Secondary School Mathematics from a Piagetian Point of View

A case will be made for rethinking how mathematics is presented in secondary schools in the light of insights which have emerged from Piaget-related research. Practicable alternative approaches to specific topics will be described.

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INTRODUCTION

High school mathematics teachers tend to see students' mathematical backgrounds as very poor and their attitudes towards mathematics as improper, irrational, and negative. On the surface, this seems to be an anomaly in view of the considerable efforts which have been expended in upgrading the elementary and junior high school mathematics programs. But could the "upgrading" have been misguided?

It is the writer's opinion that curricular reforms in school mathematics have encouraged teaching approaches which run counter to natural learning processes.

Piaget's theory of intellectual development clearly describes *natural* or *true* or *living learning* as originating from the *child* and *his own* interests and drives. Too often the child comes to view "school learning" as a process of mastering this or that skill "for the teacher" or "to get by" or "to beat the system", whereas in real life the child learns from active interaction with his environment, doing things that are vitally interesting to *him* and for *his own* welfare. He explores, experiments, and modifies his behavior and conceptions of the world by means of basic self-fulfilling drives.

It is essential that we organize the content and approaches in secondary school mathematics courses so that each student is able to build more effectively on the considerable talents he has already developed pursuing "real life" learning outside the classroom. To do so, we need to have some understanding of the ways in which our students learn and how their learning strategies are affected by their level of intellectual development at any given time.

*CONCRETE VERSUS FORMAL OPERATIONS**

Secondary school students are able, by and large, to operate with mathematical ideas at either a *Concrete operations* level or a *Formal operations* level, or both.

According to Piaget's research, the dominant thought structure for the majority of children in the age range of seven years to eleven or twelve years is that of the *Concrete operations* stage. However, the work of Lovell (and others) with British children suggests that it is only the very brightest children who progress beyond the concrete operations stage by the age of twelve when they are working with mathematical concepts. The majority of the British students do not emerge from the concrete operations stage until age fourteen or fifteen and some never do in the context of mathematical tasks (Lovell, 1971).

However, as will be pointed out later, the pegging of ages to stages is much less important than realizing that children's and adult's thinking patterns do repeatedly progress through stages like those described by Piaget.

What is *concrete-operations-stage* thinking like? As described by Hermine Sinclair (1971), one of Piaget's Genevan colleagues, in the stage of concrete operations,

> ... the child can think in a logically coherent manner about objects that do exist and have real properties and about actions that are possible; he can perform the mental operations involved when asked purely verbal questions and when manipulating objects [pp.5-6].

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*For a more complete summary of Piaget's stages of intellectual development and a discussion of implications for elementary and secondary school mathematics education, see Harrison, 1969.

The concrete operation child "... can manipulate and think about real objects but he cannot work with hypothetical entities [Sinclair, 1971, p.6]."

The operations available to the child in this stage of development include *classification* (putting objects together in a class and separating a collection into sub-classes), and *seriation* (ordering things, like numbers, or events in time).

Again as Sinclair (1971) has said:

> These operations are transformations that are reversible, either through annulment (as in the case of adding, annulled by subtracting) or through reciprocity (as in the case of relationships: A is the son of B, B is the father of A) [p.7].

Using their classification skills, concrete operational children can successfully make comparisons between a general set of objects and its subsets; they can determine elements in the interaction of given sets; they can find missing elements in double-entry tables. Their seriation notions enable them to cope successfully with transitivity arguments such as: if $A > B$ and $B > C$, then $A > C$ [p.9].

During the concrete operations stage, the types of reasoning made possible by these operations become more powerful and are applied in more and more difficult contexts, paving the way for much more general *formal operations*.

As Lovell (1971) has indicated, "from around 12 years of age in the brightest pupils and from 14 to 15 years in ordinary pupils, we see the emergence of formal operational thought [p.7]." This stage is characterized by the development of formal, abstract thought operations with which the adolescent can reason in terms of hypotheses and not only in terms of objects. Prior to this level of development, the child thinks concretely rather than reflectively, dealing with each problem in isolation and not integrating his solutions by means of any general theories from which he could abstract a common principle. In contrast, the adolescent is most interested in theoretical problems and constructing theoretical systems (Piaget, 1968). The adolescent can identify all possible factors relevant to a problem under investigation, and he can form all possible combinations of these factors, one at a time, two at a time, three at a time, and so on. He can form hypotheses, construct experiments to test the hypotheses against reality, and draw conclusions from his findings. He need no longer confine his attention to what is real but can consider hypotheses that may not be true and work out what would follow if they were true. That is to say, in addition to considering what *is*, he can consider what might be. The hypothetico-deductive procedures of mathematics and science have become open to him (Piaget, 1964; Inhelder, 1962; Adler, 1966; Berlyne, 1957).

LEARNING CYCLES

An interesting interpretation of Piagetian theory is the view that the *concrete operations* used by an individual are "concrete" in the sense that they are mental operations involving some system of objects and relations that is *perceived* as real by the person. What is "concrete" is relative to the person's past experience and mental maturity. While the kindergarten child considers the union of two beads with three beads as a concrete operation but the addition of 2 and 3 as not, the introductory algebra student considers 2 + 3 as concrete but not x + y. The student of introductory abstract algebra considers the additive group of integers to be concrete but not so the concept of an abstract group. So the progession goes, and it is evident that "concrete" operations are used not only in the concrete operations stage, in which they are the most advanced operations of which the child is capable, but also at all succeeding levels of learning. In the development of new concepts at any level it is essential to proceed from what the learner perceives as concrete to what to him is abstract (Adler, 1966). Indeed, as Ausubel and Ausubel (1966) said,

> Even though an individual characteristically functions at the abstract level of cognitive development, when he is first introduced to a wholly unfamiliar subject field, he tends initially to function at a concrete-intuitive level [p.407].

A similar point of view has been taken by Dienes (1966) in his postulation that the learning of abstract concepts can be thought of as occurring in cycles which can be regarded as microscopic copies of Piaget's developmental cycle - that is to say, the concrete operations to formal operations cycle (at least) repeats at higher and higher levels of abstract learning.

GENERAL WAYS OF KNOWING

A recurring theme through Lovell's (1971) excellent paper "Intellectual Growth and Understanding Mathematics" is the characterization of Piaget's work on intellectual development as describing the gradual development, in a person, of *general ways of knowing*. General ways of knowing have to be actively constructed by the child through active interaction with his environment. Once a child's experiences make him aware, for example, of the relationship between a class and a sub-class, he never loses that idea in mental health. The quality of a child's *general ways of knowing* determines the manner in which, and the extent to which, he will be able to assimilate any particular knowledge he is exposed to in school settings.

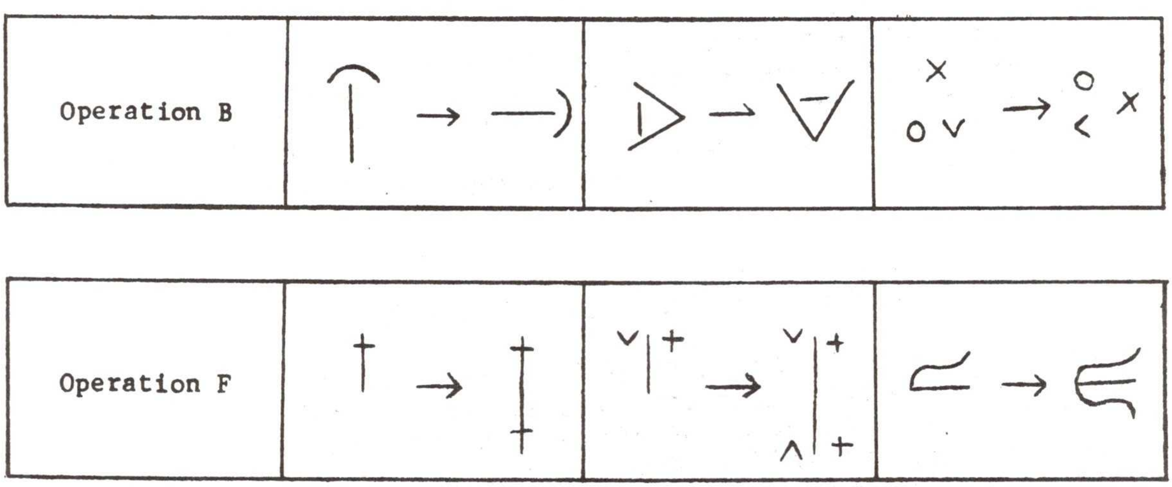
SKEMP'S REFLECTIVE INTELLIGENCE

Richard Skemp (1958), a psychologist at the University of Manchester, has stated that the chief ability required in mathematics at the secondary school level is the ability of the mind to become aware of and to manipulate its own concepts and operations, an ability he calls *reflective intelligence*. Reflective intelligence notions tie in very closely with characterizations of thinking at the formal operations stage. Skemp has devised tests to measure student abilities to reflectively manipulate concepts and operations. Consider, for instance, his operations test. In the first part of the test (SK6, Part I), the subject is required to operate on test figures using operations illustrated on a Demonstration Sheet by means of three examples. In the second part (SK6, Part II), the subject is required to demonstrate "combining", "reversing", and "reversing and combining" operations on test figures. Since to do this test, the subject must have discovered what the ten basic operations in SK6, Part I, are, the operations are explained to the subjects after the administration of Part I but before Part II. There are five "combine ", five "reverse", and five "reverse and combine" items in SK6, Part II. Some sample operations and sample items from Skemp's operations test are reproduced on the following page.

In a study involving 50 fifteen-and sixteen-year-olds, Skemp (1958) found an amazingly high correlation of 0.72 between the students' scores on this test and their scores on a general certificate of education mathematics examination (something like a C.E.E.B. mathematics examination at a younger age).

In 1966, the writer administered Skemp's tests to two classes of students at each of the Grades V through XI levels (a total of 340 students). A plot of their mean scores on Skemp's reflective intelligence test (SK6(2)) by student age levels is made in Figure 1. Overlooking the fourteen-year-olds (most of whom were frantically preparing to write external Grade IX examinations at the time tested and their mean scores were not statistically significantly lower than twelve-and thirteen-year-old means in any case) one could say that, in general, the SK6(2) mean scores increased with increasing age. Such evidence gives further support to the notion of a gradual development with age of more sophisticated levels of thinking or "general ways of knowing".

SK6: DEMONSTRATION SHEET (Sample Operations)*



*Sample Operations and Sample Items from Skemp's SK6 test are reproduced from Harrison, 1967, pp. 312, 313, 319, 320.

SK6: PART II (Sample Items)

In Part II, the problem is to combine the operations on the Demonstration Sheet, or to do them in reverse, or both. When combining operations, they are to be done in the order given (that is, "Combine C and G" means "Do Operation C first and then do Operation G.").

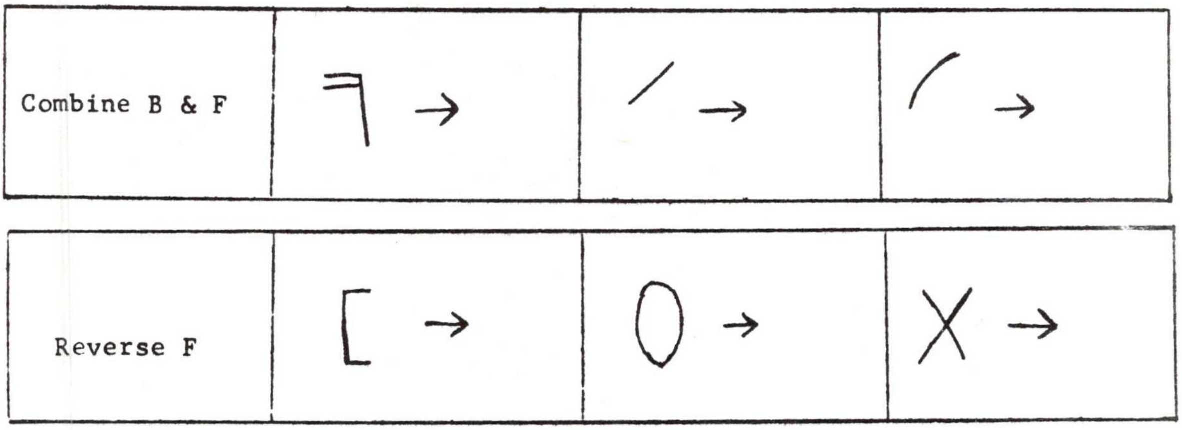
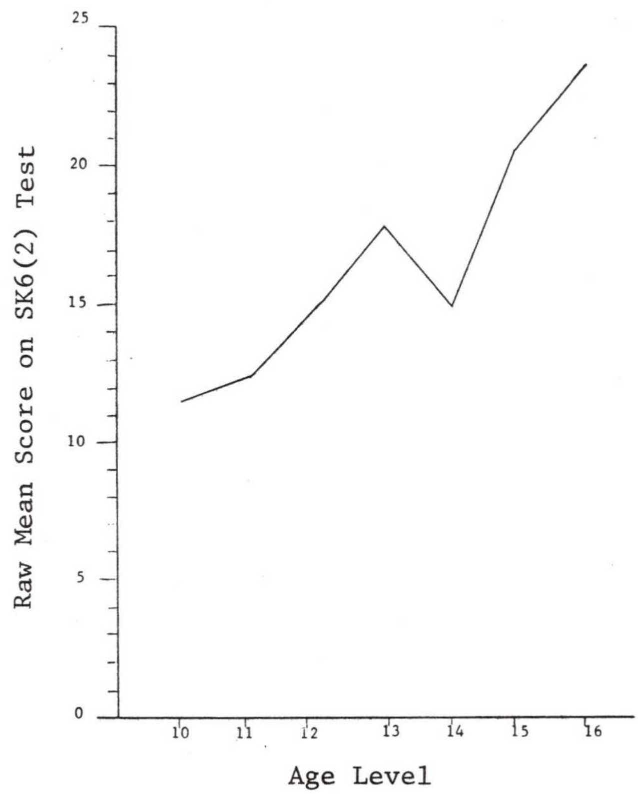


Figure 1
THE RELATIONSHIP BETWEEN MEAN SK6(2) SCORE
AND AGE LEVEL OF STUDENTS TESTED*



*Reproduced from Harrison, 1967, p. 225.

MATH CONCEPTS STUDIES

Lovell (1971) has gathered evidence from numerous research studies to support his contention "... that it is the development of ... general ways of knowing which determines the manner in which taught material is understood [p.10]."

For instance, many studies with British and American pupils have

> ... confirmed that apart from very able 12-year-olds, it is from the beginning of junior high school onwards - the actual age depending on the ability of the pupil - that facility is acquired in handling metric proportion [for example, constructing a rectangle similar to but larger than a model]. Many pupils may not be able to do this until 14 to 15 years of age and some never [Lovell, 1971, p.8].

For example, in one study cited by Lovell it has been found that it was not until the onset of formal operational thought, around 14 years of age, that the majority of British students tested "... were able to dissociate, completely, area and perimeter of square/rectangle, and realize that under certain changes area is conserved and not perimeter, while under other changes the reverse is true [Lunzer cited by Lovell, 1971, p.6].

Still another example has arisen from Reynold's investigation of "The Development of the Concept of Mathematical Proof in Abler Pupils", involving students at the equivalent of our Grades VII, IX, and XII. This study showed that answers "... that were characteristic of the concrete-operational stage of thinking appeared regularly, but the answers also indicated an increasing ability to use formal-operational thought with age [Lovell, 1971b, p.77]."

FUNCTION STUDIES

Following up Piaget's earlier studies of the development of the concept of functions involving laws of variation, H.L. Thomas, at Columbia in 1969, explored the understandings of more general mathematical functions that had been attained by very capable Grade VII and VIII students studying Secondary School Mathematics Curriculum Improvement Study (SSMCIS) materials (average age: 13 years; mean I.Q.: 125; very capable in mathematics). In the Grade VII SSMCIS materials, the concept of a function is introduced as a *mapping* of set A to set B, using the word "image" to refer to the object in B assigned to an element of A. The three essential elements of any mapping are described as "... a first set A whose members are assigned images, a second set B from which the images are selected, and a rule or process which assigns to each element of the first set exactly one element of the second set [Thomas, 1969, pp. 25-26]." In the Grade VII SSMCIS materials, arrow diagrams, rules, ordered pairs, and graphs are used in treating mappings, composition of mappings, inverses, translations and dilations, all in the context of developing the concept of function.

Thomas (1969) administered written function task tests to 201 Grade VII and VIII SSMCIS students and carried out detailed interviews with 20 selected students to assess their grasp of the notions about functions to which they had been exposed. Analyzing the responses, Thomas identified four stages in the development of the concept of a function which will be described later. Of the 201 written test subjects, 55 (27 percent) were rated as having attained an understanding of the function concept at the two highest stages, while 164 (82 percent) could be said to have attained a minimal concept of function. Thomas' (1971) assessment of these results (even though the individual interviews were more encouraging; 13 out of 20 showed mastery of the basic concept of a function as a special relation) was that

> It was ... a shock to this investigator to find that, in a group of students who had supposedly been carefully introduced to the concept of function, many could not distinguish functions from non-functions in simple and concrete situations. At the same time these students could carry out many of the processes associated with the function concept.
>
> One might speculate on this basis as to whether students should be allowed to work with the processes associated with function and only later learn to discriminate sharply those objects that are functions. This has, indeed, been a traditional route. Current thinking, however, runs counter to this approach [p.7].

Orton (1970), at the University of Leeds, carried out a cross-sectional study of the development of the concept of a function by individually interviewing 72 subjects ranging in age from 12 to 17; eight boys and eight girls in the upper half of the ability range in mathematics from each of the forms equivalent to our Grades VIII through XI, and eight very select mathematics students from the equivalent of our Grade XII. By Grade VIII these students had a background of sets, operations on sets, ordered pairs in various contexts, and graphs of ordered pairs. Beginning in Grade VIII, they were introduced to relations by means of arrow diagrams and mappings illustrating such relations as "is a brother of". Domain and range were defined and then a *function* was defined as a relation in which each member of the domain has only one image. Then graphing of functions was covered, followed by inverse functions and linear and non-linear functions (School Mathematics Project, Book 2, 1966, pp. 153-170). In each successive grade, relations and functions were repeatedly worked with.

Through tasks which had to be completed, the students were required to recognize functions, distinguish between functions and relations, and pick out the domain, range, and set of images in a wide variety of situations in which the relations considered were described by means of arrow diagrams, graphs, ordered pairs, tables and equations. A sample Orton function task employing an arrow diagram is reproduced on the following page.

Based on Thomas' stages and the information from his own interviews, Orton described the following stages in the development of the concept of a function.

STAGE I

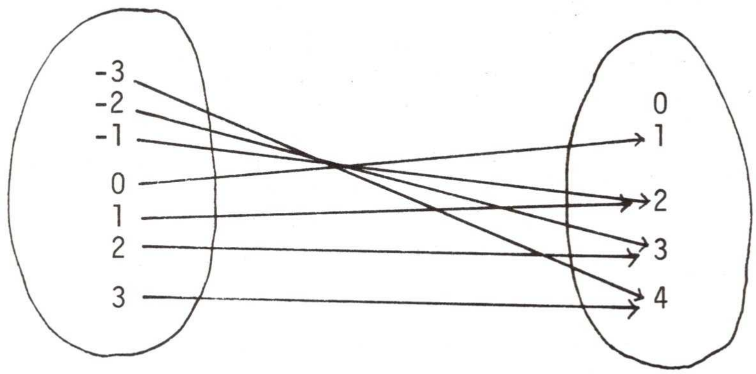
- concrete, intuitive
- can handle processes when arithmetic, or in arrow diagram or table
- concept of function as specific type of relation not mastered
- limited extension of notions in ordered pair graphs

STAGE II

- basic criterion for relation to be a function still not mastered
- good grasp of relational aspects of function concept in that able to find images, pre-images, sets of images, and domain.

SAMPLE ORTON FUNCTION TASK*

1. Study this arrow diagram for a relation which maps
   {-3, -2, -1, 0, 1, 2, 3} into {0, 1, 2, 3, 4}



(i) Write down each image of 2.

(ii) Write down each number that has 2 as its image.

(iii) Write down the domain for this relation.

(iv) Write down the set if images.

(v) Write down the range for this relation.

(vi) Is this relation a function?

STAGE III

- can identify whether a relation is a function or not in several types of representation
- mastery of basic concept of function
- care not always taken to check uniqueness of images or correct domain for inve

STAGE IV

- mastery of basic concept of function to greater degree of generality than in Stage III
- all representations of relations and their inverses classified as functions or not with precise analysis of the uniqueness criterion [Lovell, 1971a, pp. 17-

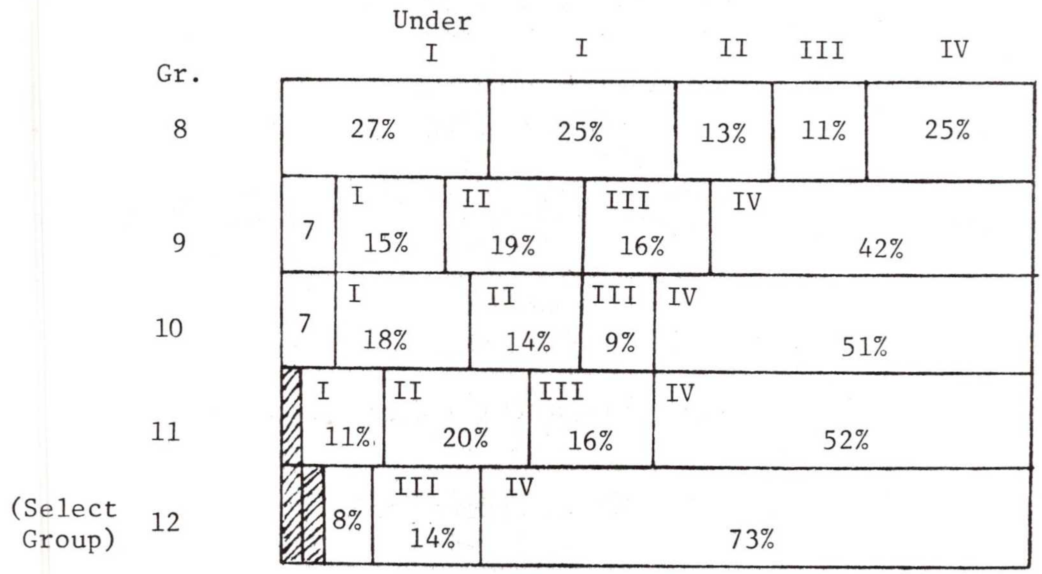
*Lovell, 1971a, pp. 25-26.

Figure 2 contains a summary of Orton's findings:

Figure 2

LEEDS STUDY (ORTON): % OF RESPONSES AT EACH STAGE BY GRADE*

Stage



The percentage of responses at the various stages at each grade level certainly supports Orton's statement that "The growth of understanding of the concept of a function takes place slowly and over a long period of time [Orton, 1970, p. 121]". At least this certainly seems to be the case in the age range sampled when the concept of a function is embodied in a fairly abstract, concise definition.

Bernice Andersen, an M.Ed. student at the University of Calgary, replicated Orton's study with 72 Calgary junior high and senior high school students. Six boys and six girls at each of the grade levels VII through XII with ability from average to above average were interviewed using Orton's tasks with minor modifications. Beginning in Grade VII, these students were given an intuitive background for the concept of function by working with sets, ordered pairs, and graphical representations of ordered pairs. They were also given, in Grade VII only, a very brief intuitive introduction to the notion of a function without the term being mentioned in the context of a very limited number of examples of one-to-one and many-to-one mappings such as these:

*Lovell, 1971a, p. 19.

$(9, 8) \rightarrow 9+8$ or 17

$(7, 1) \rightarrow 8$, $(1, 7) \rightarrow 8$, $(6, 2) \rightarrow 8$

Functions, as such, did not come up again until they were formally defined in Grade XI after a unit on relations. In Grade XI the definition presented to students was:

> A function on a set A is a relation on A such that for every element of the domain there corresponds a unique element of the range [Beesack, 1966, p. 67].

Since the Calgary students through Grade X had not been introduced to the terms "function", "relation", "domain", or "range", Orton's tasks were reworded in terms of "mapping", "set of ordered pairs", "set of first components", and "set of second components" as in the redraft of question 1, which follows.

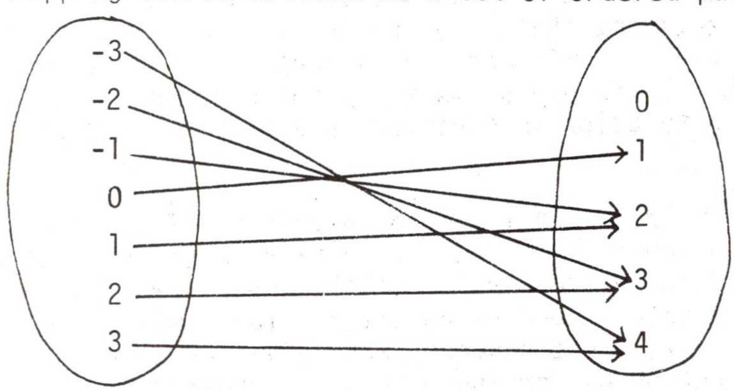
Figure 1

SAMPLE MODIFIED ORTON FUNCTION TASK*

1. Study the arrow diagram below. The arrows indicate a mapping of

   {-3, -2, -1, 0, 1, 2, 3} into {0, 1, 2, 3, 4}

   The mapping can be written as a set of ordered pairs.



Refer to Figure 1

(i) Write every ordered pair that has 2 as its first component.

(ii) Write very ordered pair that has 2 as its second component.

(iii) Tabulate the set of all first components.

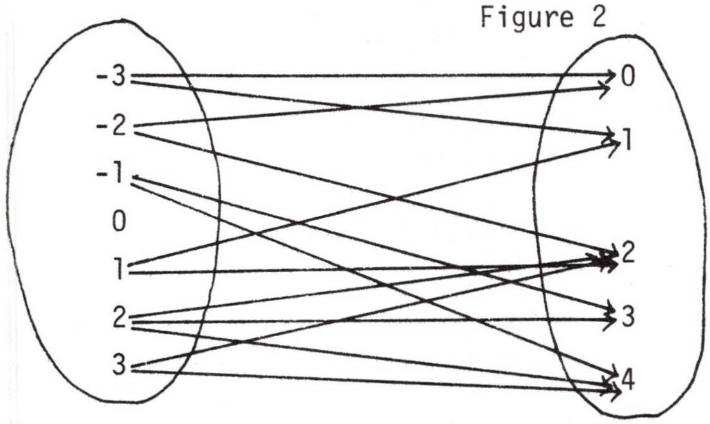
*Andersen, 1971, pp. 83-84.

(iv) Examine the set of second components illustrated by the arrow diagram. Tabulate the set of all second components that *are paired* with members of the other set.

(v) Tabulate the set of all the numbers in Figure 1 that could have been used as second components.

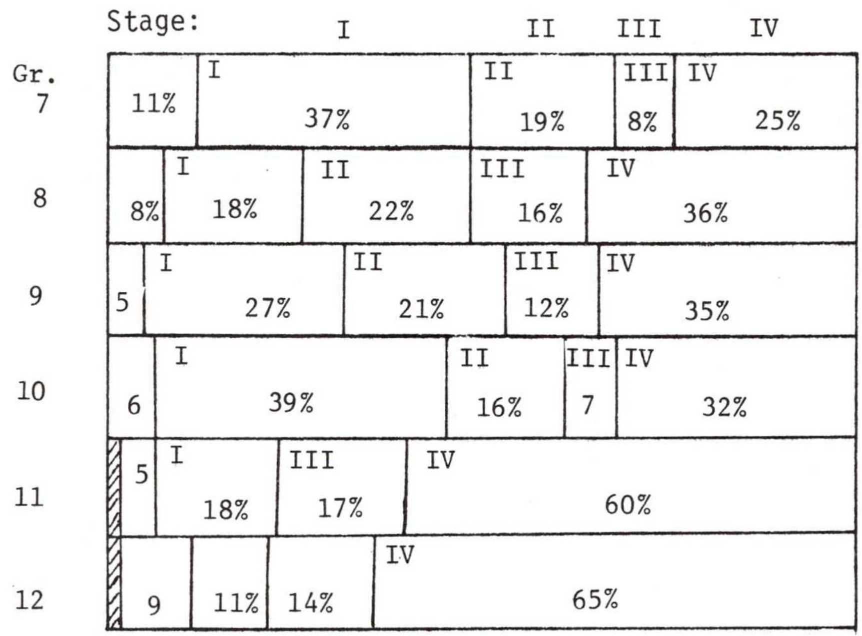
(vi) Is the set of ordered pairs indicated by Figure 2 a mapping? Discuss.

Figure 2



Orton's interview procedures, scoring and assignment of responses to stages were closely followed in the Calgary study. Figure 3 summarizes the results.

Figure 3
CALGARY STUDY (ANDERSEN): % OF RESPONSES
AT EACH STAGE BY GRADE*

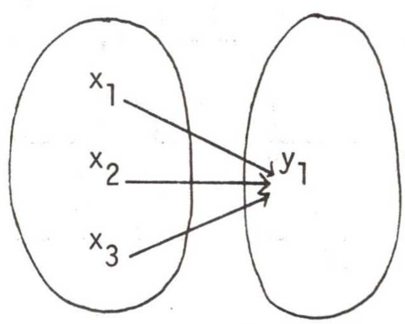


*Andersen, 1971, p. 31.

Comparing the results of the Calgary study with those of the Leeds study, focussing on the percentages of responses at the Stage IV level, it appears that the Calgary students in Grades VII and VIII (who were not introduced to a formal definition of function at all, except in the context of the modified Orton tasks) were able to sort out the function notions in Orton's tasks just as well as the Leeds eighth graders who had been "taught" a formal definition of function. There is no marked increase in the percentage of Stage IV responses in the Calgary results until Grade XI, whereas the Leeds results show a gradual growth under repeated coverage of function properties coupled with the formal definition. It is in Grade XI in Calgary schools that the formal definition of a function has been given in the past, and, as can be seen, the percentages of Stage IV responses at the Grade XI level for Calgary and Leeds students are very similar. Keeping in mind that the twelfth-grade-equivalent Leeds students were a very select group of mathematics majors while the Calgary group was not, the Grade XII Calgary results also appear comparable to those of the Leeds study. In fact, statistical comparisons between the Calgary and Leeds mean function task scores across the grade levels showed no significant differences. Admittedly, comparisons such as the foregoing are fraught with all kinds of pitfalls, but considering that, as far as could be determined, the Calgary sample was certainly not more capable academically than the Leeds group, the patterns in the findings do seem to suggest that the three years of formal work with functions prior to Grade XI might not be time well spent. Perhaps a better use of the time would be to have students explore function ideas in very concrete contexts (such as those suggested later in this paper) building a firm intuitive foundation for the later formal definition.

Many of Orton's Grade VIII students had not appreciated the basic definition of function and wanted to define function as a relation which produced a pattern when plotted as an ordered pair graph. Accordingly, he recommended that, if it is desired to use the modern definition of function, functions involving proportionality, which produce a particular kind of graphical pattern, should not be introduced too soon as pattern confuses the issue if it is mentioned before the basic definition of function is appreciated. (Interestingly enough, the Calgary Grade IX students study a section on graphs of formulas involving direct and inverse variation.)

Orton also found that many students in all of the year groups (whose notions of functions were in terms of one-to-one or many-to-one relations) confused many-to-one and one-to-many.



Some students saw the above many-to-one relation as one-to-many because there was only one arrow leaving each member of the domain, but *many* arriving at a single image. Because of this common confusion, Orton (1970) says that "it is better to attempt to define a function in terms of uniqueness of images of members of the domain, as, for example [in the routine way favored by some subjects], that in an arrow diagram, [of a function] only one arrow leaves each member of the domain [p. 140]."

The Grade VIII and IX students were generally not able to interpret the graphical representations of relations with any confidence. Difficulties were frequently encountered in the finding of images for given pre-images and, vice-versa, finding the domain and range, and converting the graph of a relation into an arrow diagram or set of ordered pairs. Orton (1970) hypothesized that " ... in the early stages of the acquisition of the conept of a function, and the concept of an (x,y) graph, a situation which involves both concepts is too difficult [p. 142]."

In discussing tasks involving the differences between a relation and a function, Orton (1970) made the following comment:

> The unsatisfactory nature of responses, the number of subjects who thought there was no difference between a relation and a function, the number who thought there was no connection at all between a relation and a function [especially at the Grade VIII and IX level], must be considered, from a teaching point of view, to leave a great deal to be desired. If functions are to be appreciated through a study of sets and relations, and then at the end of such an introduction no clear idea of the connection between a relation and a function is held, then some of the point of the approach is lost [p. 142].

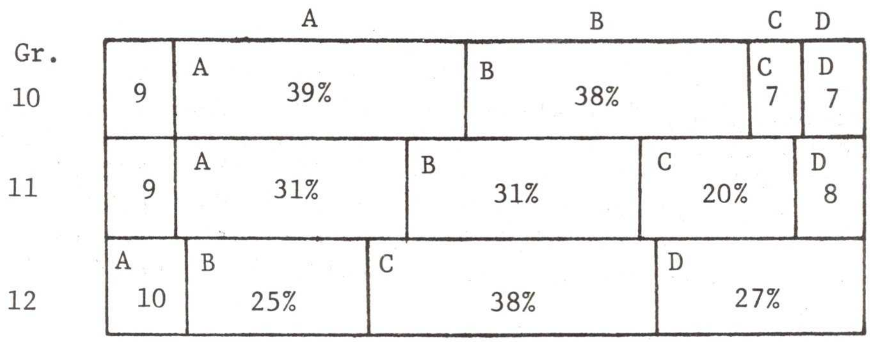
The main reason for this seemed to stem from the fact that the subtle distinction between "relation" and "relationship" was not appreciated by the younger subjects.

Many excellent insightful further observations about the difficulties encountered by students in studying functions are included in the appendix of Orton's (1970) thesis and in Lovell's (1971a) description of Orton's study. Any teacher introducing or working with functions should at least read Lovell's description of Orton's study.

The Grades X, XI, and XII students in Orton's study had been taught *composition of functions*, and they were interviewed further using tasks dealing with composition of functions and inverses. The tasks included f-notation, discrete domains, and equations. The percentage distribution of responses by grade and stage is recorded in Figure 4.

Figure 4

COMPOSITION OF FUNCTIONS
% DISTRIBUTION OF RESPONSES BY GRADE AND STAGE*



Orton (1970) interpreted the above findings in the following way: "It appears ... that the ideas associated with compositions of relations and functions and their inverses ... are not generally understood by other than the most able subjects in the [eleventh and twelfth years] [pp. 132-133]." This even after better than two years of working with functions. So, again, a gradual growth pattern is indicated when a new abstract concept is being learned. No matter how well a difficult concept is taught, we have to learn to allow for this gradual growing awareness in the student's mind about how everything fits together.

SMP, BOOKS A TO H

It is encouraging to note that an alternative version of the School Mathematics Project (SMP) series (namely, SMP Books A to H), aimed at a less select group than SMP Books 1 to 5 (which have an early formal approach to function much like that described in Orton's study), has been produced. In Grade VII in this new series, intuitive notions about relations are introduced, leading gradually into first notions about mathematical relations. This is followed by graphs of relations mapping diagrams, arrow diagrams and inverse mappings in Grade VIII. As the teacher's guide notes, the words domain, codomain and range are purposely not used at this stage since many pupils find them confusing. A *mapping* is described as " ... a special kind of relation in which each member of the starting set is related to exactly one member of the finishing set [SMP, *Book D*, 1970, p. T230]." The authors have decided in the SMP A to H series to keep the mathematical language as simple as possible; hence the use of the word function is avoided. There is apparently a very deliberate de-emphasis on verbal precision in the SMP A to H series. One cannot help feeling that the trend away from rigor toward more emphasis on preparatory, intuitive experiences at the junior high school is very healthy and certainly seems to be supported by the kinds of research summarized on the preceding pages.

*Lovell, 1971a, p. 19.

MATHEMATICS CURRICULA

The results of studies such as those to which we have referred have inescapable implications for designers of mathematics curricula. As Beilin (1971) has pointed out:

> Mathematicians who choose to teach a sequence of mathematical concepts and functions on purely *a priori* bases may encounter great difficulty having these concepts learned. Logical relations are not inevitably paralleled by psychological relations. Unfortunately, little effort has been expended in testing the relations between the conceptual systems of mathematics and the cognitive system of the child except in the most limited of circumstances [p. 118].

Indeed, it has been an all-too-common tendency for textbook authors and teachers to try to "do the whole job" in teaching an abstract concept at first exposure, when the majority of the students are really not intellectually mature enough to effectively assimilate the ideas.

Again, a warning from Beilin (which has also been given by Skemp and Lovell): "When the mathematical idea to be learned depends on a level of logical thought beyond that which the child possesses, the idea is either partially learned or learned with much difficulty and his grip on the idea is tenuous [Lovell (citing Beilin), 1971, p. 3]."

The frustrating thing about students who don't really grasp the basic patterns and ideas in the mathematics they are taught is that they learn by rote enough of what they think we expect, enough to get by on an exam, say, but they do not build the intuitive insights and understandings necessary for progress to more and more abstract ideas. The frustrating thing about teachers, especially those with strong mathematics backgrounds, is that when they find that a student is confused, they explain the idea in increasingly tidier and more abstract terms, which the student is unable to assimilate. The teacher, having the concept firmly in mind, has difficulty imagining what it would be like to assess the situation without the benefit of the concept.

In Alberta schools we have had "first generation" modern mathematics textbooks in the junior high schools with "transitional" and, more recently, "second generation" modern mathematics textbooks in the senior high school programs. A somewhat incongruous result is that the treatment given a particular topic in the senior high school text is often less rigorous and much easier to understand than the treatment of the same topic in junior high.

For example, a thorough coverage of exponents and radicals occurs in Grades VIII and IX and then re-occurs in Grade X with virtually no additional sophistication and a somewhat more straightforward approach. The Grade X teachers say they have to reteach the topic from scratch. Why?

The writer was in a Grade IX class recently in which a girl, who had transferred in from another province, was having trouble deciding what to do to simplify

$$4\sqrt{2} - \sqrt{2}$$

She was asked what kind of mathematics she had been studying in the other province, to which she replied, "modern algebra". So, the writer said, "then you know how to simplify 4x - x." She wrote 3x immediately. Then the writer rewrote $4\sqrt{2} - \sqrt{2}$ under the above expression, but she saw no connection, she just shook her head and looked bewildered.

DAVIS' APPROACH TO VARIABLES, RELATIONS AND FUNCTIONS

Our Grade IX students would be better off if they had early experiences with placeholders and variables in the way Davis (1964) approaches them in the Madison Project materials -- not just

$$2 \times \square = 5$$

but all the interesting, fun things that can be done with placeholders and functions, such as

(a) Nora's Secrets

Can you find the truth set for the open sentence:

$$(\square \times \square) - (5 \times \square) + 6 = 0\ ?$$

"Nora says she knows two secrets about this kind of equation. Do you know what she means [Davis, 1967, p. 112]?"

(b) Guessing Functions

Davis has successfully led Grade V children to develop "finite differences" strategies for coming up with rules that would generate the following tables of values. The "differences" are shown and the rules so discovered are written below the tables.

$\square$	$\triangle$	
0	2	
		) 3
1	5	
		) 3
2	8	
		) 3
3	11	
		) 3
4	14	

$$(3 \times \square) + 2 = \triangle$$

$\square$	$\triangle$		
0	3		
		) 1	
1	4		) 2
		) 3	
2	7		) 2
		) 5	
3	12		) 2
		) 7	
4	19		

$$(\square \times \square) + 3 = \triangle$$

Not only do children exposed to Madison Project materials come up with rules for tables of values of linear, quadratic and exponential functions, but they also derive rules for the patterns they see in graphical representation of these functions and explore all of the very interesting relationships and patterns that exist in the various modes in which functions can be represented - for example, a rule (or formula), table of values (or tabulation of ordered pairs), or graph. Sigurdson and Johnston (1968, 1970) provide an excellent application of this kind of approach at the Grade XI level.

(c) Madison Project "Independent Exploration Material" (often referred to as Davis' "shoeboxes"). These shoeboxes contain materials and instruction cards designed to produce data for graphs and "function guessing".

(d) Approaching functions with Cuisenaire Rods as in Davis' "Centimeter Blocks" shoebox and in the ways described by Gail Lowe (1972) (e.g., using the white rod as a stamp to cover: individual rods, rods placed end-to-end ("trains"), rods placed side-by-side, side-by-side and staggered, "pyramids", etc.).

The preceding are only a very small sample of the rich sources of ideas and materials for enabling children to explore concepts like functional relationships in a very concrete and interesting way (an excellent annotated compilation of manipulative materials currently available can be found in *Fabric of Mathematics*, Laycock and Watson, 1972).

Although this paper has focused on functions, similar cases could be made for the development of concepts such as mathematical proof (Sample Research: Reynolds, 1967; Lovell, 1971b; Sample Approaches: Davis, 1967), and proportionality and probability (Lovell, 1971c).

SUMMARY

One interpretation of the presently available evidence from classroom research conducted along Piagetian lines is that, at least until the end of junior high school for most students (and even longer for some), the main focus in presenting mathematics in schools should be on providing rich concrete experiences as a foundation for meaningful formalizations in the high school years. Children can certainly begin working with functions, for example, in elementary school but in a very concrete context, and they should have frequent access to concrete embodiments of functional relationships until each child *himself* is ready to progress to a formalized, generalized, abstract conception of what principles are embodied in the many related concrete experiences he has had.

If the reader has any doubt as to what can be accomplished under a student-oriented, active-learning approach, Davis' *Experimental course report: Grade nine* (Davis, 1964a) would make very interesting reading indeed.

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