves, it is imperative that the teacher provide encouragement of and show respect for pupils' ideas. A teacher's dismissal of a student's idea as being trivial, incorrect, worthless, and so on, will repress future ideas.

7. *Do encourage group interaction.* Discussion within, as well as among, groups can be intellectually stimulating. Encourage students to communicate with their peers and teacher. The importance of having this opportunity to tell one's thoughts, observations, and ideas cannot be overestimated. As pupils grow older, this freedom to express personal ideas is accompanied by a responsibility to defend or at least support a position, should the need arise. Some teachers fear that one student will dominate a group of peers. This may sometimes happen; however, the careful selection of group membership can keep this problem at a minimum.

8. *Do ask pupils questions.* It is often essential that certain points be called to the pupils' attention. Sometimes big ideas are missed completely. Other times one child may divert group attention to some minor or obscure point. In either case, you, as the teacher, must be prepared to ask pertinent leading questions.

9. *Do allow children to make errors.* Some may view this as heresy. However, children must have an opportunity to be wrong or to make a mistake. Often greater learning and more lively discussion follow an incorrect answer than a correct one. Besides, the natural learning process is characterized by much trial-and-error learning. To do otherwise, that is, to attempt to eliminate incorrect answers or faulty speculation, is to create a highly artificial learning situation.

10. *Do provide follow-up activities.* Discussion, correlated readings, reports, and projects, as well as replications of activities, enhance the prospects of learning. Searching questions forcing pupils to further analyze and synthesize their results can be very productive, as they encourage students to "pull together the loose ends". They might be followed by additional questions that require extrapolation from these activities and encourage speculation on the outcome of other related events.

11. *Do evaluate the effectiveness of materials after using them.* Immediately upon the completion of an activity, it can be very helpful to note particular problem areas, strengths, weaknesses, and suggestions and to define areas of needed improvement as well as possible areas of modification. A continuous reevaluation of manipulative materials ultimately results in better materials as well as more effective use of them.

12. *Do exchange ideas with colleagues.* Many new functions of manipulative materials result from actual classroom use. Sometimes pupils either consciously or unconsciously propose additional uses. At times, informal exploration with manipulative materials by either teacher or children suggests new activities, which adds to the reservoir of potential uses for this set of manipulative materials. A mutual exchange of ideas among teachers allows each to profit from the experience of the others. Perhaps you remember the fable: If I have a dollar and you have a dollar and we exchange dollars, we both still have a dollar. However, if I have an idea and you have an idea and we exchange ideas we both now have two ideas.

Now for some teacher don'ts!

1. *Don't use manipulative materials indiscriminately.* Care must be taken to ensure that these materials properly embody the mathematical concept being developed. Be sure the materials and concept are commensurate with your objectives and the pupils' level of development.

2. *Don't make excessive use of manipulative materials.* They should be used only when they represent an integral part of the instructional program and when the program could not be achieved better without the materials. One exception to this might be manipulative materials that are directed more toward recreation. There are instances where the traditional curriculum fails to reach many pupils. Often the recreational aspect of manipulative materials has attracted the attention of these youngsters and eventually paved the way to more academically-oriented activities. Some teachers fear that excessive use of manipulative materials will lead to overdependence on physical representations. There are cases where the manipulative materials are used as "crutches". However, most pupils will gradually stop using the materials when they have reached a higher level of development. Signs of boredom from the children may indicate excessive use of manipulative materials, or may suggest the need for raising additional questions or extending the concepts being explored with the manipulative materials.

3. *Don't hurry the activity.* Once the concept has been developed, most children are eager to explore other ideas. However, every pupil should have ample opportunity to use the manipulative materials, thereby convincing himself of the principle or formulating the concept. Hurrying through the activity may impose unnecessary pressure on some pupils as well as creating a very artificial learning situation. Few individuals learn well when they are rushed. Some children may formulate the concept within minutes, whereas other children may require several days or perhaps months. Rushing children as they use manipulative materials does not solve the problem but rather compounds it.

4. *Don't rush from the concrete to the abstract level.* This is a sequel to the previous suggestion. Perhaps the most frequent error in using manipulative materials is the speed at which children are rushed from the concrete stage to the symbolic level. There seems to be some myth that you can't learn mathematics unless you are actually writing something, that is, working with symbols. This is, of course, nonsense! Most good mathematics at the primary level is done without symbolization. In fact, if serious consideration were given to Piaget's research, nearly all mathematics in the primary grades would be at the concrete stage. It must be noted that symbolization occurs quite late in concept formation. Symbols are reserved for describing or making a record of the concept or mathematical principle. Hence, they can only be properly used after the concept has been abstracted. Since the process of learning a mathematical abstraction is time-consuming, it is ludicrous (at least with most elementary children) to use manipulative materials for one or two days and then move directly to the symbolic level. The wrong kind of experience may result in the children's viewing manipulative materials as toys or entertainment, in no way related to mathematics.

5. *Don't provide all the answers.* In working with manipulative materials, pupils acquire experience in abstracting from a set of phenomena or a body of data.

As each child is actively involved in this process, conflicts frequently arise. One pupil has one answer, another child has a somewhat different result. Often the first reaction of the teacher is to settle the issue by providing the correct answer. It is difficult to resist the temptation to tell the correct answer, but resist the teacher must! To do otherwise is to discourage individual thought, squash natural curiosity to search for other solutions, promote dependence on the teacher rather than independence, and preclude further discussion of the problem, as everyone now knows the correct answer. On the other hand, you may decide to ask some leading questions; you may have the pupils explain their solution; you may wish to have them replicate the activity using the manipulative materials; or you may pursue some other alternative. Regardless of the option selected, the teacher must refrain from serving as the purveyor of truth and source of all knowledge. Remember that to children and adults alike, "The art of being a bore consists in telling everything."

CONCLUSION

Perhaps the best one can do is identify those materials that best meet the criteria and then concentrate on developing effective ways of using them. This requires several steps. First, the desired learning must be clearly identified. Then manipulative materials that will aid in the learning process need to be selected. The third step requires that these materials be integrated into an organized learning sequence, so that pupils progress from the simple and concrete to the complex and abstract. Only in this way can manipulative materials be an integral part of the mathematics education program.

Remember that manipulative materials are not to be considered a substitute for teaching, - something one uses in lieu of teaching. There is not now, never has been, and, it is hoped, never will be a genuine substitute for a good teacher who knows how and what children need to learn and when they need to learn it!

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The geoboard: a versatile instructional aid

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The geoboard: a versatile instructional aid

Manipulative materials play a useful role at most levels of mathematics instruction. Some can be used to predict or check answers when solving problems which involve the four operations; others can lead to the discovery of patterns and relationships; and still others, such as games, can aid the development of problem-solving strategies. It is the purpose of this article to illustrate, with specific examples, how the geoboard could be used in each of the settings described above.

Commercial geoboards come in many shapes and sizes. However, they are relatively simple to construct from a piece of plywood (approximately 6" by 6") and 25 finishing nails. Observations seem to indicate that a 5-nail by 5-nail geoboard is more than adequate and that it is advantageous not to paint or draw lines onto the piece of plywood (between the nails) since they seem to distract from the constructions created by the children. If the spacing of the nails is a little less or a little more than one inch, even a small rubber band can be used to create various ingenious designs. Larger spacings (1 1/2" or more) seem to be unsuitable for shorter bands, tend to restrict constructions to only one area of the board, and frequently result in broken rubber bands when they are stretched across the whole geoboard. A spacing of exactly one inch seems to lead to the adoption of that unit whenever discussions arise and tends to make the students less flexible in adopting an arbitrary unit and/or name which at times could be advantageous.

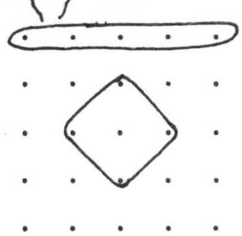
SUGGESTED ACTIVITIES

The suggestions which follow are meant to achieve two puposes:

1. Illustrate the versatility of the geoboard by describing some of its uses in a variety of topics and settings, and
2. Present some sample instructions and questions which could be used in developing activity sheets for students related to these topics.

Number - numerals

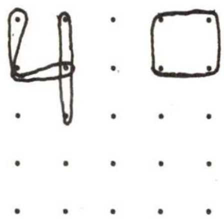
1. On your geoboard, in how many different ways can you put one rubber band around 5 () (or 6, or 7, etc.) nails? Record and display your results.
   Example:



   How are the arrangements different and how are they the same?

2. Put a rubber band around some nails and from these cards
   [2], [3], [4], [5], [6] ... choose the correct name for the number of nails enclosed.

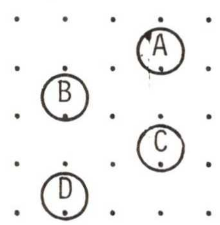
3. How many different numerals can you make on the geoboard with your rubber bands? Record and display.
   Example:



   Match the numerals with the appropriate number of nails.

Counting

1. If we call the nail labelled A, number 7, where we have started to count?



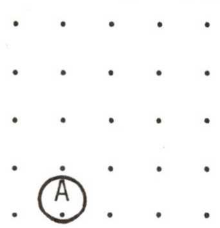
   Using the same starting point, what number would you assign to nail B?, C? and D?

2. If the number 14 is assigned to the nail labelled A above, how did we count? Where did we begin to count? What number would you now assign to nails B, C, and D?

3. Assign 18, 21, and so on to nail A and repeat the questions in part 2.

Ordinal number

The first task is to invent or agree upon a notation which would assign every nail on the geoboard a distinct number name. A variety of possibilities exist: (Row 2, Nail 4) or (R-2, N-4); (Up 2, Across 4) or (U-2, A-4) or (↑2, →4); (Down 2, Across 4) or (D-2, A-4) or (↓ 2, → 4); or an agreement is reached on the starting point and a simple number pair is used, for example, (2, 1) for the nail labelled A below.



After a notation has been agreed upon:

1. Give a name to all the nails on your geoboard.

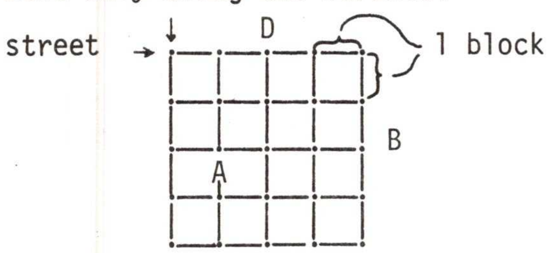
2. Use a piece of colored construction paper or wooden beads to mark the following:

   a. Both parts of the number pair names (first and last number for each nail) are even.
   b. Both parts of the number pair names are odd.
   c. The sum of the first and second number for each nail is 6.
   d. Both parts of the number pair names are the same.
   e. The second part of the number pair name is twice as large as the first part.
   and so on.

3. Use pieces of construction paper or beads to make a pattern on your geoboard. Write the number pair names for your pattern. What did you notice?

4. Pretend there are streets along the rows and columns on your geoboard.

The distance between any 2 nails (corners) is "one block". You are allowed to walk only along the streets.



a. How many blocks is it for the shortest walk along the streets from corner A to corner B? A to D? A to C? C to B?
b. How many different ways can you find to walk the shortest distance from corner A to corner B?
c. Write the address (number pair) for corner A. Write the address for B. Is there any way of finding the answer for (a) from the addresses? How? Does it work for the others? Does it work for any two corners on your geoboard? Try it.

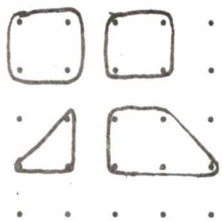
Addition/subtraction

Addition and subtraction problems involving simple basic facts could be simulated on the geoboard.

1. Put a rubber band around 8 nails. Put a rubber band around 7 *different* nails. How many nails are in both rubber bands? How do you know? Can you find another way of showing the sum? (regrouping by fives, or by tens and ones)

2. Put a rubber band around 13 nails. Put another rubber band around 6 of the 13 nails. Of the 13, how many nails have only one rubber band around them?

3. Using 2 rubber bands, how many different ways can you find to show the number 8? How are they different and how are they the same?

Example: 

Use the same technique for 9, 12, 6, and so on.

4. Repeat part 3 using 3 rubber bands.

multiplication/division

1. On your geoboard, show 7 groups of 3. What is the total?

2. On your geoboard, show 5 groups of 2. Now show 2 groups of 5. How are they different? How are they the same?

3. Put rubber bands across 3 rows on your geoboard. Take one rubber band and stretch it across one column. Count the number of nails which have 2 rubber bands around them, or the intersections, and record your results in the table.

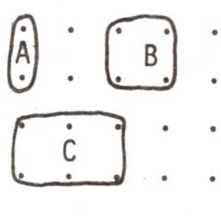
No. of Bands Across or Horizontal	No. of Bands Up and Down or Vertical	No. of Nails inside two Bands or Intersections
3	1	3
3	2	
3	0	
3	4	

Reorganize the data entered into the table (column 2 from smallest to largest). Do you see a pattern?

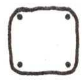
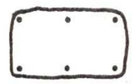
4. Put a rubber band around 16 nails. If you use 2 other bands to make 2 groups of the same size, how many would there be in each group? What if you were to make 4 groups of the same size, how many would there be in each group? How about 8 groups?

Patterns

1. Use this pattern.



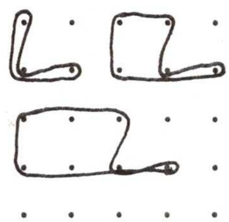
a. Can you tell what the next 3 (or 4) would look like? Construct them on your geoboard and record your results in the table. Describe the pattern.

	Sketch	Number of Nails
A		2
B		
C		
D		
E		

b. If you add any 2 numbers from the table above, can you make the sum look like the sketches? Try it on your geoboard.

c. If you were to multiply any 2 numbers from the table, will the product look like the entries in the table? Try it.

2. Use this pattern.



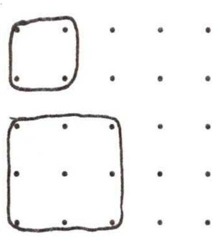
a. Find the next 3 (or 4). Construct a table.

b. Add any 2. Construct your result. What can you say about the appearance of the sum?

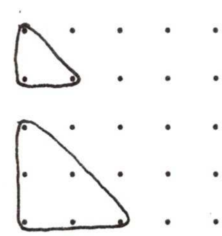
c. Multiply any 2. Construct the result. What can you say about the product?

3. Follow the same 3 steps for these patterns.

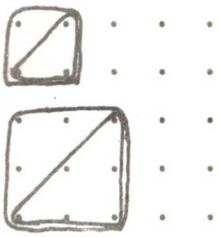
a.

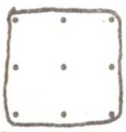
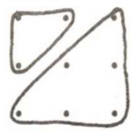


b.



4. Combining the 2 previous patterns results in the following pattern.



Sketch	Number	Triangles	Sum
	4		1 + 3
	9		3 + 6

Describe the pattern. Does the pattern work for other "square" numbers? Use your geoboard to find out.

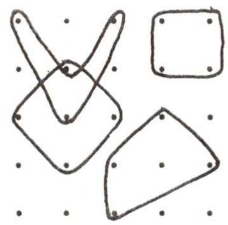
5. a. Take a number from the table in part 1 and add it to a number from the table in part 2. Display the sum on your geoboard. Can you make the sum look like an entry in the table of part 1 or part 2? Try other sums. What can you say about them?
   b. Take a number from the table in part 1 and multiply it by a number from the table in part 2. Display the product on your geoboard. Can you make the product look like an entry in the table of part 1 or part 2? Try other products. What can you say about them?

Figures

1. Familiar figures
   a. On your geoboard, make figures that look like something in your classroom; a kitchen; a store, and other places. Record, label, and display.
   b. Look at a figure someone else has constructed and try to guess what it could be after he has told you where it might be found.
   c. Does your figure look the same when you turn the geoboard? How many corners (sides) does your figure have? Can you make a figure with more (fewer) corners (sides)?
   d. Make a figure. Now use another rubber band and try to construct a figure which looks the same but is larger or smaller; longer or shorter; narrower or wider.
   e. Construct a figure which has 4 (or 3) sides which is long; short; long and wide; long and narrow; short and wide; short and narrow. Record and display your results.

2. Construct all the different 4-sided figures you can think of. Record, display and compare your results.

Examples:



3. Do the same as in part 2, but with 3-sided or 5-sided figures.

4. Construct different figures on your geoboard and record your results in the table.

Sketch	Number of Sides	Number of Corners
	3	3
⋮	⋮	⋮

What is the pattern? Does it always work?

Segments

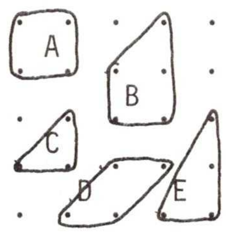
1. Try to make segments that are short; long; straight; "crooked". Record, display and compare your results.

2. Construct segments that touch; do not touch; will never touch; cross each other. Record, label, display and compare.

3. Construct various segments leading to 2 (or more) nails; which are equal in length; which are not equal in length.

Area

Call the size (area) of square A, one "unit".



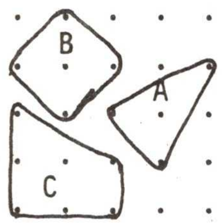
1. If figure A has an area of one "unit", use rubber bands to find the area of figures B, C, D, and E.

2. Construct "gardens" with an area of 2, 2 1/2, 3, 6, 7 1/2, and so on units. Label and display your results.

3. Construct as many gardens as you can which have an area of 8 (or 12, and so on) units. Sketch, label, and compare your results. How are the gardens different? How are they the same?

4. Construct more gardens but always leave one tree (nail) in the interior.

   Example:



   Find the size of the gardens you have constructed and record your results in the table.

Garden	No. of Posts (nails) used in the fence of garden	Size of garden
A	3	
B		
C		

   Build gardens with 6, 7, and so on fence posts. Don't forget the tree in the middle. Record your results in the table. Do you see a pattern? How could you predict the size of the garden from knowing the number of fence posts which were used? Does this work for bigger gardens? Try it.

5. Construct gardens which have no trees in the interior. Count the number of fence posts, calculate the area, and record your results in a table. Find a pattern for these gardens.

6. Construct gardens with 2 (3) trees in the interior. Count the number of posts used in the fence, calculate the area, and record your results in a table. Find a pattern for these gardens.

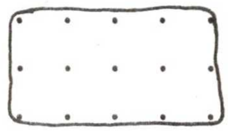
Perimeter

Call the distance between 2 adjacent nails in a row or column, one "unit".

1 unit

1. Use rubber bands to construct figures that have a fence which is 4, 6, 8, 10, and so forth units long. Record, label, display and compare your results. How are they different or how are they the same?

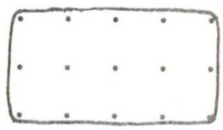
2. Construct as many figures as you can which have a fence that is 12 (or 8, or 16) units long. Record your results in a table and display.

Sketch	Length of fence
	12
	⋮

How are the figures different? How are they the same?

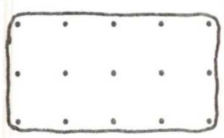
Area and perimeter

1. Construct a number of different rectangular figures with an area of 8 "units". Record your results in a table.

Sketch	Area	Length of fence or Perimeter
	8	
	⋮	

What can you predict about the area and perimeter of these figures?

2. Construct a number of different rectangular figures with a perimeter of 12 units. Record your results in a table.

Sketch	Dimensions	Perimeter	Area
		12	
		⋮	

What can you say about the dimensions and the area of these figures?

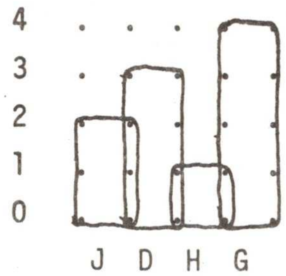
Graphs

1. John, Dick, Harry and George played hockey for the school team. The table shows how many goals each one of them scored in the last 2 games.

Name	No. of Goals
John	2
Dick	3
Harry	1
George	4

Use 4 rubber bands and your geoboard to show these results. Label your graph and display it.

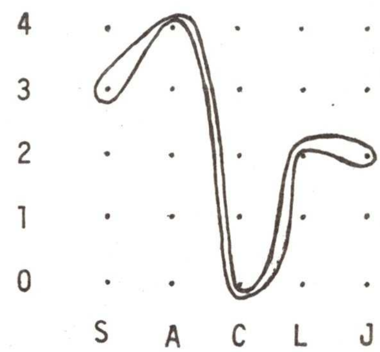
Example:



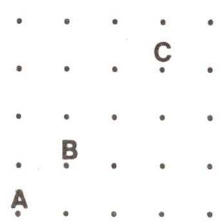
2. Collect other information (such as hair or eye color, days away from school), construct similar graphs and display your results.
Summarize the results shown in your graph by writing a story. Display the story. Which do you prefer, the graph or the story? Why?

3. Line graph. Use one long rubber band and your geoboard to show the results from the table. Label your graph and display it. Collect other information, construct similar graphs, label and display them.

Name	No. of Pets
Suzie	3
Ann	4
Caren	0
Linda	2
Joan	2



Notation



Consider the following directions ↑↓↗↙→←↖↘ and use the nail labelled A above as a starting point. Various directions can be taken to get from A to B.

Examples: A ↗
A →↑
A ↑→
A →↖→

Can you think of others? Try them.

1. Put a bead or piece of construction paper onto the nails which would be the end point of the following instructions:
   a. A →
   b. A → → →
   c. A → ↑ ↑
   d. A → → → ↑↑
   e. A → → ↑↑↑

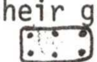
   What does this construction look like to you? Make up similar directions for a construction of your own. Write number pair names for a to e above.

2. Write as many different directions as you can for a route from nail A to nail C. Compare your answer with a friend. Which one is the shortest? Which one is the longest route?

3. Using the nail labelled A as a starting point, could you find a nail for A ← ↖ ? Why or why not? Could you find it if you were to use another geoboard? Make up some directional expressions which take you from A to a nail on the second geoboard.

4. Using the nail A as a starting point, could you find nails for A ↓ ↓ → or A ↓ ↓ ← ↙ ? Why or why not? How could you solve these problems?

Hide and seek (battleship)

Two contestants, or two teams, face each other in a game of "Hide and Seek". They construct on their geoboard, hidden from each other's view, an agreed-upon figure (ship),  , in any position they like. The task consists of each person trying to find the opponent's figure first, by taking turns and asking questions in terms of number pairs, for example, "Does your figure touch (1, 2)?" Positive responses could be marked with beads or pieces of construction paper. The person who has determined the exact location of the figure first is then declared the winner. (After each positive response try to consider all the possible locations for the opponent's figure. Try to determine which "shot" or position could give you the most information when it is your turn to ask the next question.) Try to find a champion for this game in your group or room.

Tic-tac-toe; four in a row:

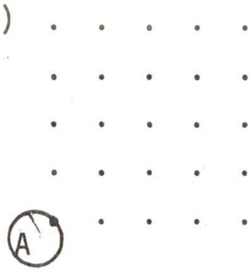
Two players take turns in placing different colored beads or pieces of construction paper onto the nails of the geoboard. The first one to get four in a row, column or diagonal is the winner. Are there any good (bad) moves? Why? Who is the champion in your group or room?

Letter - messages (codes)

1. How many different letters of the alphabet can you make on your geoboard? Are there any that cannot be made? Try as many as you can, sketch, display and compare your results.

2. Use rubber bands and follow these directions:

call A, (1,1)



a. Join (1,1) to 1, 3)
and (1, 3) to (2, 1)
and (2, 1) to (2, 3).

b. Join (3, 1) to (3, 3)
and (3, 3) to (4, 3)
and (4, 3) to (4, 1)
and (4, 1) to (3, 1).

What does a and b say? ____________________

3. Make up directions for a "secret" message of your own. Let a friend try to figure it out.

Treasure hunt

1. a. Join (2, 2) to (4, 2).
   b. Join (3, 1) to (3, 3)
   and (3, 3) to (5, 3)
   and (5, 3) to (5, 1)
   and (5, 1) to (3, 1).

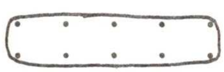
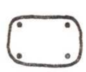
   The treasure is hidden where a and b intersect. What is its location?

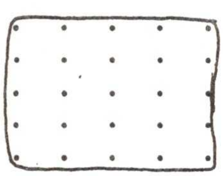
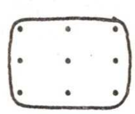
2. Make up directions for a treasure hunt of your own and let your friend try to find it.

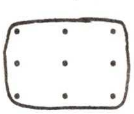
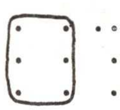
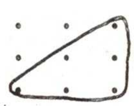
Fractions

By assigning the value of "one whole" to either a row of nails, a row of squares or to the whole geoboard, various activities can be designed to illustrate:
1. fractional notation or the meaning of fractions,
2. equivalent fractions,
3. addition of fractions.

Examples:  = Unit, then  = ? etc.

 = Unit, then  = ? etc.

 = Unit, then  = ? etc.

 = Unit, then  = ?,  = ? etc.

CONCLUSION

Some suggestions for activities have been made to illustrate that the geoboard can be used at any grade level for a great variety of topics. The aid has various other advantages: it withstands wear and can be constructed by the students, at least at the upper elementary level; concepts can be classified or reinforced through demonstrations, or the discovery method can be employed to teach these concepts; various settings can be created which could develop a student's appreciation of mathematics, develop his interest, enhance his initiative, arouse his imagination and provide recreation and enjoyment.

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Popsicle sticks as a manipulative device

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Popsicle sticks as a manipulative device

The teacher of elementary school mathematics needs a supply of varied manipulative materials for her own demonstrations and the pupils' exploration of mathematical ideas. Two important criteria for the choice of manipulative aids are availability and versatility in use. Popsicle sticks are one type of aid which meets these two criteria admirably.

A popsicle stick of the kind discussed here is 11.5 cm. long, 1 cm. wide and 2 mm. thick. It is made of moderately soft wood, in natural color, and is quite well finished. Only very occasionally will it sliver and cause inconvenience to the user. The sticks are available from dairies which produce popsicles, at a price of approximately $1.35 per thousand, or from Moyer-Vico Ltd. for $1.65 per thousand. They are also available commercially in boxes of one thousand, under the trade name "Coffee Sticks" but the cost is nearly twice as much.

There are several physical embodiments of mathematical concepts that can be demonstrated with popsicle sticks. In this article, some suggestions are made for number, numeration and geometry. The ingenious teacher will discover other ways to develop these and other concepts.

NUMBER AND NUMERATION

One-to-one correspondence is an important part of the initial development of the whole number concept. Popsicle sticks form useful sets of objects because they lie flat on a child's desk and do not make excessive noise. The sticks may be used as elements of a set to be placed in one-to-one correspondence with the elements of another set, or they may serve as connecting lines between the elements of two different sets. The teacher can use the sticks to cast shadows on a screen with the overhead projector, thus demonstrating one-to-one correspondence between two other sets of shadow objects.

For numeration experiences, the teacher can provide the children and herself with popsicle sticks and rubber bands. To demonstrate the numeral necessary to indicate the number of sticks in a set, the teacher first of all groups by 10s, with a small rubber band around each 10 sticks, then groups the 10s into groups of 10, with a stronger rubber band around each 10-10, and so on until no further grouping is necessary. The numeral for the total number is easily determined by indicating the number of each kind of group, beginning with the largest kind of group and ending with the number of loose sticks left over.

Recognition of the number properties of sets, understanding of operations with whole numbers and use of the numeration system are woven together. The teacher must use her judgment as to when the sticks should be in loose sets and when they should be bundled in 10s and so on. Some examples of situations in which the sticks are loose follow.

1. The teacher sets out 6 paper cups. She asks a child to bring her enough sticks so that she has one stick for each cup.
2. Each child is asked to hold 3 sticks in one hand and 5 in the other. "How many sticks do you have all together?"
3. "Pick up 12 sticks. Lay them on your desk in sets of 3."
4. "Pick up 16 sticks. Make two equivalent sets with them."

Situations involving addition and subtraction where "carrying" and "borrowing" occur should be demonstrated with sticks bundled according to the numeration system. In such cases, the union and separation of sets provide the basis for the operations with numbers and the grouping provides the basis for the algorithms used. Two examples should help to clarify what is meant.

1. 87 + 45 = n

   In one place, the child has 8 bundles of 10 sticks and 7 loose ones. In another place, he has 4 bundles of 10 and 5 loose ones. If the two sets are put together, how many sticks will he have? The child is helped through the actions associated with the sentence as follows:

   7 + 5 = 1-10 and 2;
   8-10s + 4-10s and 1-10 = 13-10s;
   13-10s = 1-100 and 3-10s; and finally
   87 + 45 = 1-100, 3-10s and 2 = 132.

2. 105 - 59 = r

   The child has 1 bundle of 100 (10, 10s) and 5 loose sticks He must remove 5-10s and 9 loose sticks. The actions are associated with the sentences as follows:

   1-100 and 5 = 9-10s and 15;
   9-10s and 15 with 5-10s and 9 removed
   leaves 4-10s and 6 = 46.

When equivalent sets are involved, as in either multiplicative or divisive situations, the manipulation must demonstrate the distributive principle of multiplication and division over addition and subtraction. Again, two examples should help to clarify what is meant.

1. 4 x 35 = p

   The child has four equivalent sets of sticks, each set consisting of 3-10s and 5 singles. He is to combine them into one set and determine how many sticks there are. The actions are associated with the sentences as follows:

   4 sets of 5 singles = 20 singles = 2-10s;
   4 sets of 3-10s = 12-10s;
   12-10s + 2-10s = 14-10s = 1-100 and 4-10s, or more symbolically,
   4 x 35 = 4 x (30 + 5) = 4 x 30 + 4 x 5.

2. 42 ÷ 3 = q

   The child has 4-10s and 2 singles. The task is to separate them into 3 equivalent sets. One bundle of 10 can be assigned to each of the 3 sets. Then the remaining 10 is ungrouped and combined with the 2 singles. The 12 loose sticks are separated into 3 sets of 4, and each set of 4 goes with the one bundle of 10. Thus

4-10s and 2 = 3-10s and 12;
3-10s and 12 divided into 3 equivalent sets yields
1-10 and 4 in each set, or,
$42 \div 3 = (30 + 12) \div 3 = 30 \div 3 + 12 \div 3.$

GEOMETRY

Geometrical properties of equilateral polygons can be illustrated simply with popsicle sticks. Because the sticks lie flat on a plane surface, they can be manipulated easily. If permanent figures are desired, the sticks can be glued to railroad board or some other similar paper product.

The teacher can discover activities for the children through experimentation. Here are some suggestions:

1. Lay 3 sticks on the desk to form a triangle. Can you make triangles of different shapes?
2. Lay 4 sticks on the desks to form a quadrilateral. Can you make different shapes?
3. Can you use a fifth stick as a diagonal of the quadrilateral?
4. Can you make a figure consisting of several triangles fitted about a central point?

No attempt has been made to suggest the geometrical ideas that can originate from the activities. Any formalization of ideas might well ruin the effect of the experimentations.

CONCLUSION

In the teaching of mathematics, especially in the elementary school, it is important to look to simple materials for manipulative and illustrative purposes. Because popsicle sticks are economical, easy to obtain, relatively non-hazardous, easy to store and versatile, they should be in every elementary classroom.

Numbo jumbo

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Numbo jumbo

Wooden cubes are inexpensive, readily available[1] and versatile teaching aids. They can be used in developing concepts of number, operations, geometry, measurement, and other topics.

OBJECTIVES

This article outlines 8 games using wooden cubes, designed to develop concepts of number and operations on numbers. Specifically, the objectives of these games are to:

1. reinforce number combinations,
2. consolidate the interrelation of operations,
3. reinforce the identity elements of 1 and 0, that is, multiplication by 1 and hence division by 1; addition of 0 and hence subtraction of 0; and multiplication by 0,
4. give practice in the use of grouping symbols such as parentheses, and
5. provide enjoyment.

MATERIALS

The following materials will be needed:

1. six wooden cubes (dice) - two red with numerals 0 to 5, two yellow with numerals 4 to 9, two blue with numerals 7 to 12.
2. packs of instruction cards (for Game 8),
3. counters (any small objects which can be grouped).

The dice are best made from 5/8" cubes, but one inch cubes are satisfactory.

These materials are age-fair and non-insulting to older pupils who may be having trouble with simple number combinations. The simplicity of their design allows for flexibility of use at any age level.

GAMES

Game 1

Use one die (specified by teacher according to child's ability). Throw it. Name numeral. Make a group that has that many counters in it.

Game 2

Use two dice. Throw one. Look at the numeral. Put the correct number of counters beside it. Throw the second die. Put the correct number of counters beside it. Has one group more? If so, which one? Or, are they equal in number?

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[1]Wooden cubes of any desired dimension can be easily made in a school shop. They are also available commercially from Moyer-Vico Ltd. for $5.10 per box of 100 plain or $5.95 per box of 100 colored.

Game 3

Take 2 dice. Throw them. Which number is larger? smaller?

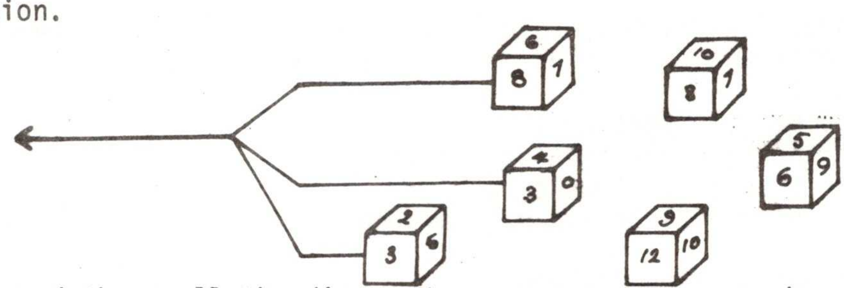
Take 3 or more dice. Throw them. Now put them in order, with the smallest on the left and the largest on the right.

Game 4

Take all the dice. Shake and throw them. Use as many as you wish to make an equation.

For example:

6 - 4 = 2



Game 5

Shake and throw all the dice. Use as many as you can to make an equation. You may use any signs such as +, -, x, ÷, or symbols such as = and (). The player who uses the most wins.

Game 6

Shake and throw all the dice. Use all of them to make an equation. Choice of symbols as in Game 5.

Note: Children become familiar with the identity properties 1 and 0 as they manipulate their dice.

Game 7

Shake and throw all the dice. Use all of them to make an equation.

A bonus is given for each different operation used, for example, 6 points for the equation; 1 point for each operation; 2 points for using a fraction.

Game 8

Use the pack of instruction cards. Shuffle it and put the pack face down. Each player in turn shakes and throws all the dice, picks the top card from the pile and follows the instructions for making his equation.

Examples of instructions:

- Include addition
- Include two different operations
- Include a fraction
- No addition.

Each pupil could make up his own set of instruction cards.

Simple computing devices for children to build and use

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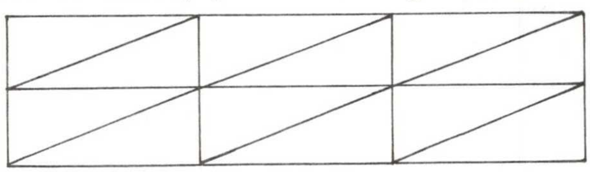
"... the conclusion is inescapable that children can study mathematics more satisfactorily when each child has abundant opportunity to manipulate suitable physical objects." (*Goals for school mathematics*, 1963, p.35)

Simple computing devices for children to build and use

The use of manipulative devices in teaching mathematics is becoming a widely accepted procedure and is supported by some research by learning theorists, and by practitioners - teachers themselves. Many teachers have long advocated that children need to manipulate a device or variety of devices before being required to abstract a concept or a model. Thus, there is available commercially an abundance of instructional materials designed to provide the concrete experiences necessary for the development of a child's concepts of number and space. One of the areas in which a large variety of aids is not available is that of computation. Yet, we live in a highly technological society in which computing devices or calculators play a major role; in addition, Man made use of computing devices even before numeration systems were fully developed. It is the purpose of this paper to outline a number of simple computing devices that elementary school children can build and use. One caution must be made: devices themselves will teach very little mathematics. It is their use under the guidance of a wise teacher that determines their effectiveness in learning.

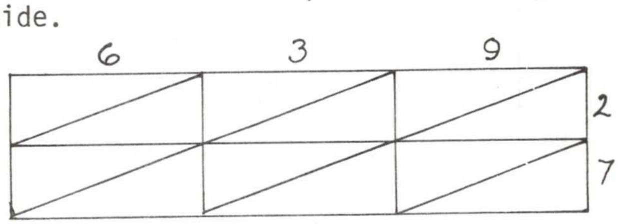
GELOSIA (LATTICE) MULTIPLICATION

The gelosia or lattice method is one of the very first methods Man used to release himself from the tedious work of multiplication. The device consists of equal sized cells, each divided into two parts by diagonals drawn from upper right to lower left. The number of cells depends on the number of digits in each factor. For example, to multiply a three digit number by a two digit number, six cells are needed:

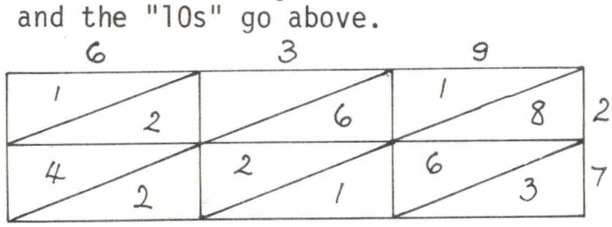


To multiply 639 by 27, follow these steps:

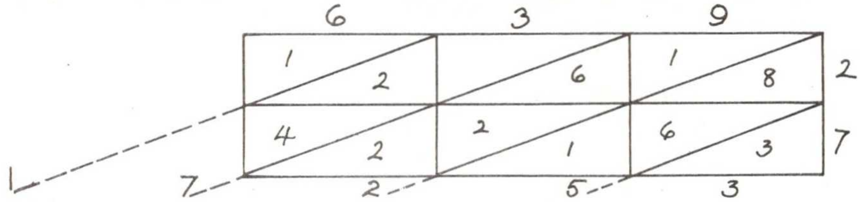
1. Place the digits of one factor at the top of each cell, and the digits of the other factor at the side.



2. Multiply each digit at the top by each digit at the side. The product is written in the cell corresponding to each pair of factors. If the product has one digit, it is written below the diagonal. If it has two digits, the "ones" go below the diagonal and the "10s" go above.



3. To find the product, add the numbers in each of the diagonals.



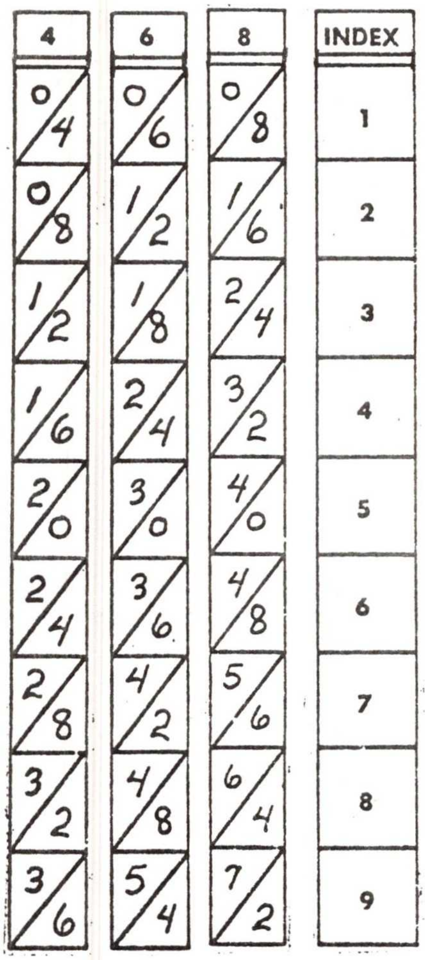
Pupils may also work on multiplying numbers having more digits, for example 368 x 472; 4562 x 835. A discussion of "why it works" should point out that the diagonals separate the numerals according to place value.

NAPIER'S BONES

In 1617, a Scottish mathematician, John Napier, developed a mechanical device that simplified the monotonous work of long multiplication. His method made use of a set of numerating rods called Napier's "bones", which were based on the gelosia or lattice method of multiplication.

To make a set of Napier's bones, use 11 strips of heavy cardboard. On 10 of them write the multiples of the numbers 0 to 9. The eleventh strip is used as an index rod and lists the digits 1 through 9. A completed set of bones is shown below. Notice that each one is actually a kind of multiplication table.

	1	2	3	4	5	6	7	8	9	INDEX
0/0	0/1	0/2	0/3	0/4	0/5	0/6	0/7	0/8	0/9	1
0/0	0/2	0/4	0/6	0/8	1/0	1/2	1/4	1/6	1/8	2
0/0	0/3	0/6	0/9	1/2	1/5	1/8	2/1	2/4	2/7	3
0/0	0/4	0/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6	4
0/0	0/5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5	5
0/0	0/6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4	6
0/0	0/7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3	7
0/0	0/8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2	8
0/0	0/9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1	9



To multiply 468 by 7, take the 4, 6, and 8 bones and the index and place them as shown at the left.

7 x 468 is shown in the seventh row. Add the numbers in the diagonals, as in the gelosia method.

7 x 468 = 3,276.

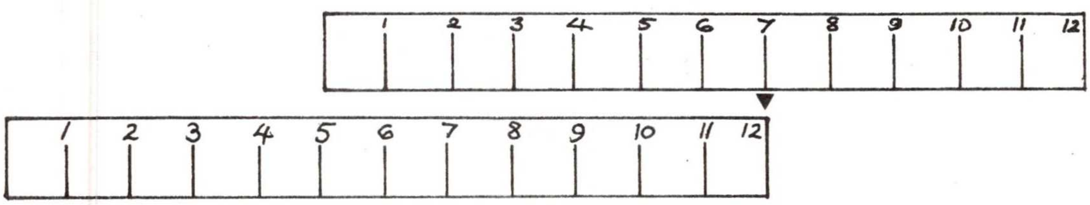
Pupils can discover how to use the bones to multiply by two and three digit numbers. The bones can then be used to check multiplication examples done the "ordinary" way.

SLIDE RULES

Addition and subtraction slide rule

Two rulers may be used as a very simple form of slide rule for addition and subtraction. Children can make their own by using two strips of heavy cardboard and marking a scale on each one, possibly using graph paper to assist them in making the scale.

To add 7 and 5, for example, place one strip above the other and "slide" the top strip to the right, until its left or 0 end is above the 5 on the bottom strip. Find the 7 on the top strip and look directly below it to the answer 12.

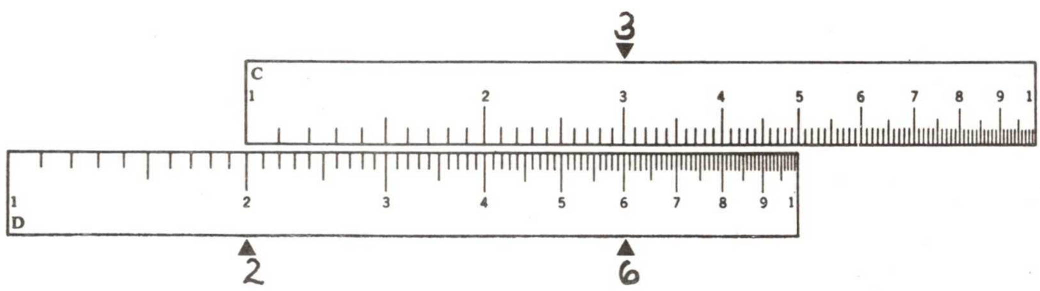


The strips can be used for subtraction by simply reversing the above procedure. Any scale may be used on these strips as long as the same one is used on both strips. For example, a scale going to 20 would allow practice on all the addition and subtraction facts through 18; a scale going to 50 would provide examples of addition and subtraction of two digit numbers through 50. Addition and subtraction of fractions and decimals may also be shown.

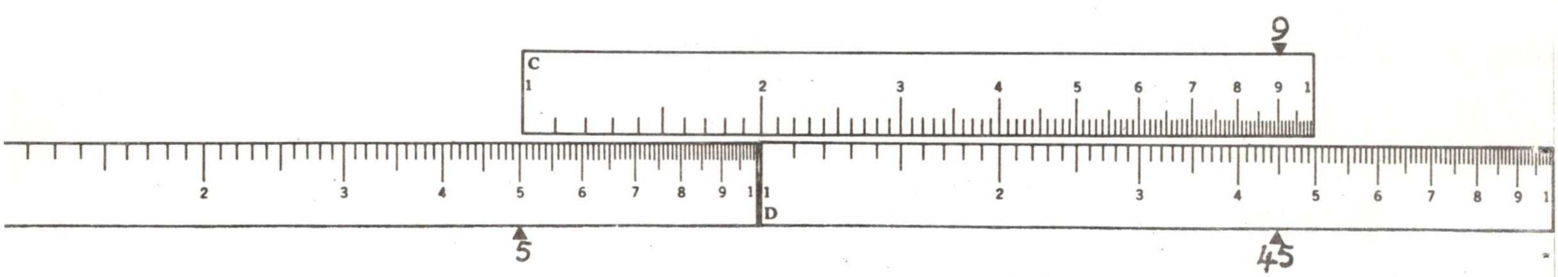
Multiplication and division slide rule

The basic mathematical idea behind this slide rule is the logarithm - an idea for which John Napier was also responsible. (A logarithm is an exponent, and exponential numbers are multiplied by adding the exponents. For example, $4^2 \times 4^3 = 4^{2+3} = 4^5$. On a slide rule, the exponents are represented by distances, and two numbers are multiplied by adding distances.)

The following diagram illustrates 2 x 3. Note that the "index" is now 1, the multiplicative identity, rather than 0, the additive identity.



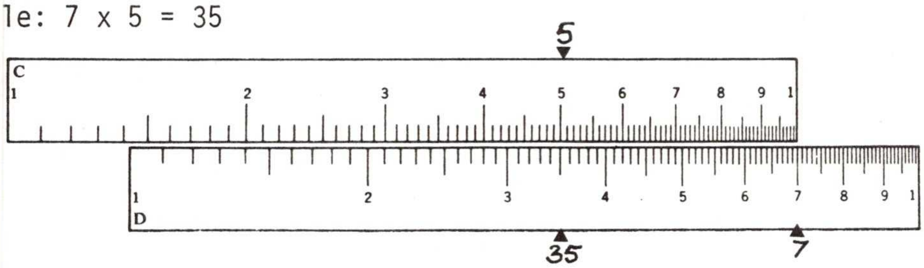
The strips show the scale for a multiplication-division slide rule. The scale goes from 1 to 10; to make a slide rule with a scale from 1 to 100, simply put two of the D strips end to end, as shown in the following example, illustrating, 5 x 9 = 45.



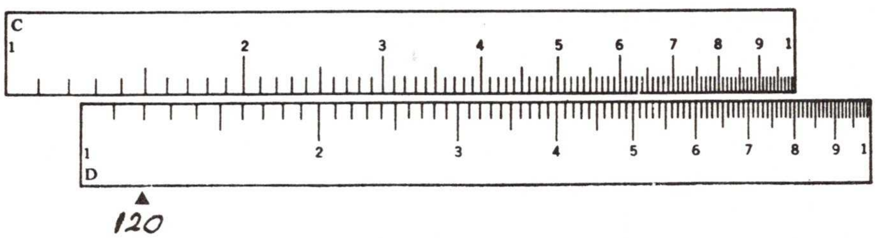
The second "D" scale can be thought of as representing the numbers from 10 to 100. To use this slide rule to provide basic fact practice, the products through 81 could be marked on the D scale.

The process of connecting the scales end-to-end to allow multiplication of any two numbers could go on forever, but it is not too practical, nor is it necessary. By sliding the C scale to the left rather than to the right, that is using the 10 on the right of the C scale as the index, all the products through 100 may be found:

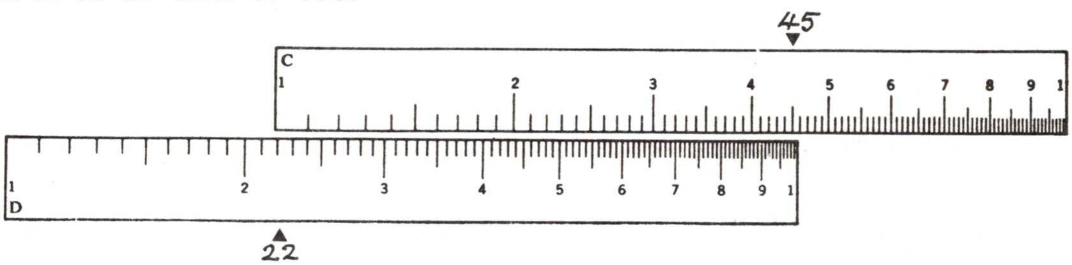
Example: 7 x 5 = 35



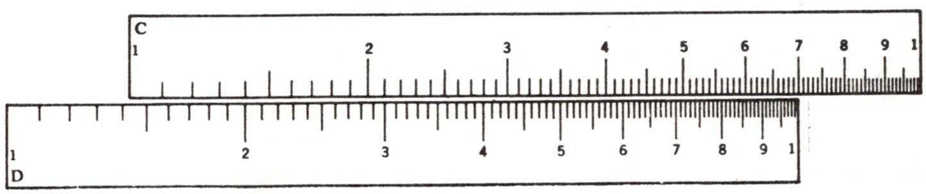
Multiplication involving a 2-digit number may be done in the same manner as above. For example, to multiply 8 x 15:



Finally, 2 2-digit numbers such as 22 x 45 may be shown. Note that the 99 is considered as 99 tens or 990.



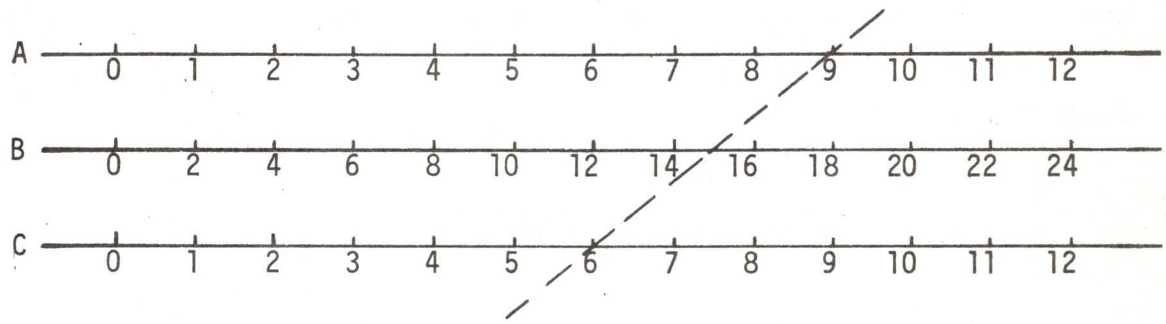
Division with this slide rule is just the inverse of multiplication. For example, to show 35 ÷ 5, put the 5 under the 35 and look above the right index to find the quotient, 7.



NOMOGRAPHS

Addition-subtraction

Devices called nomographs have been designed to allow a person to do rapid computing by just reading numbers from scales drawn on graph paper, or plain paper. One type of nomograph is shown below:



To add any two numbers, locate one on the top line and the other on the bottom line. Place a straightedge at these two points; the straightedge will cross the middle line at the sum. The diagram above illustrates 9 + 6 = 15. For subtraction, use the top and middle lines and read the difference from the bottom line. The diagram above also illustrates 15 - 9 = 6.

To make such a nomograph, start with three equally spaced, horizontal or vertical, parallel lines, A, B, and C. Mark off equal spaces on each line. On lines A and C, number corresponding marks with the same numerals. Mark each point on line B with a numerical value twice that of the corresponding marks on lines A and C. These lines are related to one another by the formula, Top + Bottom = Middle. Using this formula, you can construct a nomograph for any set of numbers you wish - naturals, integers, fractions, decimals.

Multiplication-division

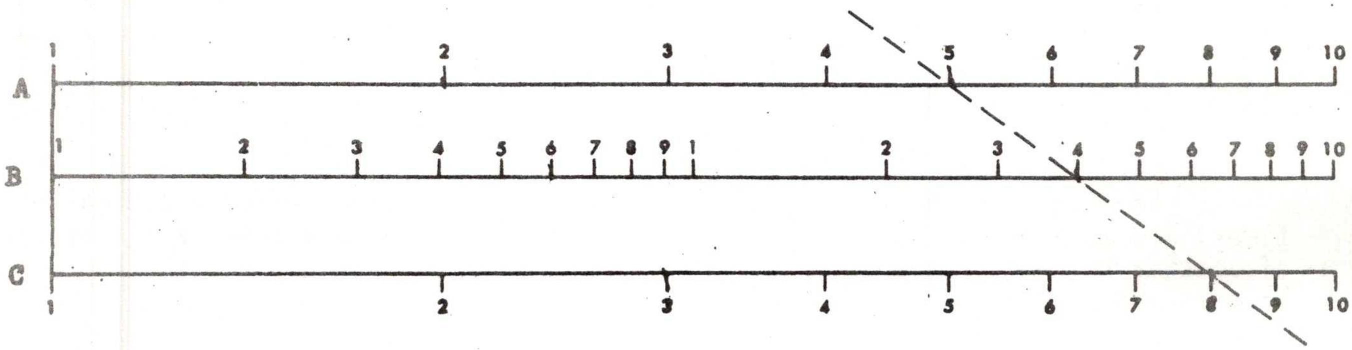
A nomograph that can be used for multiplication and division works in much the same manner as the addition-subtraction one. The diagram shows a nomograph for multiplication and division. Again there are three equally spaced, horizontal or vertical, parallel lines, A, B, and C. Lines A and C are marked off exactly like the scale of a multiplication slide rule. Line B has half the scale of the other two, thus has two slide rule scales in the same length that lines A and C have one.

To multiply any two numbers, locate one on line A and the other on line C. Join with a straightedge, which will cross line B at the product of the two numbers. These lines are related to one another by the formula, Top X Bottom = Middle. The following diagram illustrates 5 x 8 = 40. (Remember, from the slide rule, that the 4 means 4 groups of 10 or 40.)

Division on the nomograph is, of course, just the inverse. Locate the dividend on the B scale, the divisor on the A scale, connect the points with a straightedge which will cross the C scale at the quotient, the answer. Thus, the nomograph below shows 40 ÷ 5 = 8.

Note that you can find the square of any number on the A and C scale by looking at the corresponding point on the B scale, and you can find the square root of a number on the B scale by looking at the corresponding numbers on the A and C scales.

To make a multiplication-division nomograph, use the scale below or the scale from one of your slide rules. To use for basic fact practice, mark the products through 81 on the B scale.



POCKET CHARTS

Pocket charts are useful not only for computing, but also for developing place value and grouping by 10 ideas.

To make one chart with 4 pockets, use a 24" x 18" sheet of construction paper and fold as follows: measure down from the top 6 1/4", then fold UP. Then measure down from the fold, 1 1/4" and fold DOWN. Measure 3 1/4" down from the second fold, and fold UP. Continue measuring 1 1/4" and fold down, then 3 1/4" and fold up, until you have four "pockets". Staple or glue the folded construction paper to a heavy cardboard backing, 13" by 18", and trim the edges with black tape.

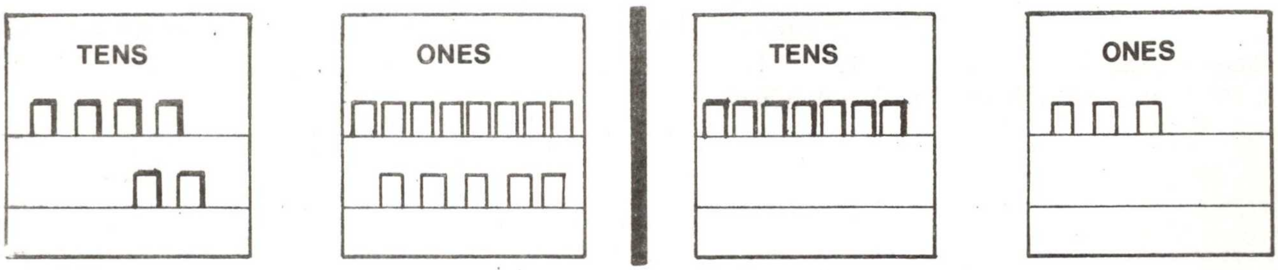
Three charts allow one each for ones, tens, hundreds; or for tenths, hundredths, thousandths if working with decimals; or for three places if working with other bases. Cards containing these titles could be made, and clipped onto the top of the chart with a paper clip.

Cut a supply of 1 1/2" x 3" cards for use as markers in the charts.

Small individual charts for each pupil are more easily made by folding a sheet of paper into thirds and labeling appropriately. Pupils can use toothpicks, popsicle sticks, or cardboard strips for markers.

The following figure illustrates an addition example and shows how the pocket charts help pupils see the meaning of regrouping. The example, 48 + 25, is shown with 4 bundles of 10 in the 10s place, and 8 ones in the ones place. 25 is shown with 2 bundles of 10 in the 10s place, and 5 in the ones place. The pupil puts all the ones together, then takes 10 of them and makes a bundle of 10 which he puts in the 10s place. Thus he sees 73 as the sum.

Borrowing or regrouping for subtraction can be similarly shown, as can multiplication and division.

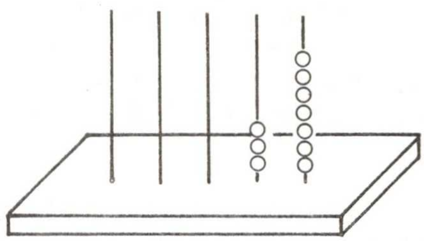


ABACUS

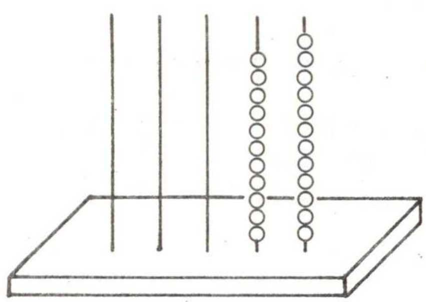
Some form of the abacus has been used as a computing device since earliest times, and is still in use in some countries today. Although there are many types of abaci available, perhaps the easiest for children to make and use is the spike or open-end abacus.

A 1" by 3" by 5" board or piece of styrofoam may be used for a base. Pieces of coat hanger wire can be pushed into the styrofoam, or put into holes drilled in the board. The wires should be uniformly spaced; 3 wires are sufficient for primary grades, and 5 or 6 for intermediate grades. Wooden or plastic beads are used to put on the wires.

To do an addition example such as 37 + 85, 7 beads are put on the ones wire and 3 on the 10s wire.



Then 5 beads are put on the ones wire, and 8 on the 10s wire.



Ten beads are then removed from the ones wire, and exchanged for one bead which is put on the 10s wire. Similarly, 10 beads are removed from the 10s wire and exchanged for one bead which is put on the 100s wire and exchanged for one bead which is put on the 100s wire. Pupils can see the result - 122.

Subtraction may be shown by reversing the steps for addition.

The wires can also represent decimal places, and illustrate computation with decimals in the same manner as above. If upper elementary students are working with bases other than base 10, the abacus can again be used to show place value, regrouping, and computation.

Some pupils may be interested in looking into the history of the abacus, and becoming familiar with the types of abaci still in use today in countries like Japan.

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What can you do with thumbtacks?

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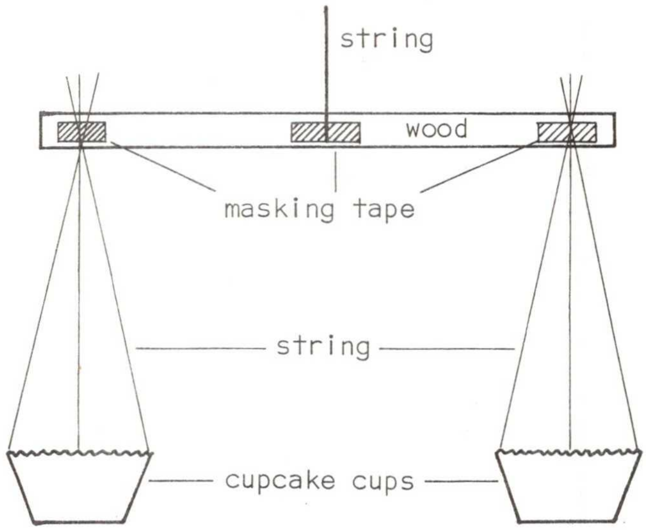
What can you do with thumbtacks?

Many common objects which teachers usually have in abundance in their desk drawers can become very useful manipulative aids. What can you do with thumbtacks in a mathematics class? Here are some ideas.

WEIGHT

In the early grades, initial experiences with any kind of measurement should be in terms of nonstandard units. Thumbtacks can serve as a unit of weight. The following type of activity card might be used in Grades II to IV. Children should work in groups of 2 or 3 children each. A balance is required for this activity. If your school does not have a pan balance you can easily construct one with string, foil cupcake cups, masking tape, and a piece of wood 15" to 18" long. The following diagram illustrates the assembly of the balance.



TACK WEIGHT

Objective: To provide initial experiences with measurement of weight.

Materials: Box of thumbtacks, balance, penny, chalk, pencil, other light objects.

Directions:

1. How many thumbtacks does it take to balance the penny?
2. One penny weighs about __________ thumbtacks.
3. Guess how many thumbtacks each of the following weigh:
   (a) a piece of chalk.
   (b) your pencil.
4. Weigh the chalk and your pencil and compare the weight with your guess.
5. Find some other small objects and weigh them with thumbtacks.

You may want your students to record their findings in a table. Pupils should discuss the weight of their pencils. Why were there differences here?

If you want to develop the use of standard units, you could ask the students to find the weight of one thumbtack. Metric weights (grams) work best. Unless you have milligram weights, this becomes a problem-solving activity because children will have to find out how many tacks are required to balance the one gram weight and then divide to find the weight of one tack.

PROBABILITY

Thumbtacks can also serve as a useful aid in working with probability at the upper elementary or early junior high school level. A laboratory activity like the following could be used.

THUMBTACK PROBABILITY

Objective: To determine the probability of the different ways a thumbtack may land.

Materials: 10 identical thumbtacks, paper or styrofoam cup.

Directions:
1. Toss one thumbtack several times. What are the possible ways the tack can land?
2. Estimate how many times a thumbtack will land in each position if tossed 100 times. Record your estimate in the table below.
3. Put the 10 tacks into the cup. Shake and dump the tacks onto the table. Count how many landed in each position. Record your results. Repeat 9 more times.

Outcome (How landed)	Estimate	Experiment number 1	2	3	4	5	6	7	8	9	10	Total

4. How close was your estimate to your experimental result?
5. Find a thumbtack with a smaller head. Will this make a difference? Experiment to find out.
6. Obtain a thumbtack with a longer stem. Experiment to see if this makes a difference.
7. What other things might change the probabiltiy?

PROBLEM SOLVING

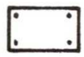
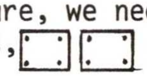
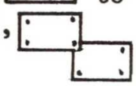
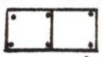
We are constantly looking for ways to place children into meaningful problem-solving situations. The following problem[1] could arise from an art lesson when the children want to hang their product. The problem is based on the assumption that all the art is done on identical rectangular paper. This problem could be attempted by students from Grade I through Grades VI or VII.

PICTURE HANGING

Objective: To provide an interesting problem-solving opportunity.

Materials: Thumbtacks, rectangular paper, equilateral triangular paper.

Directions:

1. To hang one picture, we need 4 tacks,  to hang 2 pictures, we could use 8 tacks,  although 7,  , would do and even 6, 
2. Every picture must be fastened by 4 tacks, one through each corner; no drawing may be covered by another except along the edge.
3. Complete the following table:

Number of pictures	Number of ways to hang	Smallest number of tacks needed	Largest number of tacks needed
1			
2			
3			
4			
5			
6			

4. Suppose your pictures were shapped like an equilateral triangle,  . Make a table like the one in number 3 for triangular pictures.

[1]This problem is adapted from Ernest R. Ranucci, "Thumb-Tacktics", *The arithmetic teacher*, 1969, 16, pp.605+.

FIGURATE NUMBERS

When your class is studying some topics from number theory, you could have one group of 2 or 3 children outline with tacks the triangular number on the bulletin board. (See Figure 1) Another group could illustrate the square numbers (Figure 2). If you have students who show a keen interest in figurate numbers, you could challenge them to find the series of pentagonal and hexagonal numbers and represent these on the bulletin board with thumbtacks.

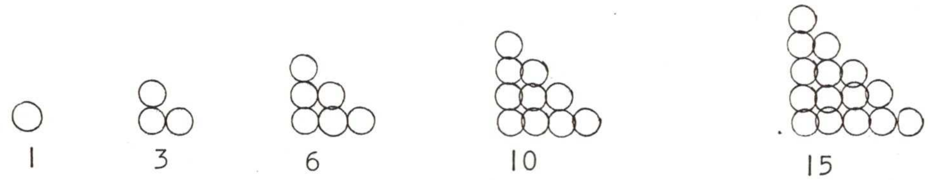


Figure 1 TRIANGULAR NUMBERS

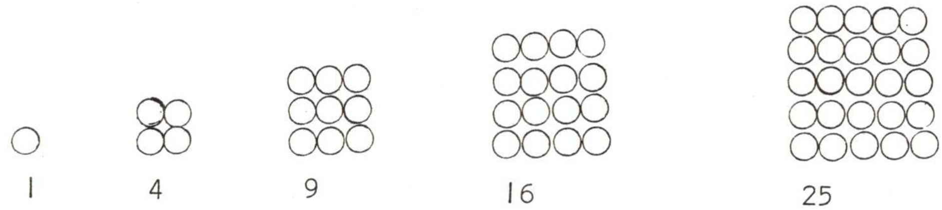


Figure 2 SQUARE NUMBERS

Manipulative aids for developing concepts of three-dimensional space

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Manipulative aids for developing concepts of three-dimensional space

In space or 3 dimensions, geometric shapes or figures are determined by surfaces which may be curved (for example, a sphere) or plane (flat, for example, a cube) in which case they are called *faces*. If the figure is determined by faces, these meet in *edges*, which are line segments. These edges may be used to determine the figure, which is then bounded by the plane, or flat faces through the edges. In the case of both 2- and 3-dimensional figures, the intersections of the edges fix the vertices of the figure. If the 3-dimensional figures are closed, then they enclose a region or volume of space [Elliott, MacLean, and Jorden, 1968, p.63].

Aids such as straws emphasize the edges and vertices of a 3-D figure. Paper models emphasize the faces of a polyhedron. Unit cubes are useful for emphasizing the filling of space. Many experiences are needed by children to help them develop the concept of volume.

<u>*STRAWS AND PIPE CLEANERS*</u>

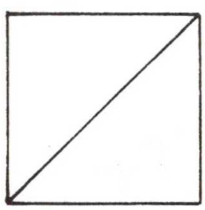
Experience 1

Each child is given a box of straws, pieces of covered wire or pipe cleaners and scissors.

1. Use 3 straws of equal length. What can you make using pipe cleaners to hold the straws together? Is it rigid or flexible? It is rigid. All triangles are rigid figures.

2. Use 3 more straws of equal length. Try to make a tent-like structure on your first figure. Is it rigid or flexible? This figure is called a tetrahedron and is rigid because it is made of triangles.

3. Use 12 straws of equal length. Make the skeleton of a cube. Is it rigid? To make a quadrilateral rigid insert a diagonal. How many diagonals must you insert in your skeleton of a cube to make it rigid? Try it and compare your results with your neighbor's. If you added a diagonal to each of the 6 sides, you have stiffened your cube. It is now divided into tetrahedra. How many?



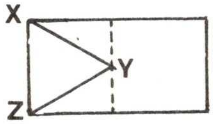
4. Use 8 straws of equal length. Use four of them to make a base. Is this base rigid? Cut another straw and insert it to make the square base rigid. Now use the other 4 straws to make a pyramid. Why is the figure rigid?

PAPER AND TAGBOARD

Experience 2

Give each child an envelope, scotch tape, and colored tagboard.

1. Johnson and Kipps (1970) suggest the following procedure for making a tetrahedron from an envelope. Seal the envelope. Draw an equilateral triangle, XYZ, on the end of the envelope using the end edge as one side. Draw a line through Y and perpendicular to the edge of the envelope. Cut along the dotted line. Fold along XY and ZY and bend back and forth several times. Pull out the open end of the envelope. XYZ will be one side of the tetrahedron. Use scotch tape to fasten the open edges together. Is the tetrahedron rigid? Why?



2. Cut out 16 equilateral triangles of the same size from colored tagboard. Make a tetrahedron. How many triangular shapes did you use? Draw a net or pattern for a tetrahedron. How many different patterns can you find?

3. Now use 8 of the triangular shapes. Fit them together and fasten them with scotch tape to make an octohedron. Make a net or pattern for an octohedron.

4. Cut out a square-shaped base which has a side equal to the side of the equilateral triangles. Make a pyramid using this square base and some triangular shapes. Draw a net or pattern for a square based pyramid.

5. Cut out 5 squares of equal size. Put them together to form an open cube. Try to draw 8 different nets or patterns for an open cube.

CARDBOARD CONTAINERS

Experience 3

Interesting cardboard containers can be found in supermarkets. J-cloths and straws sometimes are sold in hexagonal prisms. Certain types of rolled oats are sold in cardboard cylinders.

A most successful experiment for learning the relationship between the dimensions and volume of rectangular prisms was suggested by Marshal Bye at a Calgary conference. A child who has completed this experiment is not likely to forget it.

Children may work in groups. Each group is given a Rice Krispie carton. Each child is to make a box which has dimensions half as large as the dimensions of the carton. Each child guesses how many of the small boxes will be needed to fill the Rice Krispie carton. Each child records his guess before making his box. After the boxes are made, the children place as many of the small boxes as possible in the carton. How does the volume of each small box compare with the volume of the original Rice Krispie carton? This activity should be given after pupils have studied division of fractions.

Experience 4

Walter (1970) cut milk cartons to make open cubes. Make the height equal to the length and width of the base and cut off the top of the carton.

Numbered diagrams of the 8 possible patterns for an open cube are shown to the children. Each child decides which pattern he wishes to make and records its number. He must think how he should cut the carton to produce the pattern or net which he chose. The diagrams may be removed before the children cut the cartons. Have extra cartons available for children who are unsuccessful on the first attempt.

Experience 5

Mark off colored tagboard in square inches. Have each child make several open boxes so that you have a supply of boxes having the following dimensions in inches:

Box	Length	Width	Height
A	2	2	9
B	4	1	9
C	4	3	3
D	6	6	1
E	6	1	6
F	9	2	2
G	18	2	1

Fasten the edges with scotch tape. Have the squared side of the tagboard on the inside of each box. These boxes will be used in an activity described under unit cubes.

UNIT CUBES

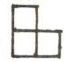
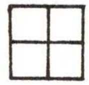
Cubes can be placed together to fill space completely. For this reason they are a valuable aid in the development of the concept of volume as the amount of space a solid occupies and as the amount of space in a hollow container.

Conservation of volume does not occur usually until a child reaches the age of 11 or 12 years. Informal experiences which a child has with unit cubes before that time will help him to reach conservation.

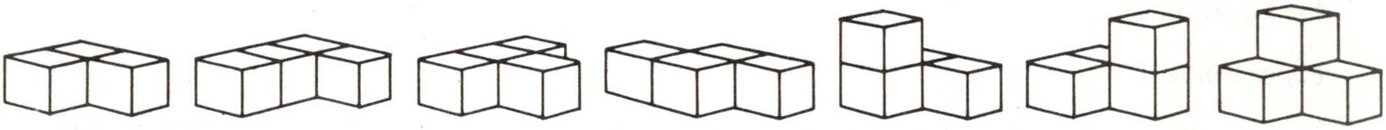
Experience 6

For this experience, each child needs about 33 unit cubes, 24 pieces of circular wood dowling, and 24 square-based wood prisms.

1. Try to build a wall of two thicknesses with each of the materials provided. (Nuffield, 1967). Record your results.

2. Patterns with cubes - How many patterns can you make with 3 cubes? Turn them around and see if they are really all the same. Call this Pattern 1:——→  How many different patterns can you make with 4 cubes? Try to find 6, 2 of which are mirror images of each other. If you have time, draw a picture of each of your patterns. These patterns may be found by beginning with Pattern 1 and adding one more cube in 6 different ways. The pattern  where the fourth block forms a square is not used.

In the next activity, these 7 patterns which you have found will be used again. Check yours with the illustration below:



1 2 3 4 5 6 7

Experience 7

The 7 pieces described in the second part of experience 6 form the SOMA puzzle. There are over 200 different ways to arrange the 7 pieces in a 3 x 3 x 3 cube. In order to determine whether a solution is the same as or different from others, always turn your 3 x 3 x 3 cube so that the L shape or Pattern 2 is on the top layer. Now place the L in one of the 4 positions shown below:

2	-	-
2	-	-
2	2	-

A

2	2	-
2	-	-
2	-	-

B

-	2	-
-	2	-
2	2	-

C

-	2	-
-	2	-
-	2	2

D

There are many different Type A solutions. To prove that you have found different solutions, make 3 grids for each solution and indicate in each square the number of the block found there.

Top Layer

Middle Layer

Bottom Layer

See how many different solutions you can find for Types A, B, C, D. Record each one and compare them with your neighbor's solutions.

Experience 8

Filling boxes with cubes.

Each child should have 40 blocks. The 7 boxes having the following dimensions were made in an earlier activity (Experience 5):

Box		Dimensions
A	-	2 x 2 x 9
B	-	4 x 1 x 9
C	-	4 x 3 x 3
D	-	6 x 6 x 1
E	-	6 x 1 x 6
F	-	9 x 2 x 2
G	-	18 x 2 x 1

Give each child the following table:

Box	Estimate of Number of Cubes Needed to Fill Box	Actual Number of Cubes Needed to Fill Box
A		
B		
C		
D		
E		
F		
G		

Have each child fill in all the estimates first, then use blocks to check. No mention need be made of dimensions. This experience highlights the fact that boxes of varied shapes may hold the same number of unit cubes.

TIN CANS

A variety of tin containers can be assembled by the children. These may include pop cans, small, medium and large fruit juice tins, salmon tins, squat size bean tins, large and small coffee tins, lard tins. Label each tin A, B, C, and so on. The following activity will assist children to understand another concept of conservation of volume, that is, the conservation of displaced volume (Copeland, 1970).

Experience 9

The following activity is adopted from Sawyer and Srawley (1957). Materials needed: 9 labeled cylinders, a large basin of water, balance and weights.

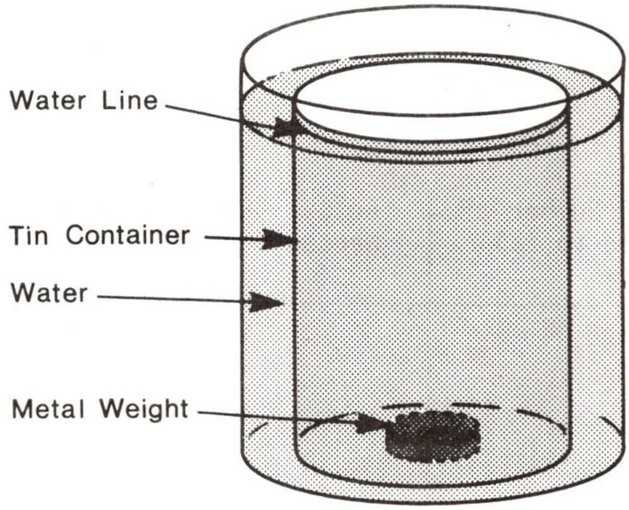
1. Fill each cylinder with water to the water line (see diagram below). Weigh each full cylinder and record the weight in grams in the table.

2. Empty each can and put enough weights in it until it nearly sinks in the basin of water. Mark the level of the water on each can. Record for each cylinder the weight in grams which was needed to just keep the tin afloat.

3. Measure and record the diameter of each cylinder.

4. Measure and record the height of each cylinder up to the water line.

5. Find the connection between the volume of each tin and the weight that almost sinks it. Do you need to consider the weight of the cylinder? Yes?



Cylinder	Wt. in Grams of Empty Cylinder	Wt. in Grams of Cylinder Filled with Water to Water Line	Total Wt. in Grams Needed to Nearly Sink Cylinder	Diameter of Cylinder	Height of Cylinder	Vol. in c.c. of Cylinder
A						
B						
C						
D						
E						
F						
G						
H						
I						

STYROFOAM

Hobby shops sell styrofoam in a variety of 3-D forms such as spheres of various sizes, cones, truncated cones (top cut off) and rectangular prisms.

Experience 10

Materials needed: 5 styrofoam cubes and a sharp knife. One size of cube which works well is a 3" x 3" x 3" but other sizes are possible.

1. Leave cube 1 intact.
2. On Cube 2, measure from each vertex, 3/8" along each side. Cut off the 8 corners as marked. This will be the first truncated cube; TC1.
3. On Cube 3, measure 3/4" from each vertex along each side. Cut off the 8 corners as marked. This will be truncated cube, TC2.
4. On Cube 4, measure 1 1/8" from each vertex along each side. Cut off the 8 corners as marked. This will be truncated cube TC3.
5. On Cube 4, measure 1 1/2" from each vertex along each side. Cut off the 8 corners as marked. This will be a cuboctahedron.

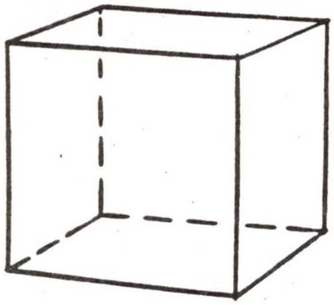
By observing the cube, the three truncated cubes, and the cuboctahedron, complete the following table:

Polyhedron	Shape of Faces			Number of Vertices	Number of Edges
	Square	Hexagonal	Triangle		
Cube					
Truncated Cube 1					
Truncated Cube 2					
Truncated Cube 3					
Octahedron					

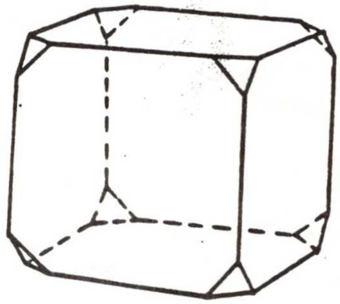
The diagrams on p.56 indicate that it is possible to begin with an octahedron and by paring off the eight vertices, to progress through truncated octahedra back to a cuboctahedron. (Guy, 1968.)

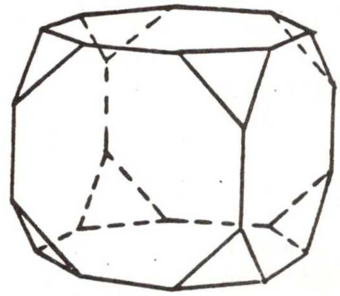
SUMMARY

The 10 activities described have emphasized the idea of the 3-dimensional object or shape occupying a space; the amount of space inside a container; conservation of volume and comparison of volume and capacity. These experiences are performed at the intuitive level as background for later work.

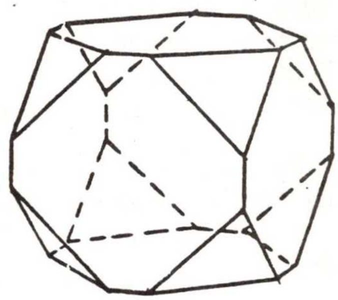


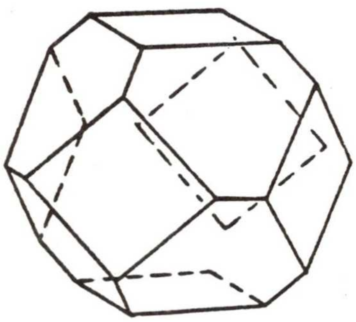
Cube

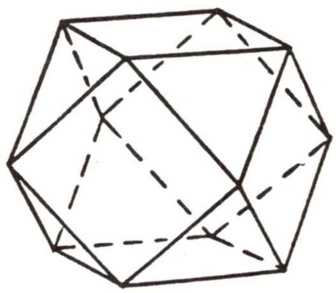




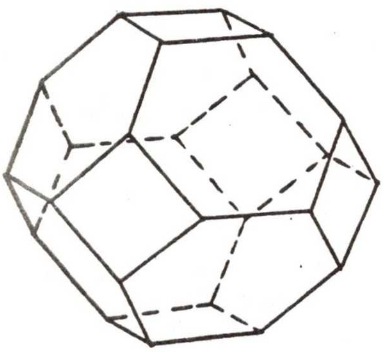
Truncated Cube



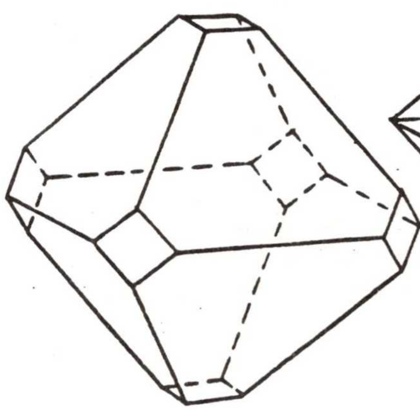


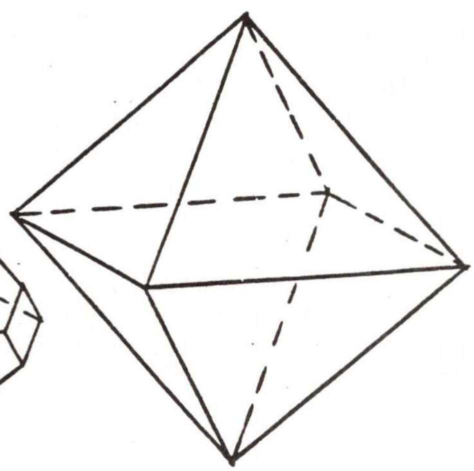


Cuboctahedron



Truncated Octahedron





Octahedron

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Anne Bernadette's tile

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Anne Bernadette's tile

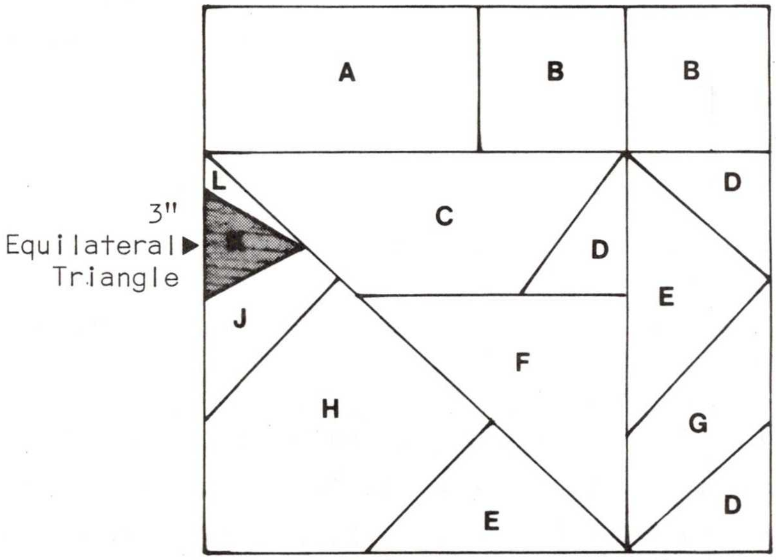
Anne Bernadette's Tile came into being when a young teacher from the area of business education was presented with a Grade VII mathematics class which did not seem to appreciate textbook teaching. Each pupil was asked to acquire a tile which he ruled and cut, a useful mathematical exercise in itself. The students enjoyed manipulating their sets of shapes as they learned about classification of shapes, area, perimeter, the identification and measurement of angles and algebraic equations. As a bonus, the set quite fortuitiously contained a 5-piece tangram.

The Tile has since been used by teachers from Grade I right through elementary school, to give children experience in matching and sorting, ordering, value relations, fractions, dissections as well as the topics previously mentioned. There seems to be no end to the variety of activities arising from the use of this simple set of shapes.

<u>*MATERIALS*</u>

A 12" x 12" floor tile, cut to the pattern below.

For the purpose of identifying the various pieces for the reader, the shapes have been labeled A to L.



SOME CHARACTERISTICS OF ANNE BERNADETTE'S TILE

1. The oblong (A), largest triangle (F) and the trapezoid (C) are equal in area.
2. The square (B), triangle (E) and parallelogram (G) are equal in area.
3. The triangle (E) together with two of the smallest triangles (D) will make up the oblong (A), the largest triangle (F) and the trapezoid (C).
4. The square (B) together with two of the smallest triangles (D) will make up the oblong (A), the largest triangle (F) and the trapezoid (C).
5. The parallelogram (G) together with two of the smallest triangles will make up the oblong (A), the largest triangle (F) and the trapezoid (C).
6. Two of the smallest triangles (D) will combine to form the square (B), the triangle (E) and the parallelogram (G).
7. The triangles (D,E,F) are similar.
8. The three shapes (J,K,L) can be rearranged to show the difference between area and perimeter.
9. The angles of the shape (L) are 120°, 45° and 15° or 1/3 turn, 1/8 turn and 1/24 turn.
10. The angles of the shape (J) are 45°, 90°, 105° and 120° or 1/8 turn, 1/4 turn, 7/24 turn and 1/3 turn.
11. The following pieces form the 5-piece tangram: square (A), triangle (E), parallelogram (G), 2 of the smallest triangles (D).

SOME ACTIVITIES

1. Take a big piece. Cover it with smaller pieces. Can you do it a different way? Is there still another way you can do it?
2. Start with any piece. Next to any side, place another piece so that the 2 edges are matched in length. Try to make your trail as long as possible. Example:

3. Share the pieces with a friend, and play the last game with him taking it in turns.
4. Sort (classify) the shapes according to number of sides, number of equal sides, number of square corners (right angles).
5. Choose a value for the small triangle (D). Work out what all the other shapes are worth (leave out J, K, L). Now choose another value for (D) and repeat.
6. Write equations using the numbers you worked out for the last activity.
7. After you know something about fractions, let the oblong (A) be worth one. Can you work out what some of the other shapes are worth? Do the same exercise when the trapezoid (C) is worth one.

8. Let the pentagon (H) be worth one; what are the other shapes worth? Can you write equations using these fractions as numbers?
9. Find 2 shapes equal in area but different in perimeter.
10. Make 2 shapes equal in perimeter but different in area.
11. Find pairs of shapes equal in both area and perimeter. In what respect are they different?
12. Order the shapes according to area from smallest to largest.
13. Order the shapes according to perimeter from shortest to longest.
14. Take the parallelogram (G). Give an accurate statement about its perimeter.
    [2 long (sides) + 2 short (sides)]
    [2 longs + 2 shorts]
    [2 L + 2 S]
    Using these terms, give accurate statements about the perimeter of the other pieces. (L and J are regarded as too difficult.)
15. Take the 5-piece tangram. Form these pieces into a square, a triangle, a parallelogram, a rhombus, a trapezoid, a pentagon, a hexagon.
16. Using any 3 of the tangram pieces at a tin, try to make the shapes listed in the previous activity. How many of the shapes can you make with any 4 of the pieces? Any 2?
17. Take the triangle (E) and the parallelogram (G). Make them into a 4-sided figure. How far around is this shape? Make the same 2 pieces into a 5-sided figure. How does the perimeter of this shape compare with that of the 4-sided figure? Now make a 6-sided figure with the same perimeter.
18. Make a dot on a piece of paper. Arrange some pieces so that their corners fill in a complete turn at that point. Try other combinations of pieces.
19. Find 4 corners of equal size, which together will make a complete turn around a dot. How much of a full turn does each corner measure?
20. Find 8 corners of equal size which together will make a complete turn around a dot. How much of a full turn does each corner measure?
21. Can you work out the fractions of a full turn (or revolution) that each corner on all the shapes measure?
22. Write an equation to show how much all the corners on any shape add up to. Can you see a pattern in your answers?
23. Use the following pieces to form a square: C, D, E, F, H, J, K, L.
24. If you tried to make the shapes left over from the previous activity into a square, would you succeed? What shape must you add?
25. What is the biggest triangle you can make? Use as many pieces as you like.

COMMENTS

1. The activities range from easy to difficult, so that Grade I pupils could do some, while junior high school students would be challenged by others.

2. The tile described is listed as 12" x 12". Now that Canada is going metric, a better dimension for the square may be 40 cm. Thus the equilateral triangle (K) and the square (B) would have 10 cm. sides.

3. A 7-piece tangram can be formed by adding to the 5-piece tangram, 2 of the largest triangle (F).

The use of attribute blocks: K - XII

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The use of attribute blocks: K - XII

THE MATERIALS

Attribute blocks or logic blocks can be obtained commercially[1] or constructed from wood[2] by the teacher or school shop personnel. A set based on four attributes - shape (triangular, circular, square, and oblong), color (red, yellow, and blue), size (large and small), and thickness (thick and thin) - would consist of 48 blocks, one for each possible combination of the variables (Figure 1). Obviously, sets with more or fewer pieces could be easily formed. For instance, by eliminating thickness as an attribute, the set described above would be reduced to 24 blocks. Or, if another shape for instance, pentagon, were added, the set would consist of 60 blocks. For certain activities, a set for young children might contain only 6 blocks (based on 3 shapes and 2 colors).

GAMES AND ACTIVITIES

The attribute blocks can be used to provide a physical setting for a variety of experiences[3] which develop or illustrate principles of logical reasoning. Many of the suggested games can be played at various levels of sophistiction, and are therefore appropriate for learners ranging in ages from 5 to 16 (and up). Young children can discover and use intuitively certain valid modes of reasoning, while older students may be able to analyze strategies, consider all possibilities, and develop proofs using words and symbols.

The blocks can also serve as the universal set for a number of problems in basic counting and probability.

Sorting

Purpose: To acquaint students with the structure of the materials; to develop classification skills.

Instructions: The blocks are placed randomly on a table (or on the floor), and the students are asked to sort them, that is, to place together those which they think are alike in some way. The result, depending on the level of the students, might be several disjoint sets (such as red, blue, and yellow pieces) or a two or three dimensional matrix (as in Figure 1). Although young children generally require and enjoy a fair amount of free play with the materials before settling down to a directed task, they will often begin sorting naturally when confronted with the blocks.

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[1]Most suppliers of materials and aids for school mathematics and science handle attribute blocks. In Edmonton, Moyer-Vico Ltd. sells various sets of blocks along with guidebooks.

[2]For mimeographed instructions describing how to make your own attribute blocks out of wood, write to: Dr. Dossey, Mathematics Department, 313 Stevenson Hall, Illinois State University, Normal, Illinois 61761.

[3]Many of the activities dealing with logic are discussed in Z.P. Dienes and E.W. Golding, *Learning logic, logical games* (New York: Herder and Herder, 1966).

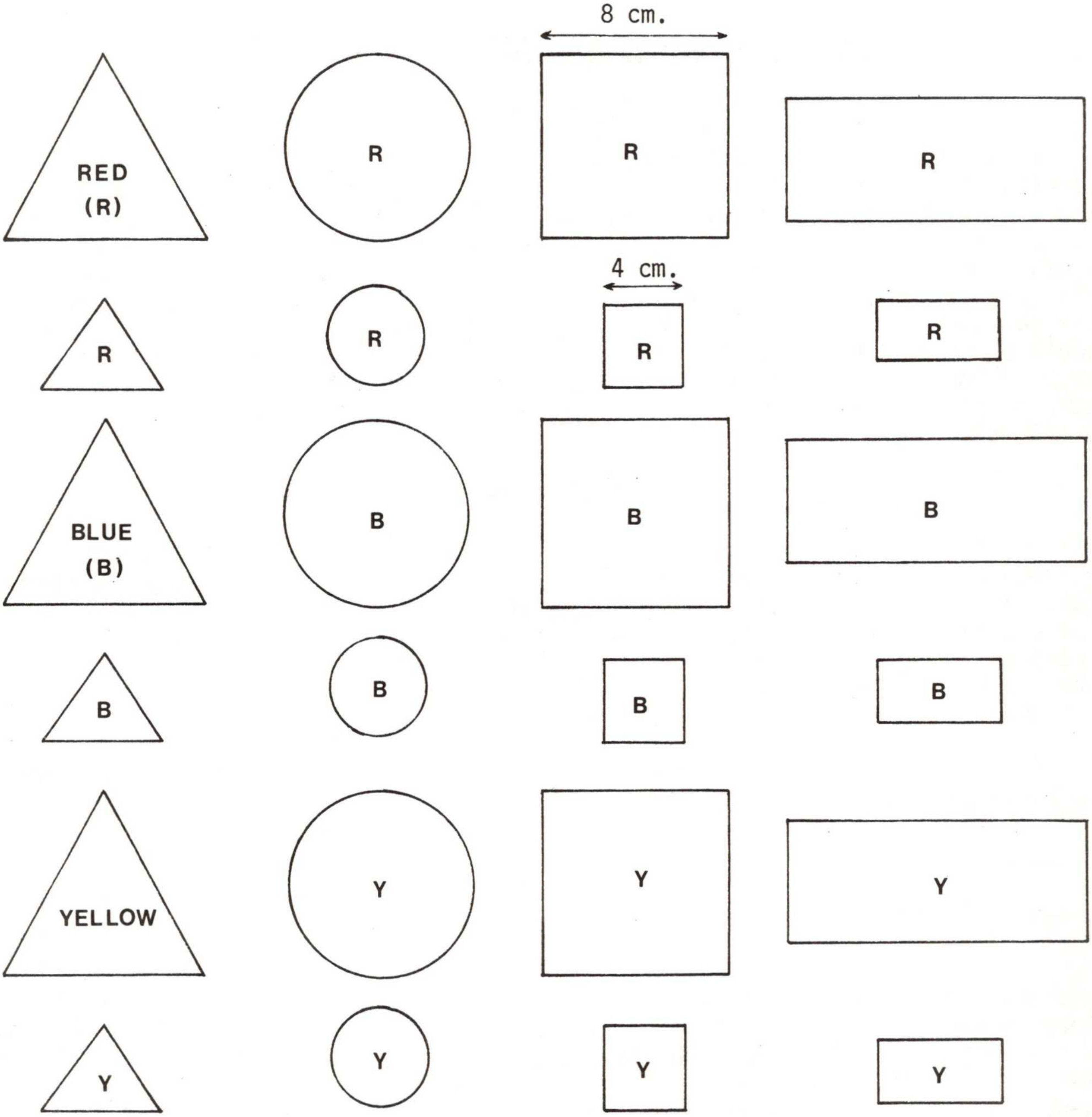


Figure 1. The Attribute Blocks: A set based on 4 shapes, 3 colors and 2 sizes. Two thicknesses of each of the above depicted blocks would result in a set of 48 pieces.

Difference games

Purpose: To further familiarize the learner with the relationships among the pieces; to focus on detecting similarities and differences; to develop strategies involving consideration of all possibilities.

Instructions: The one-difference game.

The first player chooses any block. The second player must select a block which differs from the first in exactly one way (shape, color, size or thickness). The next player then finds a block differing from the second in one way, and so on. For example: 1st player - small thick red triangle, 2nd player - small thick red circle, 3rd player - small thin red circle, and so on. Points are obtained by corectly choosing a block or by challenging a block incorrectly played by someone else.

The two-difference game.

Played as above except that a piece played must differ from the previous one in exctly 2 ways.

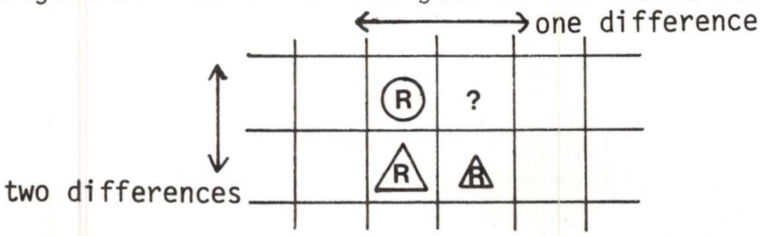
The three-difference game.

Pieces played must differ in exactly 3 ways. Students often discover that this can be viewed as a "one-same" game and that using this (logical) strategy simplifies the task.

Two dimensional difference game.

A grid is needed. One version of the game would require pieces to differ by one attribute in one direction and by two attributes in the other direction. In Figure 2, the square marked, ?, could be correctly filled in several ways (such as thin red large circle; thick red small circle). Players might wish to formulate a hypothesis concerning the relationships of correctly played diagonal pieces in this game. A scoring system might be instigated to encourage the filling of squares bordered by 2, 3 or 4 pieces. Suppose that, in attempting to fill such a square, a player decides that he is unable to find a block satisfying all required conditions. This may be either because all suitable pieces had been previously played or because the conditions make the existence of such a block logically impossible (see Figure 3). Determining (proving) that a space cannot be filled would be scored higher than simply correctly playing a block. Strategies of exhausting or considering all possible cases would be developed in such investigations.

Figure 2. The difference game in two dimensions. (The three blocks are all thin.)



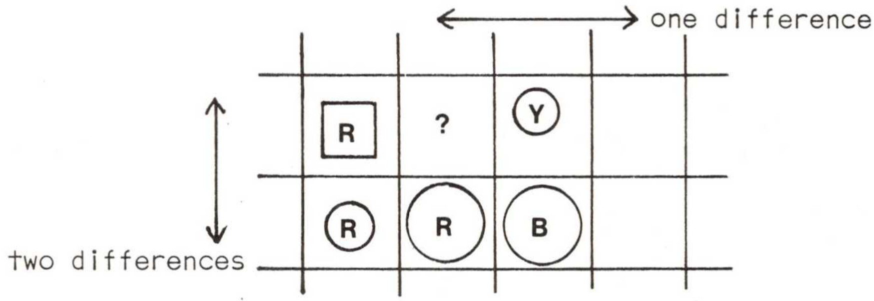


Figure 3. Filling the ? square is logically impossible. (All blocks depicted are thin.)

Guessing games

Purpose: To develop skills in asking relevant questions and in utilizing efficiently information obtained from the answers; to acquaint students with the use and power of the logical terms "not", "and", "or"; to illustrate a valid mode of logical reasoning.

Instructions: The teacher or game leader selects (in his mind) one of the 48 blocks. The players attempt to identify this block by asking questions which can be answered by "yes" or "no". For instance, if the answer to the question "Is it thin?" is "No", the players learn that they need not ask the question "Is it thick?" since this can be determined through logic. The situation can be analyzed as follows:

If thin or thick		$p \vee q$
and not thin	or symbolically	$\sim p$
then thick		q

An examination of the truth table of the related compound statement

$$[(p \vee q) \wedge \sim p] \rightarrow q$$

shows that the statement is a tautology and hence that this mode of reasoning is valid. Thus for the above argument, if the premises are true, the conclusion must always be true. To assist in the thinking process, a chart similar to the following might be kept by students as they play this game.

Answer	Deduction
not thick	thin
not red, not blue	yellow
square	square
...	

The meaning, advantages, and limitations of "and" and "or" questions may be brought out during the course of this activity. It should be discovered that questions such as "Is it the small, thin, red square?" or even "Is it large and blue?" are not generally productive or helpful. On the other hand, "or" questions provide the basis for more interesting discussions. One problem which can generate considerable interest is stated as follows: "Which is the better question to ask (would yield the most information)?" (1) Is it red?; or (2) Is it blue or yellow?" It is usually only after some debate that participants generally (not always unanimously) agree that the two questions are logically equivalent. Logically, if a block is either red or blue or yellow, then it is red or blue if and only if it is not yellow. Symbolically, if r, b, and y are mutually exclusive,

$$(r \vee b \vee y) \rightarrow [(r \vee b) \leftrightarrow \sim y)].$$

Now, what is the maximum number of questions required to identify the unknown block in the guessing game? The answer is 6 questions - one for size, one for thickness, 2 for color, and 2 for shape. Shape can be determined in two questions if the first one is an "or" question. For instance:

Question 1	Is it circular or triangular?
Question 2	(Answer yes) Is it circular?
	(Answer no) Is it square?

In a far more difficult version of the guessing game, the objective is to determine a subset of the blocks defined as the conjunction of 2 attributes. Suppose, for example, that the game leader is thinking of the "thin circles". As the players point to various blocks one at a time, the leader must indicate whether each is or is not in the set he had in mind. A partitioning of the blocks in 2 classes is thus commenced. What strategy should be used to most efficiently identify the set? First, one block belonging to the set must be determined; suppose it is the *large red thin circle*. These attributes must then be varied one at a time while the other 3 are held constant. For example, the large red thin *square* might next be selected. Since this piece is *not* in the set, the conclusion would be that shape (circular) *is* a defining characteristic. On the other hand, since the *small* red thin circle *is* an exemplar of the set, size would *not* be a defining characteristic. Similarly, the relevance of color and of thickness could be tested.

Hoop activities

Purpose: To create Venn diagram-type illustrations of logical connectives and relations; to show the relation between sets and logic.

Instructions: Hoops (or rope or string) are required. Sample activities are as follows:

1. Place one hoop on the floor (or table). Put all the red blocks inside the hoop. The blocks outside the hoop then would be the "not-red" blocks. If the "not-red" blocks were placed inside a hoop, the blocks outside the hoop would be the "not (not-red)" blocks which are the red blocks, illustrating that $\sim(\sim r) \leftrightarrow r$.

2. To illustrate disjunction and conjunction, 2 (or more) hoops are required. The task might be to place the blue blocks in one hoop and the square blocks in another. To accomplish this, one must overlap the 2 hoops. The set of blocks is thus partitioned into 4 classes, each identified by the conjunction of 2 attributes as indicated in Figure 4. Such activities help to clarify the logical meanings of "and" and "(inclusive) or". One of DeMorgan's laws may

be illustrated as follows: Since the blocks inside either of the 2 hoops are "blue or square", those outside are "not (blue or square)". Hence not (blue or square) if and only if not blue and not square; or

$$\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q).$$

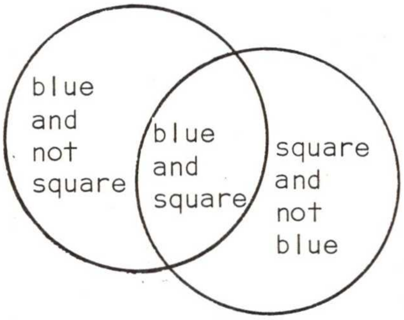


Figure 4. The conjunction of two attributes.

The number of elements in the intersection and union of two sets

Purpose: To allow students to discover methods for determining the number of elements in the intersection and union of 2 (non-disjoint) sets.

Instructions: Answers to the following questions can be obtained by counting. Students should be encouraged to discover methods (formulas) for finding answers to similar questions without counting. Answers obtained through using formulas arrived at inductively can then be checked by counting.

1. What fraction and how many of the blocks are

square?	Answer 1/4 (12)
red?	Answer 1/3 (16)
small?	Answer 1/2 (24)
thick?	Answer 1/2 (24)

2. How many blocks are red and square?
   Answer: By counting - 4.
   Methods:
   (a) 1/3 of 48, or 16, are red.
   1/4 of these, or 4, are square.
   (b) 1/3 are red and 1/4 are square.
   1/3 x 1/4 = 1/12 are red and square.
   1/12 x 48 = 4

3. How many blocks are either square or thick?
   Answer: By counting - 30.
   Method:
   # (square or thick) = # (square) + # (thick) - # (square and thick)
   = 12 + 24 - 6 = 30

Probability

Purpose: To determine empirically and theoretically the probabilities of simple events, independent events, and composite events (A or B, when A and B are not mutually exclusive).

Questions: If a block is selected at random, what is the probability that it is 1. blue? 2. red and square? 3. square or thick?

Instructions: To determine the probabilities empirically (to find the relative frequencies): For each of the 48 blocks, list the shape, color, size, and thickness on a separate slip of paper. Put the 48 slips in a box. Draw a slip at random; note the (1) color, (2) color and shape, (3) shape and thickness; and replace the slip. Repeat this procedure 100 times (or more).

The relative frequency is the number of times (1) blue, (2) red and square, (3) square or thick was noted divided by the total number of draws.

To determine the probability theoretically:

1. $\text{Prob (blue)} = \dfrac{\text{No. of blue blocks}}{\text{Total no. of blocks}} = \dfrac{16}{48} = \dfrac{1}{3}$

   Answers to the next 2 questions (and similar questions) can be found, first by counting or otherwise determining the number of blocks satisfying the given conditions. Formulas can then either be developed inductively or verified in the specific cases.

2. $\text{Prob (red and square)} = \dfrac{\text{No. of red and square blocks}}{48} = \dfrac{4}{48} = \dfrac{1}{12}$

   $= \text{Prob (red)} \times \text{Prob (square)} = \dfrac{1}{3} \times \dfrac{1}{4} = \dfrac{1}{12}$

3. $\text{Prob (square or thick)} = \dfrac{\text{No. of square or thick blocks}}{48} = \dfrac{30}{48} = \dfrac{5}{8}$

   $= \text{Prob (square)} + \text{Prob (thick)} - \text{Prob (square and thick)}$

   $= \dfrac{1}{4} + \dfrac{1}{2} - \dfrac{1}{8} = \dfrac{5}{8}$

Answers obtained to the above questions using empirical and theoretical procedures should be compared and any differences discussed.

Getting the 'rect'-angle on mathematical activity

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Getting the 'rect'-angle on mathematical activity

There has been a large amount of literature and perhaps an even larger amount of activity packages developed for and by teachers and students of mathematics over the last 5 years. The basis for much of this development came from the assumption of the value of individualized instruction and of such adages as "doing produces understanding", "go from concrete to abstract", and "discovery". The teacher, in face of the proliferation of such educational slogans and complex masses of material, must search out fundamental personal reasons for using activities. Upon finding convincing reasons for having students engage in such activities, the teacher might well ask, "Can this be done without large expenditures of time, money, or both?" It is one purpose of this paper to briefly discuss some issues surrounding the "Why activities?" question. The major portion of the paper will try to cope with the "how" problem by using the simple mathematical creature, the rectangle, and its sub-species, the square.

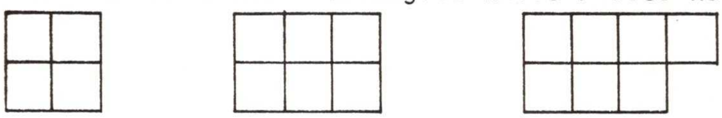
PLAYING AROUND WITH MATHEMATICS

There are many potential benefits of activities in mathematics. From a developmental point of view, it is considered that students, probably through junior high school age (Lovell, 1971) are capable of logical thinking about real or potentially real situations, but not very able to deal with completely hypothetical situations. Thus physical models or pictorial images provide a necessary grist for the logical- mathematical thinking of children perhaps up until the age of 14 or 15.

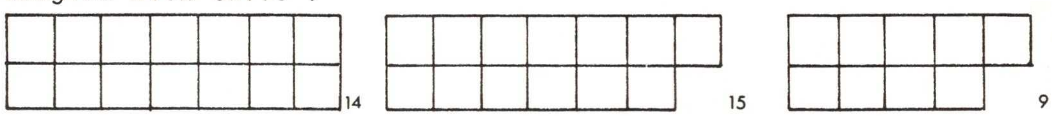
Perhaps as important as the notion of using concrete or pictorial models as starting points for mathematical ideas, is the notion that students can profitably play with such models. For the preschool child, 2 major modes of changing his picture of the world exist, playing and imitating. The latter affords the child the opportunity to reshape his thought to accommodate some phenomenon in the world. Thus a child watching hockey on T.V. finds a model he can imitate in handling a hockey stick, an action which may have been completely foreign to him previously. Playing allows the child to impose his already made ideas on new phenomena. For example, our young "hockey player" finds he can use balls, bottle caps or plastic discs as "pucks" and many things for sticks, goals and even rinks. Thus playing allows for creating powerful general ideas. In school, and perhaps this practice increases with the grade level, we tend to take a one-sided view of acquisition of ideas. Imitation is taken to be the tool, while play becomes a frivolous activity or one which perhaps can take place as a student practices with an idea or skill learned through imitation. In mathematics, this tends to mean that learning becomes being told the "right" way to do something and then verifying through practice that this way is useful. It denies the more playlike mathematical processes such as looking for quantitative or spatial aspects in a situation, looking for relationships, and guessing at patterns. In "playing" with physical materials and pictorial images, the student can bring his own ideas to bear and extend them to include new notions.

If the above arguments are convincing, the teacher is tempted to order a lot of materials or open the mathematics laboratory manuals and get on with the business of playing with mathematics in any form. But the business of learning mathematics is not that simple. Whether the activity is solving a problem, proving a theorem or applying mathematics to everyday life, mathematics means successfully working with symbols. Thus in our "play" (presenting activities-oriented experiences to students), we must ascertain that it will contribute to later symbolic activity. In particular, we must be certain that the student will not have to unlearn what he learned from his activity work in effectively dealing with mathematics symbolically. To insure this, the teacher must see that the mathematical form in the activity should be at least analogous to the later symbolic form. Perhaps an example will be useful.

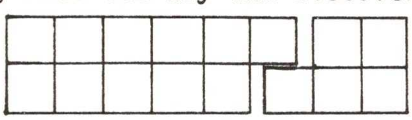
An activity which one can do with congruent square tiles is to try to make rectangles 2 units on one side out of sets of tiles. For example, sets of 4 and 6 admit such rectangles while 7 does not.



Quickly the student sees numbers as falling into 2 sets, the evens whose tile sets make into "2 x n" rectangles and the odds whose sets make "rectangles with tails".



Once this classification is made, the student might "add odds" in the following concrete way and discover that an "odd plus an odd is an even".



11 + 7

Now of course the mathematical activity would not have to involve physical materials. Students who could divide by 2 could make the original classification and by the following type of exercise:

$$11 + 7 = 18, \quad 19 + 13 = 32, \quad 3 + 5 = 8$$

induce that "an odd plus an odd is even".

Since this symbolic activity is so easy, why all the fuss about concrete activity? The answer is "form". Adding 11 and 7 physically in the manner above is powerfully suggestive of the symbolic proof of the theorem:

$$\underset{\text{odd}}{(2n + 1)} + \underset{\text{odd}}{(2m + 1)} = 2n + 2m + \underset{\text{even}}{\text{"2"}}$$

$$= 2(n + m + 1).$$

Adding 11 and 7 or 101 and 93 symbolically makes no contribution to the form of the mathematics. Thus though both activities allow for discovery, the physical activity allows for seeing more mathematics than the symbolic activity.

In answering the questions, "Why use physical and pictorial models?", "Why use labs?", or "Why use activities?" a teacher might consider the following guidelines which summarize the section above.

1. Activities with models provide an appropriate setting for development of mathematical ideas for students in elementary and junior high school. They can provide an appropriate bridge to the world of ideas and symbols.

2. Activities provide one opportunity for effective "play". During such "play", students can exercise such processes as seeing the mathematics in a situation, observing possible relationships and guessing and testing personal mathematical ideas.

3. To realize the above in a way which most contributes to further mathematics learning, the teacher must choose activities which best "form" mathematical ideas.

TO POLYNOMIALS AND BACK AGAIN

Once one has a basis for using activities, the question of how to do so effectively arises. Can one use activities in a variety of contexts and can one do so without lots of fancy materials? What follows is an answer to these questions.

The activities designed below relate to mathematics which has traditionally been in the curriculum for Grades IV to X. All of the activities are based on the following physical materials:

1. A large number of squares (1/2 to 3/4 inches on a side) of either oak tag or plastic.

2. A large set of cubes (1 cm. to 3/4 inch on a side).

3. Coordinated sets of squares and rectangles (such as 7 x 7 squares, 7 x 1 rectangles, unit squares) of wood or tag-board.

4. Grid paper.

5. Sets of colored oak-tag rectangles (1" x 2").

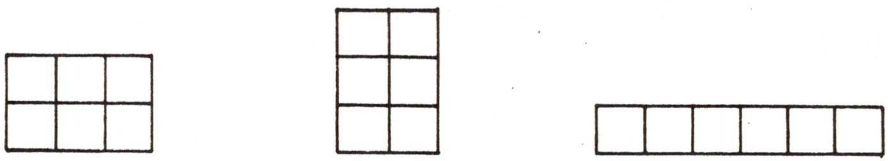
Several of the activities are just described; others are given in the form of activity cards. The methods of use can vary. The whole class can work individually on the same activity interspersed with teacher-direction or class discussion. Or the class may work in small groups, each group working on an independent activity. Above all, the activities are merely suggestive of things you can do to enrich the mathematical experience of your students.

Building up factors

Materials: A set of 25 to 40 squares of plastic or tag for each student or group.

CARD A1

How many rectangles can you make from 6 squares? Here are some.



3 by 2   2 by 3   6 by 1

The label below the rectangle tells how we describe each rectangle.
Complete the following table:

Number of Squares	Rectangles	How many Rectangles
1		
2		
3		
4		
5		
6	6 by 1, 3 by 2, 2 by 3, 1 by 6	4
7		
8		
9		
10		
11		
12	(table can be extended to suit your class)	

CARD A2

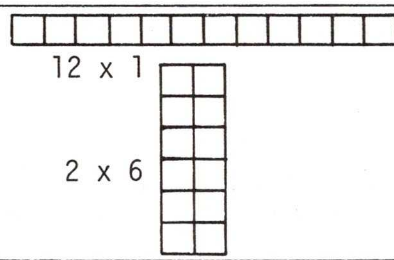
Explorations

1. For what number can you make the fewest rectangles?
2. List those numbers for which you can make only 2 rectangles.
3. Are there any numbers for which you can make squares?
4. Tell the number of squares used in these rectangles:

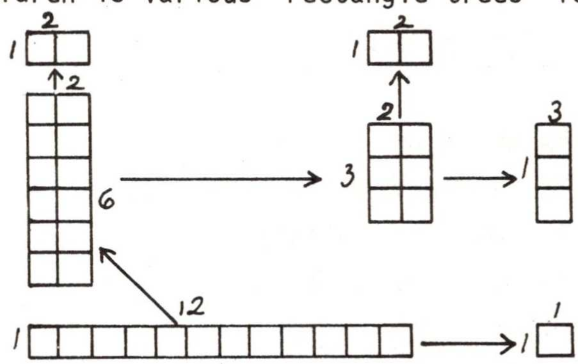
   (a) 4 by 2 ______________ (d) 7 by 3 ______________
   (b) 1 by 5 ______________ (e) 4 by 5 ______________
   (c) 12 by 8 ______________ (f) 30 by 20 ______________

   How do you find the number?
5. There are many other things which you can explore. Collect information or make charts on the following:
   (a) Are the number of squares and the number of rectangles you can make related?
   (b) For 4 squares, the numbers used in describing the rectangles are 1, 2, 4. The sum of these is 7. For 5 squares the numbers are 1,5. This sum is 6. Make this kind of sum for all the numbers in your chart and tell about any patterns you see.

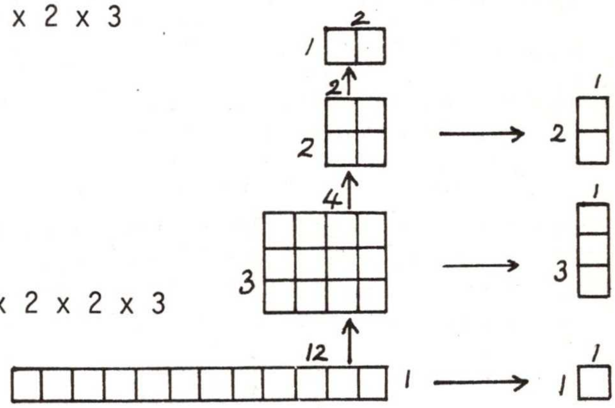
Note: There are many uses for Card A2. Clearly the exercises mentioned in point 5 are more sophisticated and could be used as projects. In 1-4, little verbalization is called for. Yet these activities and their results have the form usable in the symbolization of factors and usable in the definition of primes or perfect squares. An alternative representation for the chart in A1 is a class display board to which students contribute correct rectangles:

Number	Rectangles
12	
13	

This chart could then be used for later discussion. Another display device built by children is various "rectangle trees" for numbers:



12 = 1 x 12 = 1 x 4 x 3 = 1 x 2 x 2 x 3

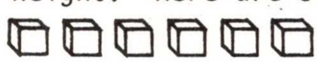


12 = 1 x 12 = 1 x 2 x 6 = 1 x 2 x 2 x 3

An Alternative to Card A1 would use cubes.

Piling Cubes

If you have 6 cubes you can pile them in several ways so that the piles are the same height. Here are some.

1 pile of 6   6 piles of 1  2 piles of 3 

Fill in the chart.

Number	Ways of Piling
1	
2	

This problem,[1] although equivalent, seems simpler than the rectangular problem and can be worked on successfully by children even as young as 6 or 7.

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[1]As a teacher, you could give this assignment over a 3-day period and then collect the results on a chart on the board. Try to collect as many different ways as possible for each number.

Prime bag:

The game, prime bag, stretches the concept of using squares, but it is a simple game dealing with primes. The game can be used with a whole class or with small groups.

Each student gets a small bag containing squares, each representing a prime number. There are 5 - 2s; 4 - 3s; 3 - 5s and 2 of each other prime up to and including 23.

Contest: How many numbers from 1 to 100 can you make up using the numbers in the bag and the operation of multiplication only? Make a list of all of your "successes".

Example: 30 = [5] x [3] x [2] (5 x 3 x 2)

35 = [5] x [7] (5x7)

Questions:

(a) How many different ways are there to construct each number? (This leads to the Fundamental Theorem of Arithmetic.)
(b) Are there any numbers you can't build? If so, how can you describe these?
(c) What is the largest number you can build using the squares in the bag and multiplication?

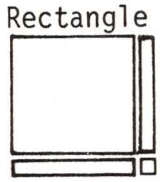
Polynomial puzzles

Materials: For each group a set like the following:
3 blue squares - 12 cm. x 12 cm.
10 blue rectangles - 12 cm. x 1 cm.
10 red rectangles - 12 cm. x 1 cm.
30 blue squares - 1 cm. x 1 cm.
30 red squares - 1 cm. x 1 cm.

All of the puzzles have the same direction. Given a certain subset of the set given above, make a blue rectangle. For example: (in the diagrams, unshaded will represent blue; shaded, red.)

Given

Rectangle



$$(12 \times 12) + (2 \times 12) + 1 = (12 + 1) \times (12 + 1)$$

or $$12^2 + 2 \times 12 + 1 = (12 + 1)^2$$

or (if the big square is considered x^2, the rectangle x and each unit 1)

$$x^2 + 2x + 1 = (x + 1)^2$$

These puzzles can be done by individuals guided by an instruction sheet, or the 2-person game Rectangl-it, may be played.

Rectangl-it:

Materials: Like those described above.

Rules:

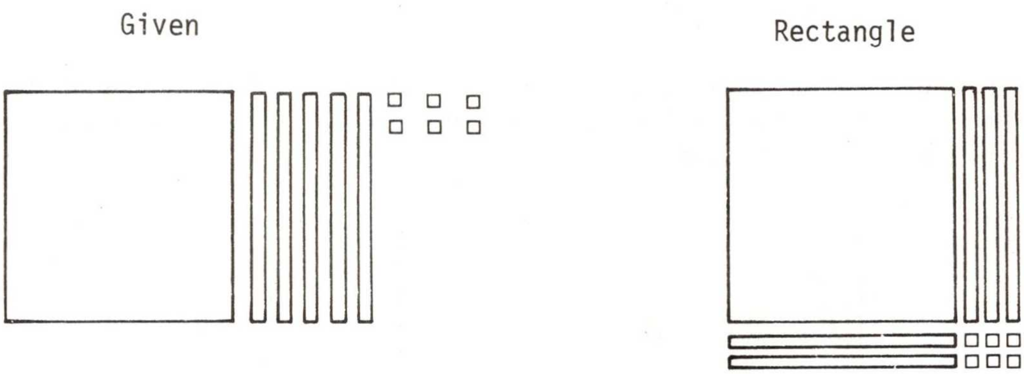
1. There are 2 positions - Setter and Maker.
2. On each play, the Setter sets the given subset of the playing set and acts as timer.
3. On each play, the Maker attempts to make a rectangle within 3 minutes.
4. Scoring: If the Maker completes a rectangle in less than:
   1 minute: 3 points
   2 minutes: 2 points
   3 minutes: 1 point

   If not, the Maker either gets no points or calls "no rectangle". If he can show that no rectangle can be made, he gets 3 points. If he calls "no rectangle" and one can be made he loses 3 points.
5. In each round, each player is the Setter and the Maker once.
6. A game is 4 rounds long.

More puzzles:

Given below are some puzzles, solutions and records. It is important that students, whether individually or in a game setting, keep accurate records of their attempts. From studying the diagrams and symbolic records, the student will be able to see the forms useful in factoring polynomials.

PUZZLE 1:



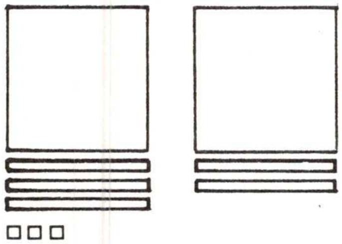
Record

$$12^2 + 5 \times 12 + 6 = (12 + 3)\ (12 + 2)$$

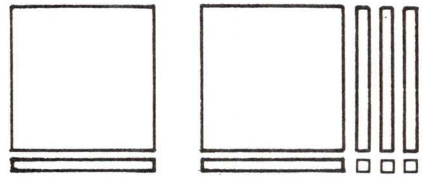
or $$x^2 + 5x + 6 = (x + 3)\ (x + 2)$$

PUZZLE 2:

Given



Rectangle



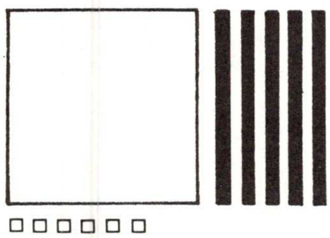
$2x^2 + 5x + 3 = (x + 1)(2x + 3)$

Note: Doing only a few such puzzles gives insufficient experience. Doing a large number allows the students to find relevant ideas such as "the factors of the constant are important". It is important for students to note that just as factoring numbers involved building rectangles, factoring polynomials involves building rectangles. This enables this "puzzle" activity to significantly contribute to polynomial problems.

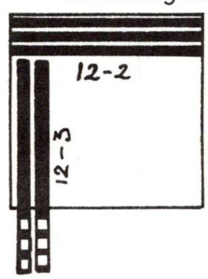
PUZZLE 3:

In this puzzle, the shaded areas stand for red and the unshaded for blue.

Given



Rectangle



Record

$12^2 - 5 \times 12 + 6 = (12 - 3)(12 - 2)$

$x^2 - 5x + 6 = (x - 3)(x - 2)$

Note: This puzzle illustrates the use of negative coefficients and illustrates that these puzzles can and should be challenging puzzles in their own right. It should be noted that red covers blue in these puzzles and that equal numbers of red and blue rectangles can be added without changing the character of the polynomial.

These puzzles represent an activity which is preliminary to symbolic factoring and which allows students to play while preserving the form of later symbolic activity. If the teacher wants a more guided activity, she can construct cards such as the following which makes use of 3 sets of materials like those described at the beginning of this section on polynomial puzzles except based on 5 cm., 7cm., and 12 cm.

CARD P1

1. Choose one card representing 5 x 5; 2 representing 5 x 1; and one unit.
   Build a rectangle.
   What are its dimensions?_________, ____________.
   This can be represented by the following sentence.

   $5^2 + 2 \times 5 + 1 = (5 + \_)\ (5 + \_)$

2. Build a rectangle from one 7 x 7; 2, 7 x 1; and one unit.
   Dimensions: ___________, _________.
   Complete the following sentence.

   $7^2 + 2 \times 7 + 1 = (7 + \_)\ (7 + \_)$

3. Build a rectangle from one 12 x 12; 2, 12 x 1, and one unit.

   Dimensions: ________, ________.

   Complete the following sentence

   $12^2 + 2 \times 12 + 1 = (\qquad)\ (\qquad)$

4. Suppose you had to build a rectangle from one, 30 x 30; 2, 30 x 1 and one unit.

   Dimensions: _________, ________.

   $30^2 + 2 \times 30 + 1 = (\qquad)\ (\qquad)$

5. Complete the following chart:

One	Two	One	Dimensions	
5x5	5x1	unit	_ _, _ _,	$5^2 + 2 \times 5 + 1 = (\quad)(\quad)$
40x40	40x1	unit	_ _, _ _,	$40^2 + 2 \times 40 + 1 = (\quad)(\quad)$
1000x1000	1000x1	unit	_ _, _ _,	$1000^2 + 2 \times 1000 + 1 = (\quad)(\quad)$
nxn	nx1	unit	_ _, _ _,	$n^2 + 2n + 1 = (\quad)(\quad)$
yxy	yx1	unit	_ _, _ _,	$y^2 + 2y + 1 = (\quad)(\quad)$

6. Complete the following:

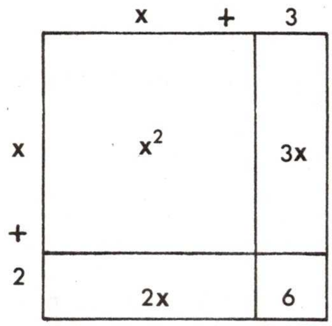
   $x^2 + 2x + 1 = (\qquad)^2$

Note: This card varies the same puzzle over several dimensions or numerical variants. It deliberately attaches the physical activity to the symbolic activity in a tightly prescribed form. It would best be used at a time when you really wished to concentrate on polynomial factoring and special forms. The previous puzzles and games might be profitable earlier. That is, this latter activity would best be used in Grades IX or X while the former could also be used in V, VI, VII or VIII. In order that this kind of activity be effective, cards for other factoring problems such as difference of squares would have to be used. Physical activity as a prelude to symbolic activity is not highly successful on a one-shot basis.

Squares, rectangles and computing

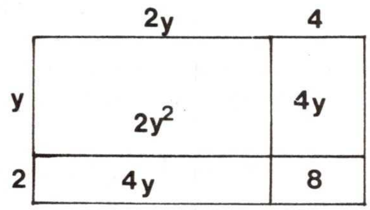
Teachers are probably familiar with rectangular pictures as models for binomial multiplication. These models are based on an area interpretation of multiplication. Some examples are given below.

1



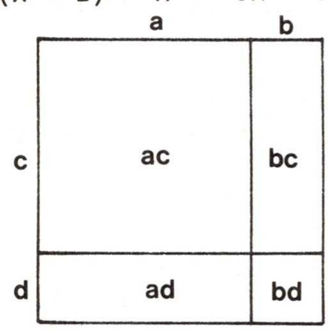
$(x + 3)(x + 2) = x^2 + 5x + 6$

2



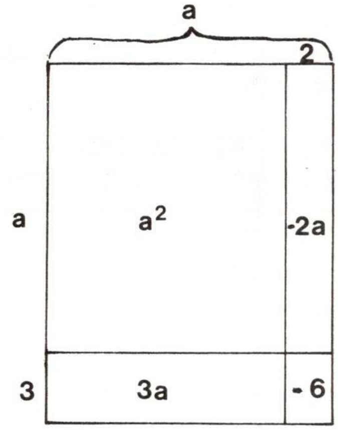
$(2y + 4)(y + 2) = 2y^2 + 8y + 8$

3



$(a + b)(c + d) = ac + bc + ad + bd$

4



$(a - 2)(a + 3)$
$= a^2 - 2a + 3a - 6$
$= a^2 + a - 6$

Making up activities such as these in which students create pictures of binomial multiplication is a means of better understanding the use of the distributive property. As is seen in example 4, this activity is interesting in itself.

This model is even more interesting with polynomials of higher order.

	x^2	+ 2x	+ 3
$2x^2$	$2x^4$	$4x^3$	$6x^2$
+ x	x^3	$2x^2$	3x
+ 5	$5x^2$	10x	15

$(x^2 + 2x + 3)(2x^2 + x + 5) = 2x^4 + 5x^3 + 13x^2 + 13x + 15$

One useful pictorial activity is to have students diagram 10 such multiplications. They may do this rather blindly but very likely they will reduce this activity to a kind of algorithm and instead of using rectangles will simply use a grid as on p.82.

	x^2	$+ 2x$	$+ 1$
$3x^2$	$3x^4$	$6x^3$	$3x^2$
$-5x$	$-5x^3$	$-10x^2$	$-5x$
$+7$	$7x^2$	$14x$	7

$$(x^2 + 2x + 1)(3x^2 - 5x + 7)$$
$$= 3x^4 + x^3 + ox^2 + 9x + 7$$

After a few such examples, they will discover that like terms lie on the diagonals and hence invent a neat multiplication algorithm.

The following card might be made up:

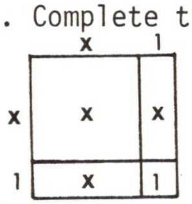
> CARD PD1
>
> Using the grid picture, make up a method for dividing polynomials.

Note: This is an interesting open-ended activity especially if the polynomials do not divide evenly.

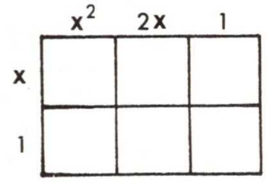
Another interesting pictorial activity is illustrated by the following card. This activity might be most useful in high school mathematics.

CARD BT1

1. Complete the following diagrams.



$(x + 1)^2 = x^2 + 2x + 1$



$(x + 1)(x^2 + 2x + 1) = (x + 1)^3$

2. Continue this process through $(x + 1)^{10}$.

3. Study the rectangles in each of the stages above. How many squares does each contain? Diagonals?

Stage	Squares	Diagonals
$(x + 1)^2$	4	3
$(x + 1)^3$	6	4
$(x + 1)^4$		
"		
"		
$(x + 1)^{10}$		

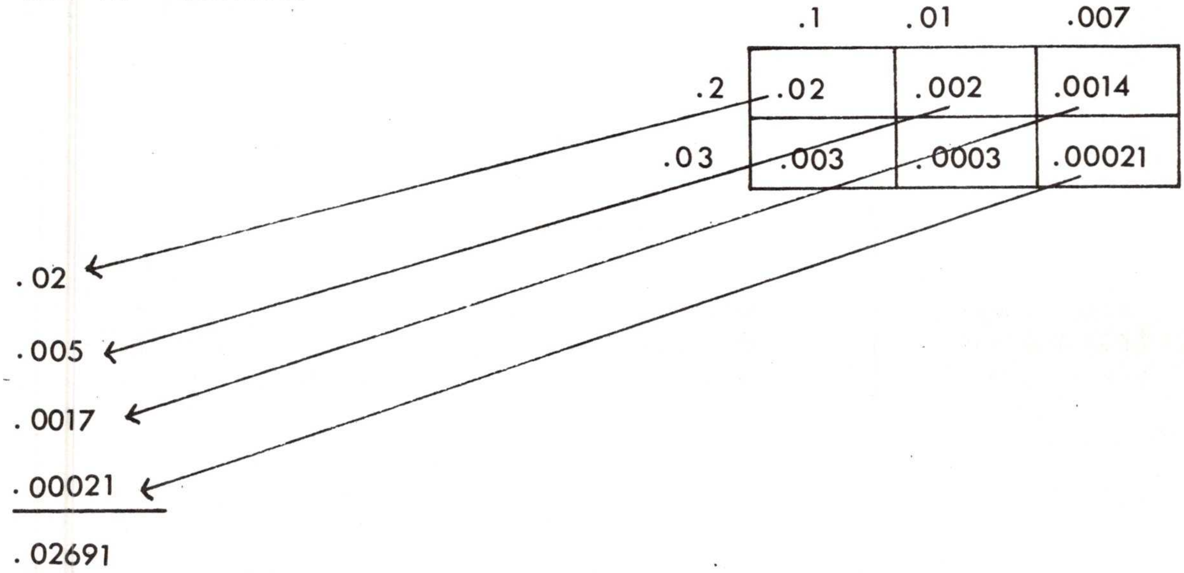
4. How many squares and diagonals would $(x + 1)^{20}$, $(x + 1)^{100}$, $(x + 1)^n$ have?

   What does this tell you about the number of terms in the product $(x + 1)^n$?

Back to the world of numbers

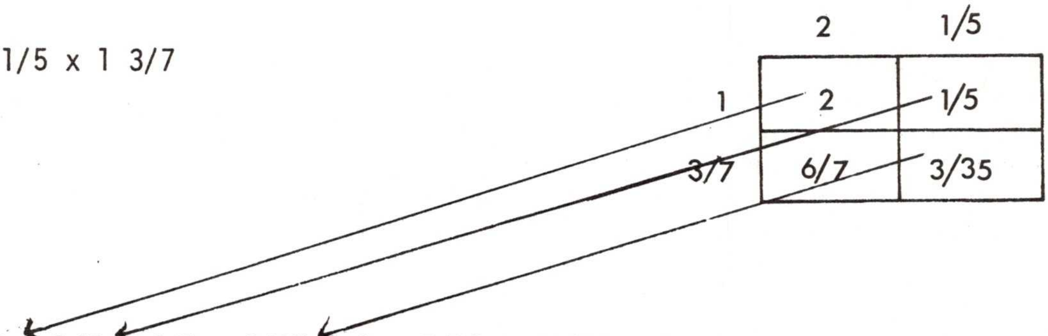
The last section showed how area presents a nice algorithm for multiplying polynomials. Since numerical representations are polynomials of a sort, it should not be surprising that pictorial algorithms hold here as well. Given below are several illustrations of pictorial multiplication activity.

CARD M1 - Decimals



CARD M2 - Rationals greater than 1

2 1/5 x 1 3/7



2 + 1/5 + 6/7 + 3/35 = 2 + 7/35 + 30/35 + 3/35

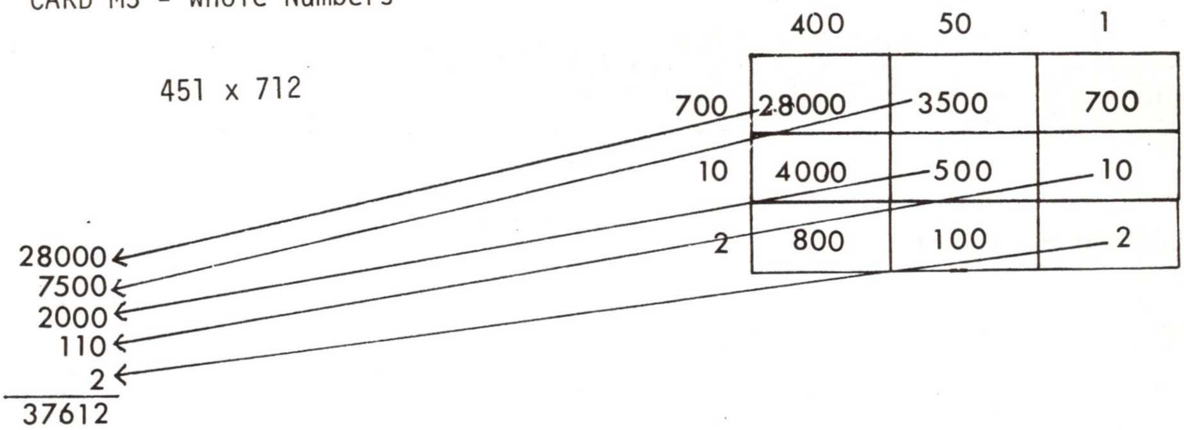
= 2 + 40/35

= 3 + 5/35

= 3 1/7

CARD M3 - Whole Numbers

451 x 712



Note: These algorithms represent one "understanding" approach to computation. From the last example, one can see that these are just models for a partial products approach. If one wished to add more realism to the problem, the student could construct "scale drawings" of the numbers for the sides of the computational rectangle.

SUMMING UP

The excursion using rectangles from numbers to polynomials and back was done for 2 purposes. Most importantly, it illustrates how physical and pictorial activity contribute to mathematics learning at several grade levels through the use of playing with "form". The second purpose was to illustrate that physical-pictorial activity was easy and inexpensive to use.

From the above, a teacher could expect to use activity extensively and at little cost. If such activity proved effective, it would clearly be cost-effective. There is no case from hard data that can be made that such activity represents a universal success. Yet it is hoped that the range of simple activities suggested above will give you the "rect"-angle on the use of mathematical activity in your classroom. From these suggestions, you may see many more ways to simply design active mathematical experiences which will be fun and productive for your students.

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Toying with TAD

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Toying with TAD

Psychologists doing research in the area of human problem-solving have discovered a phenomenom called "functional fixedness", which refers to the tendency humans have to identify objects with some specific role or function. In many cases, this identification is so strong that it impedes productive problem-solving. This occurs when problem-solvers cannot see some given object as being a tool which might help them because they have fixated on the standard function which the object serves and this use differs from the one required to solve the problem.

Some mathematics teachers sometimes exhibit a form of functional fixation with respect to teaching aids. In this variation of functional fixedness, one particular use of a teaching aid is so strongly identified with the aid that it tends to block out its other potential uses. Hence, for example, Cuisenaire rods are employed almost exclusively to teach basic number operations to young children and rectangular grids are seldom used except for purposes of "graphing". To say that some mathematics teachers tend to functionally fixate with regard to some of their teaching aids is, in some sense, to say that they are not getting as much "mileage" out of these aids as they might.

Of the many types of mathematics teaching aids, perhaps the worst "mileage-getters" of all are mathematical games. Many teachers use mathematical games as a form of reward: "If you answer all your questions correctly, then you can go to the back of the room and play 'Cube-Fusion' or 'Hi-Q'." and so on. While this is certainly one valid use of these aids, it is likely that fixating on the reward function may well lead a teacher to overlook some of the other functions this type of aid might serve. This would be most unfortunate since some of these mathematical games offer extremely rich frameworks for significant mathematical activity on the part of students.

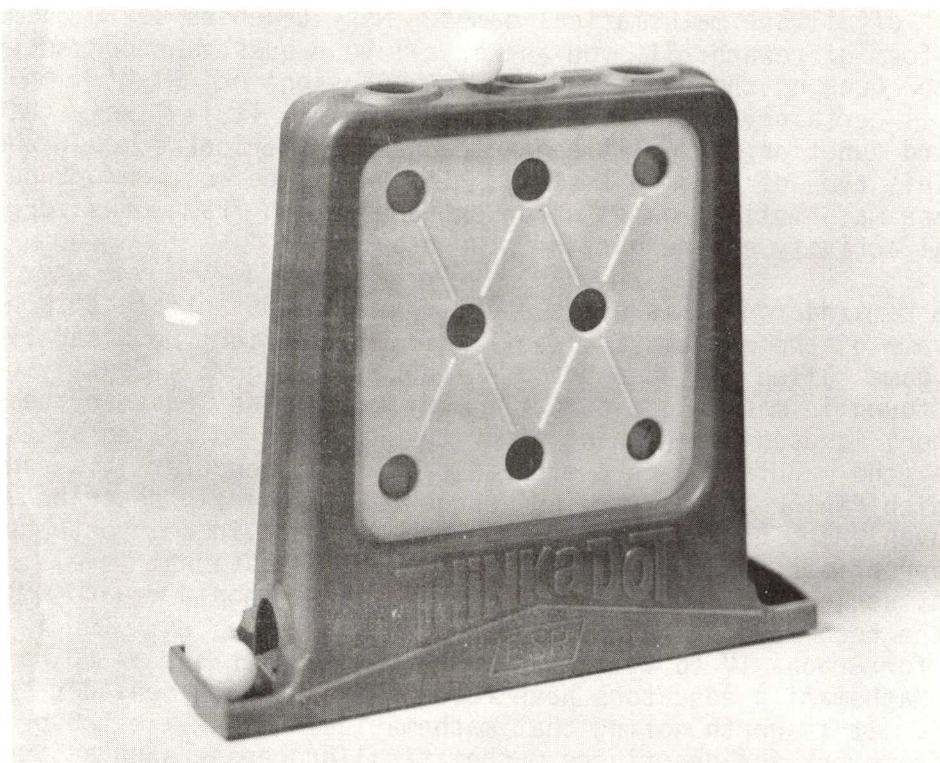
The remainder of this article attempts to substantiate this position by outlining some of the mathematical activities which might be generated by the structural game called "Think-a-Dot". Although this game is easily obtainable[1] and can be found in many classrooms, it very seldom (at least in the experience of the author) is used as anything other than a toy. Following a description of the game (the manufacturers prefer to call it a "computer"), a range of activities which "Think-a-Dot" (or TAD) suggests are briefly described. For the sake of convenience, these activities have been subdivided into those which might be appropriate for learners at 4 different educational levels: elementary, junior high, senior high and university. (Teachers should regard with some suspicion the recommended age levels for mathematical games. TAD is usually prescribed for Grades IV to VIII with almost no suggestions as to how it might be used.) Mathematics educators have recently started to consider "process objectives". It is worth noting that mathematical games like TAD provide an excellent framework for practicing mathematical processes such as "generalizing", "proving", "symbolizing" and "clarifying" (Morley, 1973).

---

[1]"Think-a-Dot" is available from Western Educational Activities Ltd., 10577 - 97 Street, Edmonton, for $3.75.

TAD is a small plastic box with dimensions approximately 6" x 5" x 1 1/2" (see photograph). On one face, there are 8 "windows" arranged in a 3-2-3 pattern, behind each of which the color yellow or blue appears. At the top of the box there are three "holes". When a marble is dropped in any of these holes, some of the windows "change color" and the marble emerges on either the right or the left side of the box. Inside the box there is a "flip-flop" inclination changing the color of its window at the same time. Certain positions or "states" can be set on the face by either tilting the box or by changing the color of each window individually.

The following activities/games/problems are certainly not the only ones that are suggested by TAD nor are they necessarily either the most obvious or the best. They serve only to indicate some of the possible areas open to investigation. For any learner, the most interesting questions are the ones he himself poses. Students should be encouraged to generate, and to work on, their own problems. The job of the teacher in this case is to help students learn how to attack these problems. Conjectures should be formulated, tested and modified. In some cases, it may be possible to construct proofs. Some problems, such as those relating to symbolization and notation, will be found at all age and grade levels. However, it is to be expected, for example, that older students will have more sophisticated systems of representation and terminology.



"Think-a-Dot"

ELEMENTARY LEVEL ACTIVITIES

With elementary school students, one may wish to play various sorts of *prediction games*.

Exit I

One prediction game which any number of children could play is Exit I. In this game, each player chooses an "exit side" and drops marbles until a marble exits on the "wrong" side. The winner is the player having the highest number of correct exits after some fixed number of turns. A variation on this basic theme can be introduced by limiting the number of drops that any player can make in a row in any one hole.

Exit II

In Exit II, each player predicts the side that a marble, dropped in a particular hole, will exit on. The winner is the player having the most correct predictions after, say, 10 drops. A more difficult version of Exit II is one in which the blank face of TAD, rather than the window face, is facing the players.

One-Change

A type of two-player game suitable for this level is the One-Change game. In this game, a player chooses one of the 8 windows and challenges his opponent to have it changed in color after, say, 3 drops. After perhaps 5 turns, the player who has most frequently been able to meet his opponent's challenge is the winner. Variations can be introduced by limiting the position of the challenge holes or by increasing the number of holes to be changed.

JUNIOR HIGH LEVEL ACTIVITIES

At the junior high level, students should be able to work on problems such as those relating to a binary representation of the states of TAD. How many different states are there in TAD? How can these states most conveniently be represented? According to the representation(s), what characterizes all states which have a blue window in the upper left-hand corner?

Given one state, how many drops are required to change TAD to another given state? If it is possible to move from one state to another by a series of drops (Note that this isn't always possible!), is this number of drops unique? (It isn't, but what can you say about it?)

Competitive games appropriate for this age group include: the *Ratio game* (from a given state, one player challenges another to make the ratio of blue windows to yellow windows say 3:5, in as few drops as possible); the *Maximum-change game* (from a given state make, say, 3 drops and change the colors of as many windows as possible); the *Symmetry game* (in as few moves as possible produce, say, a top row which is color-symmetric, or anti-symmetric, to the bottom row); and the *Lines game* (from a given state, produce after, say, 3 drops as many blue or yellow "lines" - three windows in a row - as possible).

SENIOR HIGH LEVEL ACTIVITIES

Senior high students might like to address themselves to TAD problems such as "accessibility", "operator analysis", "proof" and "duality".

Accessibility

From a given state, only certain other states are *accessible*. These are the states that can be reached after a series of drops. How many of these accessible states are there for any given state? Characterize them. Does your representation method give you any insights into the problem of accessibility? What does "parity" have to do with accessibility? Put an upper limit on the number of drops required to transform a given state into some other accessible state.

Operator Analysis

Consider any finite sequence of drops to be an *operator*. (Use some method of distinguishing the 3 different types of drops. We will use "l", "m" and "r" for drops in the left, middle and right holes respectively.) How many "essentially different" operators are there? (There are 128.) Given any operator (say lmlrmmrllmrllrll), can you find its "canonical" representation? To what extent are operators independent of states?

Proof

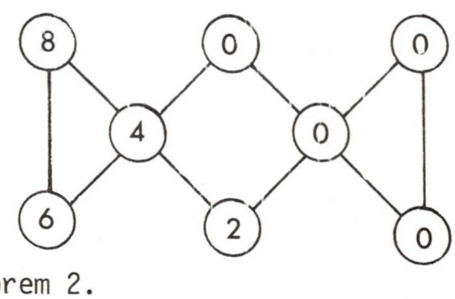
Prove the following "theorems" about operators:

1. $l^8=m^8=r^8=1$ (the identity operator).

2. $l^2m^2r^2=1$.

3. lm=ml, lr=rl, mr=rm.

For example, to prove that $l^8=1$, consider the following diagram which shows the number of "window changes" brought about by 8 drops in the left hand hole.



Generalize Theorem 2.

Prove one part of Theorem 3 in 2 different ways.

Duality

Call states which differ in color in every window *dual states*. Call an operator which transforms a given state into its dual, a *dualizing operator*. Is the dual of a given state always accessible from that state? (Yes. Proof?) What is the minimum number of drops in a dualizing operator? Give an example of a state which dualizes in this number of drops. How many of these states are there? Characterize them. How many states can be dualized in 4 drops? Generalize. Prove. Define "inverse-operator". Which operators are self-inverse? What is the relation between self-inverse operators and dualizing operators?

UNIVERSITY LEVEL ACTIVITIES

The following activities involve concepts not usually encountered at the secondary school level. They might, however, provide an introduction to such concepts for the capable high school student. In fact the number of quite sophisticated mathematical concepts embodied by this so-called toy is surprisingly high.

TAD as computer

From a mathematical viewpoint all digital computers are finite state machines or automata. A *finite state machine* is "a five-tuple [A, S, Z, u, v] where A is a finite list of *input signals*, $A = a_0, a_1, \ldots, a_n$; Z is a list of *output signals*, $Z = z_0, z_1, \ldots, z_m$; S is a set of *internal states*, $S = s_0, s_1, \ldots, s_r$; u is a *next-state function* from SZA into S and v is an *output function* from SZA into Z (Birkhoff & Bartee, 1970, p.68)".

Can TAD be considered a finite state machine? If so, what are the values of n, m and r? Is it possible to construct a state diagram and a state table here? (See Berkhoff and Bartee, 1970, especially Chapter 3, for an elaboration of this topic.)

Group-theoretic aspects of TAD

Does the set of operators, G, form a group? If so, what is the order of the group and what properties does it have? Consider the set of self-inverse operators, H. Is H a group? Can you find a subgroup of H? Is this a normal subgroup? Why? What special set of operators is a subgroup of H?

Consider the Abelian group, M, which has a presentation a,b,c:
$a^8=b^8=c^8=a^2b^2c^2$. Is M isomorphic to G? (See Macdonald, 1970, especially Chapter 8 for an elaboration of this topic.)

Computer-simulation of TAD

TAD presents many opportunities for computing science students to practice their programming skills. One good project, particularly for students who have access to some form of visual display apparatus, is the programming of a computer simulation of TAD. It may also be interesting to consider TAD-like systems which differ in only a few ways from TAD.

What *Boolean* algebra aspects does TAD have? How is the question of accessibility related to the concept of *equivalence classes*?

CONCLUSION

In the preceding sections, an attempt has been made to substantiate the claim that there is more mathematical potential in some common teaching aids than is usually recognized. Although TAD may be a particularly rich situation, similar activities can be created centering on other aids. The mutual formulation and investigation of such activities is, in the opinion of the author, a most worthwhile pursuit for both mathematics teachers and mathematics students.

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Representing a reflection in the plane

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Representing a reflection in the plane

In reading mathematics education journals or attending conferences, one is struck by the increasing popularity of motion geometry as a way of bringing geometry back into the mathematical mainstream and as a natural way of reviving student interest in geometry.

There are many specific reasons for including motion geometry in the mathematics curriculum.[1] This author will not review these arguments for motion geometry but will pass directly to a review of some easy ways to model a line reflection in the plane. The line reflection, after all, is the basic building block of all motion geometry.

Mathematically, the line reflection in a plane has the following definition:

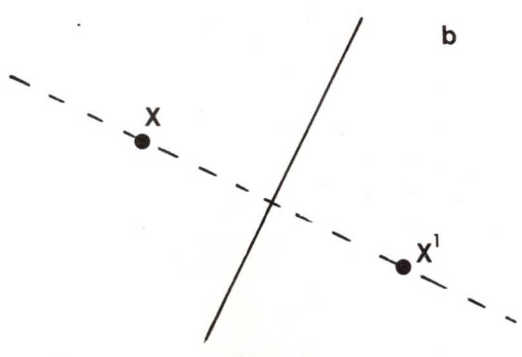
> Line Reflection of a Plane: Given a point x and a line b, x^1 is the reflection of x in line b if and only if b is the perpendicular bisector of the line segment determined by x and x^1.

In the remainder of this article, 5 easy ways of modelling a line reflection will be catalogued.

METHODS OF LINE REFLECTION

Construction

There is an obvious way of finding the reflection of a point in a line simply by constructing it with a compass and straightedge. In the diagram below, a line perpendicular to line b and passing through point x has to be constructed. Then x^1, the reflection of x in line b is constructed on this perpendicular so that it is in the opposite half-plane of b from x. Furthermore, x^1 must be the same distance from b as x is from b.



[1] *The mathematics teacher* since 1968 has had a number of articles which make a strong case for motion geometry.

Paper folding

In the diagram below, x is to be reflected in line b.

1. Mark the point x heavily with a soft-leaded pencil.
2. Fold the paper along line b, as in Figure 2.
3. After the folding, one should be able to see the markings for point x through the paper. On the backside of the paper, mark the position of point x with a pencil.
4. Open the paper and one will find that some of the marking from x will have been transferred to another point. This marks point x^1, the reflection of point x in line b.

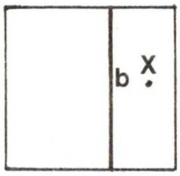


Figure 1

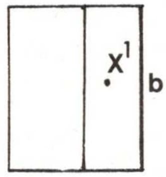
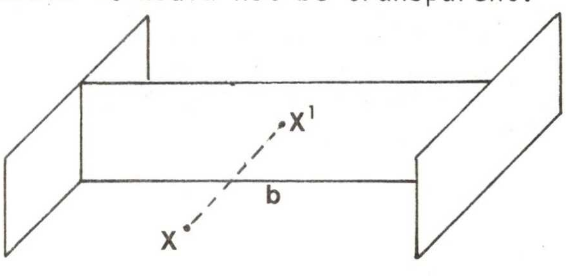


Figure 2

Mira

Mira[2] is a commercially available device that can be used to illustrate a line reflection. Incidentally, this device could be nicely used in a physics class when dealing with reflections there. The device is pictured below. It is made of some type of red plexiglass material. Again we have a point x which is to be reflected in line b. Now the bottom edge of the Mira is placed along the line b. By looking at the Mira one can see the reflection of point x. By reaching behind the Mira, one can mark the point x^1 in line with the reflection of point x. The point x^1 is the reflection of x in line b, as before. The plexi-glass material is both transparent and also reflective. A silvered mirror would not work because it would not be transparent.



Transposition

Again we have a point x which is to be reflected in line b (Figure 1).

1. Mark an arbitrary point z on line b (Figure 3).
2. Lay another sheet of paper down on top of the original sheet. Mark point x, point z and line b on this second sheet.
3. Turn the second sheet over using line b as an axis. Lay it down on top of the first sheet so that line b and point z in the two sheets correspond. Point x in the second sheet now corresponds to point x^1 in the first sheet. x^1 is the reflection image of x.

---

[2]Mira is sold by Moyer-Vico Ltd., 10924 - 119 Street, Edmonton. The device sells for $2.95 individually or $2.36 each in classroom lots of 32 or more. A brochure is included which explains uses.

4. The position of x^1 can be marked by sticking a pin through point x in the second sheet. If a soft-leaded pencil is used on the second sheet then x^1 could also be determined in a manner similar to that of method two above.

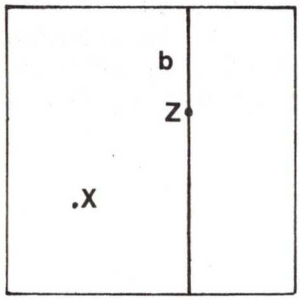


Figure 3

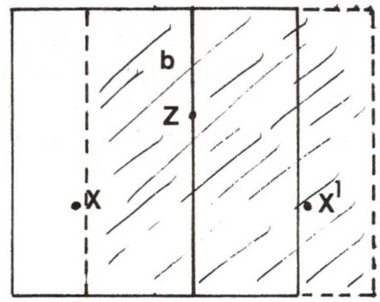


Figure 4

Line reflections in the coordinate plane

Performing line reflections in the coordinate plane is another easy and valuable experience in motion geometry. As an example, let the line of reflection be the graph of x = 4. Then the reflection of (7,3) will be (1,3). The points and lines of reflection can be varied to provide much valuable experience in motion geometry and also in the coordinate plane.

CONCLUDING REMARKS

In the examples given above, a single point was reflected in a line. Mathematically, a reflection in a line is defined point-to-point. However, in the classroom different kinds of geometric figures should be reflected in lines besides single points.

What conjectures can be made when a geometic figure is reflected in one line and that image is then reflected in a second line? This is an example of an open-ended question that would furnish a good introduction to motion geometry.

Reflections in a line lead naturally to symmetry conditions. Mathematically, symmetry can be defined only in terms of reflections.

Invariance is a key idea in motion geometry. What properties or characteristics of some geometric figure also hold for its reflection image? This is the essential question in experiments with invariance.

The 3 methods - paper folding, Mira, and transposition - mentioned above appear to be more valuable as mathematical experiences than do construction and line reflections, which both are less illustrative of the intrinsic properties of a line reflection. These judgments should be considered in choosing classroom models for a line reflection.

In working with the Mira method, does the relative position of the eye, with respect to point x and line b, make any difference? A good amount of mathematics and experimentation is involved with that question.

In this article, some models for a line reflection in the plane have been described along with some questions and ideas that lead on from such an introduction. The author hopes that classroom teachers will be able to try out some of the ideas described here. The author would enjoy hearing from anyone who has tried any of these methods or others.

An aid to 'uncovering' mathematics: a select bibliography

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Some books are to be tasted, others to be swallowed,
and some few to be chewed and digested.

— Francis Bacon

An aid to 'uncovering' mathematics: a select bibliography

Some mathematics teachers have traditionally thought of their work in terms of "covering topics". Given a class and a certain number of periods per week, one took a textbook and proceeded to cover the material. It was all quite straightforward, but there were two problems. The first was that for all too many students, one might just as well have been covering "Geometric Constructions", or whatever, with a blanket. The other problem was that over the years the danger of slipping into a pedagogical rut was very great.

Many teachers now find it helpful to think, not just in terms of "covering" topics, but also in terms of "uncovering" them as well. Instead of just pulling students through a well-travelled but narrow groove, they attempt to open up to students a number of alternative areas related to a given topic. To work effectively in this manner, however, one needs to have a knowledge of a considerable range of source materials. The writer has found the following books to be of help in "uncovering" mathematics, both for himself and for his students. Hopefully they will be of use to other mathematics teachers as well.

Three particular groups of teachers have been kept in mind during the compilation of the bibliography. They are:
- those who are in a position to suggest library purchases in their school;
- those seeking sources for "option" or "enrichment" lessons;
- those wishing to increase their own mathematical knowledge.

Mathematical soundness was the only essential criterion that the books had to meet. The bibliography is by no means exhaustive but does include a large percentage of what the writer believes to be the best works in the field. To maximize information while minimizing space the following coding system has been employed:

* recommended	$ good value for money
** highly recommended	$$ especially good value for money
*** most highly recommended	hb hardback edition
	pb paperback edition

The "reading level" for each book has been estimated on a scale ranging from (1) to (6). The scale is roughly linear, with books graded (1) being readable by the least capable junior high students and those graded (6) being readable by the most capable senior high students.

Nearly all the titles should be obtainable through any good bookstore and most of the books will be available through the larger libraries. Book prices have been obtained from the 1972 edition of *Books in print*. A question mark following an entry signifies that the information was not verified.

A few titles are available only in the United Kingdom; these have U.K. following the publisher in the entry. Several titles are significantly cheaper in their U.K. editions and are priced U.K.£__. W. Heffer and Sons Ltd., 20 Trinity Street, Cambridge, England CB2 3NG, have an efficient and friendly mail-order service. They can also be paid in Canadian funds by cheque; £1.00 is approximately $2.52.

For convenience, the bibliography has been divided into five sections: Texts and teachers' handbooks, Special topics, General surveys, Recreational and activity, and Associations, journals and bibliographies.

TEXTS AND TEACHERS' HANDBOOKS

Banwell, C. *et al*, *Starting points*. Oxford University Press, 1972, 246 pp.
▸ ***, (3), pb $7.75 (U.K. £2.75). Described as a "collection of suggestions for the teaching of mathematics", this highly imaginative, lively, handbook has sections on methodology, situations and materials. Appropriate for junior high school and upwards; British terminology.

Del Grande, J.J. *et al*, *Math, book 1*. Gage Educational, 1971, 342 pp.
▸ ** (2), hb $5? The first text in a new series written in Ontario. The second volume, *Math, book 2*, for Grade VIII, appeared in 1972; others are to follow. A smooth integration of standard topics such as ratio with non-standard ones such as Papygrams and Pick's Theorem. Colorfully illustrated and well produced.

Hess, A.L., *Mathematics projects handbook*. D.C. Heath, 1962, 60 pp.
▸ (4), pb $2? Although somewhat dated, this pamphlet still has a number of worthwhile suggestions for the teacher considering project work. Topic list and bibliographies.

Jacobs, H.R., *Mathematics: a human endeavor*. W.H. Freeman, 1970, 529 pp.
▸ **, (4), $, hb $8.50. Subtitled "A textbook for those who think they don't like the subject", this book is just that. A painless introduction to functions, logs and statistics with well-chosen, interesting exercises. Generously, aptly and humorously illustrated. A must for "Peanuts" fans and libraries.

Johnson, D.A. and G.R. Rising, *Guidelines for teaching mathematics, second edition*. Wadsworth, 1972, 544 pp.
▸ (5), hb $9.95. Probably still the best of the "Math Ed" texts. Sometimes rather stuffy but complete. Very good appendices on instructional aids, enrichment materials and publications.

Paling, D. *et al*, *Making mathematics 1, second edition*. Oxford University Press, 1971, 95 pp.
▸ **, (1), $, hb $1.50? (U.K. £0.50). The first of a four volume series intended for the "non-academic" secondary school student in Britain. Simplicity without condescension. Workbooks and topic books are available to accompany the course.

Wheeler, D.H. (ed.), *Notes on mathematics in primary schools.* Cambridge University Press, 1969, 340 pp.
▸ **, (3), $pb, pb $5, hb $10? (U.K. pb £1.50, hb £3.15). By members of the Association of Teachers of Mathematics in the U.K. Don't be misled by the title; this is a treasure-trove of starting points from elementary school to senior high. Well illustrated; bibliography.

SPECIAL TOPICS

Adler, I., *Magic house of numbers.* Signet, 1957, 126 pp.
▸ (2), $pb, pb $0.60 (hb $3.69, John Day). One of the better collections of number curiosities, calculating tricks and games.

Bell, E.T., *Men of mathematics.* Simon and Schuster, 1963?, 596 pp.?
▸ (4), $pb, pb $2.95, hb $7.95 (U.K. pb £0.60, 2 volumes, Penguin). Biographies of some 30 mathematicians from Zeno to Cantor. An entertaining presentation which emphasizes the social context of the mathematician's work.

Budden, F.J., *An introduction to number scales and computers.* Longmans (U.K.), 1965, 192 pp.
▸ *, (4), pb £0.65. A comprehensive survey of "bases" with applications ranging from the elementary to the complex. Good exercises with answers; bibliography.

Cundy, H.M. and A.P. Rollett, *Mathematical models, second edition.* Oxford University Press, 1961, 286 pp.
▸ **, (5), $, hb $6.50 (U.K. £1.50). The classic work in this area. Models of all sorts: wire, wood, perspex; linkages, knots, curvestitching, polyhedra. Just the thing for the "meccanno"-ly minded. Lots of "real" mathematics here; bibliography.

Davis, P.J., *The lore of large numbers.* Random House, 1961, 165 pp.
▸ *, (4), pb $2.50. A title in the SMSG monograph series. A potpourri of number lore at many levels. A good reference to have around; excellent problems and appendices.

Fielker, D.S., *Topics from mathematics.* Series. Cambridge University Press, 1967 and on, 32 pp.
▸ *, (1), $, pb $1 (U.K. £0.30). Fielker has written four booklets in this series - "Cubes", "Computers","Towards probability" and "Statistics". The series (See also J. Mold.) is straightforward and well written. Junior high students would enjoy working through some of them for a project.

Gardner, M., *Mathematics, magic and mystery.* Dover, 1956, 176 pp.
▸ * (3), $$, pb $1.50. One of Gardner's earlier books but of the high quality we have come to expect from him. Just the thing for the back-corner blackjack players or the nimble-fingered teacher.

Golomb, S., *Polyominoes.* Scribners, 1965, 181 pp.
▸ ** (3), hb $6.50. Golomb invented polyominoes when he was a graduate student at Harvard in the early 1950s. This "space filling" situation (the

3-dimensional version is marketed as "Soma") has appeal at all levels. Some fairly high level combinatorial theory can be painlessly taught here.

Johnson, D.A. and W.H. Glenn, *Exploring mathematics on your own.* Series. McGraw-Hill, 1961, 64 pp.
▸ *, (3), pb $1.80 per booklet. There are 16 titles in this series: "Adventures in graphing", "Basic concepts of vectors", "Computing devices", "Curves in space", "Finite mathematical systems", "Fun with mathematics", "Geometric constructions", "Logic and reasoning", "Number patterns", "Numeration systems", "Probability and chance", "Pythagorean theorem", "Sets", "Shortcuts in computing", "World of measurement" and "World of statistics". They make good companion or enrichment booklets to many areas.

Mold, J., *Topics from mathematics.* Series. Cambridge University Press, 1967 and on, 32 pp.
▸ * (1), $, pb $1 (U.K. £0.30). Mold's titles in this series are: "Circles", "Rolling", "Solid models", "Tessellations" and "Triangles". See comments on D. Fielker.

Stover, W., *Mosaics.* Houghton-Mifflin, 1966, 34 pp.
▸ *, (5), pb $2? A small but fully-packed booklet on nets/tiling patterns. Exercises, project suggestions and bibliography.

Walter, M.I., *Boxes, squares and other things.* NCTM, 1970, 88 pp.
▸ *, (3), pb $1.80. Subtitled "A teacher's guide for a unit in informal geometry", this booklet describes a mathematical journey from carton folding to group theory. Suggestions for extension of work and an excellent bibliography.

Wenninger, M.J., *Polyhedron models.* Cambridge University Press, 1971, 208 pp.
▸ *, (4), hb $15? (U.K. £5). This book presents 119 polyhedra, from the tetrahedron to the great dirhombicosidodecahedron. Photographs, nets and advice for every polyhedron; bibliography. (Caution - may be addictive.)

Wisner, R.J., *A panorama of numbers.* Scott Foresman, 1970, 176 pp.
▸ **, (4), $, pb $2? A delightful introduction to number theory. The complete preface to this book reads, "I wrote this book because I wanted to." We should be pleased that he did.

GENERAL SURVEYS

Bergamini, D., *Mathematics.* Time-Life Books, 1963, 200 pp.
▸ **, (3), hb $7.60. A volume in the Life Science Library with 8 well-written chapters. Superbly illustrated; another library must.

Hogben, L., *Mathematics in the making.* MacDonald (U.K.), 1960, 320 pp.
▸ *, (5), hb £3.50. A well-written and profusely illustrated view of the development of mathematics. Despite the generally advanced nature of most of its topics, junior high readers could browse with benefit.

Kasner, E. and J. Newman, *Mathematics and the imagination.* Simon and Schuster, 1964?, 380 pp.?
▸ *, (5), $U.K., pb $1.95, hb $4.50 (U.K. £0.40, Penguin). Some quite original material in this classic. Chapters on puzzles, paradoxes, topology and calculus are written with flair and are well illustrated.

Kline, M. (ed.), *Mathematics in the modern world.* Freeman, 1968, 409 pp.
▸ **, (6), $pb, pb $6.50, hb $10. A collection of 50 readings taken from *Scientific American* over a period of 20 years. Sections on biography, foundations and applications. Compact, well written articles, most by first-class mathematicians, make this a valuable reference volume.

Newman, J.R. (ed.), *The world of mathematics.* Simon and Schuster, 1956, 2,537 pp.
▸ **, (6), $pb, pb $15, hb $30. Subtitled "A small library of the literature of mathematics from A'h-mose the scribe to Albert Einstein presented with commentaries and notes", this massive 4-volume set is brilliantly edited, and contains much material that is almost impossible to obtain elsewhere, including original papers and large sections of out-of-print books. The volume titles are: *1. Men and numbers*, *2. World of laws and the world of chance*, *3. Mathematical way of thinking*, *4. Machines, music and puzzles.*

Sackett, D., *The discipline of numbers: foundations of mathematics.* S. Low, Marston (U.K.), 1966, 128 pp.
▸ *, (3), $, hb £1.70. A volume in the Foundations of Science Library, this book has an applications orientation to most of the standard secondary school topics. Very well illustrated.

Sawyer, W.W., *Introducing mathematics.* Series. Penguin.
▸ *, (5), $. This 3-volume set contains a wealth of teaching methods and suggestions. Sawyer has no peer as a popularizer of traditional mathematics. The volume titles are: *1. Vision in elementary mathematics*, 1964, 346 pp., pb $1.75 (U.K. £0.40), *2. The search for pattern*, 1970, 349 pp., pb $1.95 (U.K. £0.40), *3. A path to modern mathematics*, 1966, 224 pp., pb $1.25 (U.K. £0.30). Two earlier books of considerable merit also are *Mathematician's delight*, 1943, 238 pp., pb $1.25 (U.K. £0.20) and *Prelude to mathematics*, 1955, 214 pp., pb $1.25 (U.K. £0.20).

Stein, S.K., *Mathematics: the man-made universe, second edition.* Freeman, 1969, 415 pp.
▸ *, (6), hb $8.50. Subtitled "An introduction to the spirit of mathematics", this book catches the flavor of mathematical research in several of the chapters. High-powered but not overwhelming; extensive exercises and references.

RECREATIONAL AND ACTIVITY

Ball, W.W.R., *Mathematical recreations and essays, 11th revised edition.* Macmillan, 1939, 418 pp.
▸ **, (5), $$pb, pb $2, hb $6. The granddaddy of the mathematics recreation books, with the first edition in 1892. This edition is revised by H.S.M. Coxeter. Crammed full of material; quite a few "new" games can be found if one is diligent.

Domoryad, A.P., *Mathematical games and pastimes*. Pergamon (U.K.), 1963, 298 pp.
▸ **, (5), hb £2.05. The Russian view of most of the standard mathematical recreations, which frequently differs significantly from the western approach.

Gardner, M., *Scientific American book of mathematical puzzles and diversions*. Simon and Schuster, 1959, 178 pp.
▸ ***, (4), $pb, $$pb U.K., pb $1.45, hb $5.95 (U.K. 0.30, Penguin).

————————, *Second Scientific American book of mathematical puzzles and diversions*. Simon and Schuster, 1961, 253 pp.
▸ ***, (4), $pb, $$pb U.K., pb $1.95, hb $4.95 (U.K. 0.30, Penguin).

————————, *New mathematical diversions from Scientific American*. Simon and Schuster, 1966, 253 pp.
▸ ***, (4), $pb, pb $2.95.

These books are the first 3 collections of Gardner's excellent monthly column, "Mathematical games", in *Scientific American*, which has been running continuously since 1956. Gardner writes with great clarity. His column now serves as a meeting point for some of the best mathematical minds of the day.

Kraitchik, M., *Mathematical recreations, second revised edition*. Dover, 1953, 330 pp.
▸ *, (5), $, pb $2.50. Another classic, this book is based on articles from the recreational mathematics magazine, *Sphinx*, which was published in the 1930s.

Lukacs, C. and E. Tarjan, *Mathematical games*. Pan, 1970, 192 pp.
▸ *, (2), $$pb, pb $1, hb $4.95 - Walker. A Hungarian view of recreational mathematics. Of particular value are some analyses of popular games such as Solitaire (Hi-Q).

Merrill, H.A., *Mathematical excursions*. Dover, 1957, 145 pp.
▸ *, (3), $, pb $1.50. Subtitled "Side trips along paths not generally travelled in elementary courses in mathematics", this is a particularly clearly written exposition of several recreational stalwarts.

Pearcy, J.F.F. and K. Lewis, *Experiments in mathematics*. Longmans (U.K.), 1966-67, 64 pp., 3 "stages".
▸ *, (3), $, pb £0.35. A collection of activities for a lab-type approach. Appropriate for the junior high level.

Steinhaus, H., *Mathematical snapshots, second edition*. Oxford University Press, 1969, 311 pp.
▸ *, (4), hb $7.50. A collection of several quite original recreational mathematics topics from a noted Polish mathematician. Particularly good use of photographs.

ASSOCIATIONS, JOURNALS AND BIBLIOGRAPHIES

Association of Teachers of Mathematics (ATM). *Mathematics Teaching* is the stimulating quarterly journal of this very active U.K. association. Overseas membership (at $8 per annum) includes a subscription. The ATM also publishes a number of excellent pamphlets such as Dick Tahta's *Pegboard games* ($0.75) and *Examinations and assessment* ($1.25). Write ATM, Market Street Chambers, Nelson, Lancashire, England BB9 7LN.

Hardgrove, C.E. and H.F. Miller, *Mathematics library: elementary and junior high school.* NCTM, 1968, 50 pp. $0.80. Annotated bibliography.

Mathematics Council of The Alberta Teachers' Association (MCATA). *Delta-K* is the quarterly newsletter of MCATA, which is affiliated with NCTM. Membership of $5 per annum includes subscription to *Delta-K* and the monographs as published. Write MCATA, Barnett House, 11010 - 142 Street, Edmonton, Alberta T5N 2R1.

National Council of Teachers of Mathematics (NCTM). *The Mathematics Teacher* and the *Arithmetic Teacher* are the two major journals of NCTM, having, respectively, secondary and elementary school orientations. The subscription fee, which includes NCTM membership, is $9 for one journal or $13 for both, per annum. Eight numbers of each journal are published per annum. NCTM also publishes *The Mathematics Student Journal* four times per annum, which is $0.60 for NCTM members or $2.50 for 5 subscriptions, and many other very good pamphlets and books on mathematics education. Write NCTM, 1201 - 16 Street NW, Washington, D.C. 20036.

Schaaf, W.L., *A bibliography of recreational mathematics.* NCTM, Volume 1, fourth edition, 1970, 148 pp.; Volume 2, 1970, 191 pp. $3 and $4 respectively. An essential for anyone seriously interested in recreational mathematics.

________, *The high school mathematics library, fourth edition.* NCTM, 1970, 86 pp. Annotated bibliography.

Sawyer, W.W. (ed.), *Student Mathematics, the Canadian student journal.* Published annually in September (1972 issue was No.3). Send $0.10 per copy and a stamped, self-addressed envelope (at least 9"x4") to Student Mathematics, Room 373, College of Education, 371 Bloor Street W., Toronto 181, Ontario.

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