When is the Thing the Thing?

It is important to recognize what things can do and what things can't do. Things can build images, but they can't teach the students to compute. They can provide experiences, but they can't make generalizations. Examples from the elementary classroom will illustrate when the thing's the thing.

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INTRODUCTION

I want to relate to the idea of using things to teach mathematics. I also want to clarify a few things related to these things. There is, for example, the idea of a mathematics laboratory which has two meanings, one of which is a physical place. I believe in the other meaning. Any classroom can be a mathematics laboratory if we provide the proper atmosphere, the feeling of doing things rather than talking about them. It is an attitude, an approach, but it is not an approach which will solve everybody's problems. It is an approach that can do certain things for us and do those things very well. We seem to try to find one solution to all of our dilemmas, but that solution does not exist. It is very important to look at what we are doing and try to point out those things which the laboratory approach can do well and those things which it cannot do well at all.

Another matter I want to clear up is the belief that a mathematics laboratory necessitates some kind of individualized instruction program. While you could operate an individualized program and use a laboratory approach to teaching some of the concepts, they are not dependent. The laboratory approach does not necessitate an individual approach, nor does an individual approach mean that you have to use a mathematics laboratory. They are just different kinds of things.

It is useful to establish some stages in developing concepts with children. The stages which help me design a program or organize curriculum are image, symbols, organization, generalization, practice, and application.

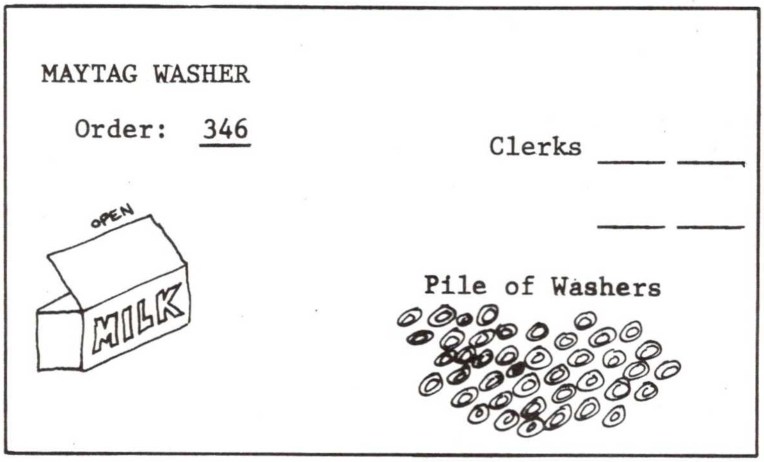
We need to be sure that all children, whether they are in Grade VIII, XII, or I, will think with images rather than just symbols. When a child sees 2 x 5, he must have some way to relate to that. When he sees a probability, he should have had some kind of experience to which he can attach that probability. When I am talking about congruent triangles, I want the child to have done things with congruent triangles so that mathematics becomes more than just maneuvers with symbols. However, I must get more sophisticated than that if I am going to develop learning very efficiently. So as I build a concept, I want to move into some kind of development which provides the student with an opportunity to arrive at some kind of generalization. Practice using the developed ideas is essential for fixing learning, and applications are of great significance to the student. If mathematics doesn't have a real meaning to him, then we are in trouble. We have to do a lot more with application.

BUILDING IMAGES

When I write down table, chair, apple, and so on, the word brings an image to your mind. It may not be the same image that I have, but you do think in terms of the images they present. In teaching reading, those words which do not have images are particularly difficult for the student. There is nothing to tie it to except the symbols. We have had difficulty with those things in mathematics also for which we do not have imagery. Everything we do in mathematics should have some kind of image. Most images can be built through physical experiences in a laboratory setting. Pictures just are not as good as the objects themselves.

Many image-building activities happen at the primary level. If they don't happen there, you will find out fairly quickly.

Figure 1



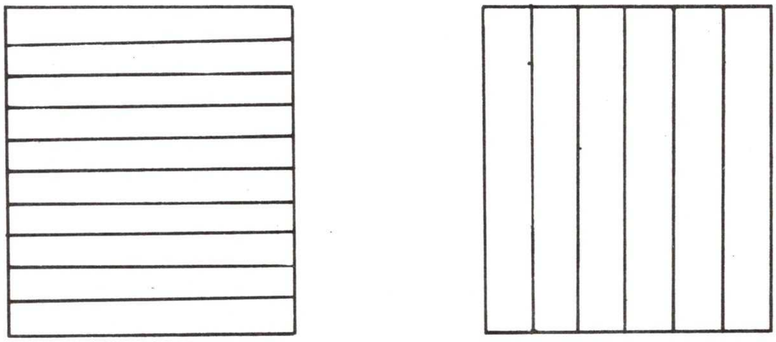
If we want to build some concepts of numeration, we can use the activity in Figure 1. We do a fairly good job talking about small numbers with children, but we "cop out" when it comes to talking about 346, for instance. Many children in Grades V and VI could not count 346 and be sure that they had 346. The Grade III activity in Figure 1 is very useful. The class is divided into two sections. One section consists of clerks - they will be in small groups of four or five students, and their job is to put 346 washers in the boxes. One student may have 45, another 52, another 85, and so on. They usually conclude that they don't know how much that is. Often some student will say, "Let's put them in groups of 10", so everybody puts them in groups of 10; and somebody else says, "let's put 10 tens together and put them in the box. That's 100." By the time they have filled that order and maybe two or three other orders, they're beginning to build an image of the importance of the numeration system in terms of 10s and 100s. It is much different than telling the students that this means three hundreds, four tens, and six. If I ask one of my students to tell me how many tens there are, I want him to tell me that there are 34, because I want him to relate to the number involved.

Students in the other half of the class are inspectors who check to see if the first students have correctly counted the order. I get the students started ahead of time by filling a few boxes with handfuls of washers. This usually results in the order not being correct. Not being correct provokes more arguments than being correct, because the students are not sure that they counted correctly. Very often they go through the first order two or three times. They can't imagine that the teacher would give them a thing entitled 346 washers without there being 346 washers in it. Both of these groups are developing images for numeration. You are going to have to do things to build imagery for fairly large numbers if you expect children to operate with them.

Some of the things I'm going to show do not require a lot of laboratory equipment. For example, a book is a good image-building device, and it is available in every classroom. I'm on page 37, I turn ahead 10 pages, what page am I on? I'm on page 52, I turn back 5 pages, what page am I on? This device may even be better than the number line.

Another example involves finding the number of intersections when these transparencies are put one over the other. There are nine lines on one and five lines on the other.

Figure 2



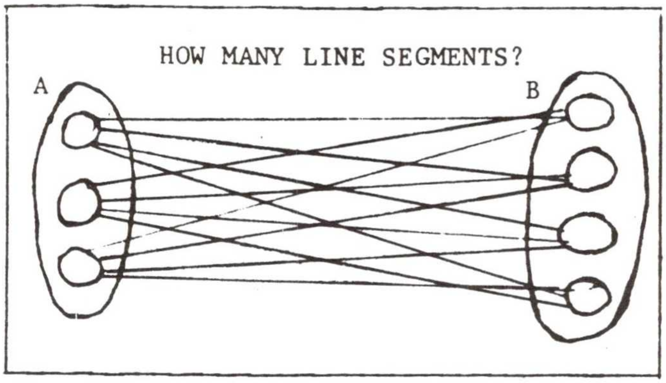
This can also be used to build an image for multiplication by 0, which is essential. Simply use a blank transparency for one of the factors. How many points of intersection will I get after I have established the model? That's far better than the teacher I had in the fifth grade who carried fruit jars from the basement as the image for multiplication. Going to the basement three times and not bringing up any fruit jars each time didn't make any sense to me. But it made better sense than the time I didn't go to the basement at all and brought up three jars each time I didn't go.

Another way to build images for multiplication is through the use of an array. A picture of an array in a textbook isn't as important as some physical objects the child can arrange. At the second or third grade level you could give students some objects and ask them to arrange them so you can see how many you have. Many students will automatically arrange them in an array. Asking them to describe the array helps translate that particular physical experience into symbols. They will tell you that there are six in each row and four rows. They will use their language, but they will have a better image than if you gave them the picture of the thing because you can't maneuver and manipulate the picture.

One important characteristic is that physical things have built into them a "forgiveness" factor. You can maneuver them, you can correct your mistakes, you can do things with them without being brought to a fatal conclusion which often happens on paper.

Not all the things that build images have to be physical. Figure 3 illustrates a perfectly good way to build images for multiplication.

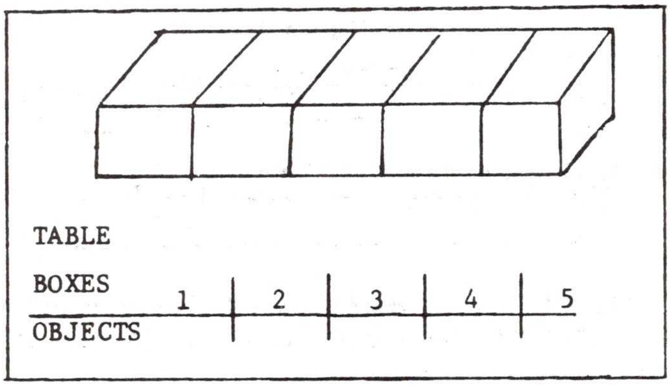
Figure 3



Using the two sets of points, Set A and Set B, how many line segments can I draw which have one end point in A and the other end point in B. The students will draw them, rather than look at a picture. You can explain that for each of the points in A there will be four segments (4 times 3). Multiplication has to take on a variety of images because it is used in a variety of ways.

Figure 4 illustrates a repeated addition model. There is the same number of objects in each of the boxes. Have the students complete the table.

Figure 4

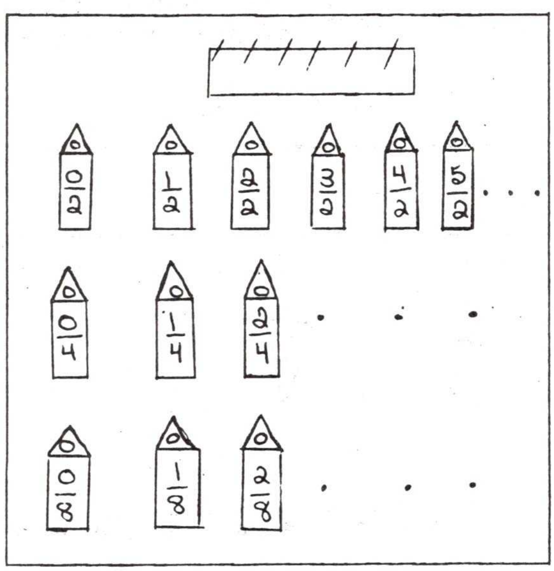


Another multiplication model is just a ditto sheet with a lot of dots on it. Ask the children to circle, say, 23 times 45. How many do you have? This can be used to provide an image for the multiplication algorithm, or, if the children have had some experience with the algorithm, they can relate the algorithm to what they are going to draw.

You should follow up the image you have built with some kind of paper-and-pencil-activity so that students can translate what you do in the physical world into something that makes sense in the symbol world.

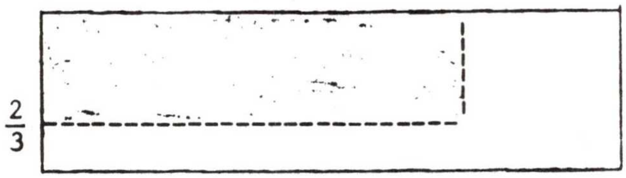
Each time we attach a new concept with children, we should have in mind some good image-building activities. Equivalent fractions is one concept for which we often ignore image-building. One model for equivalent fractions consists of some nails and tags as shown in Figure 5. What other tags go on the same nail? Seeing tags tacked up seems to have a different effect than simply talking about equivalent fractions.

Figure 5



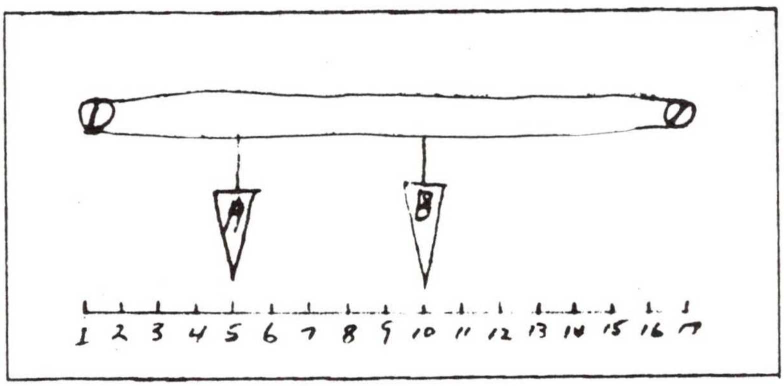
An image you might use for multiplication of fractions is shown in Figure 6.

Figure 6



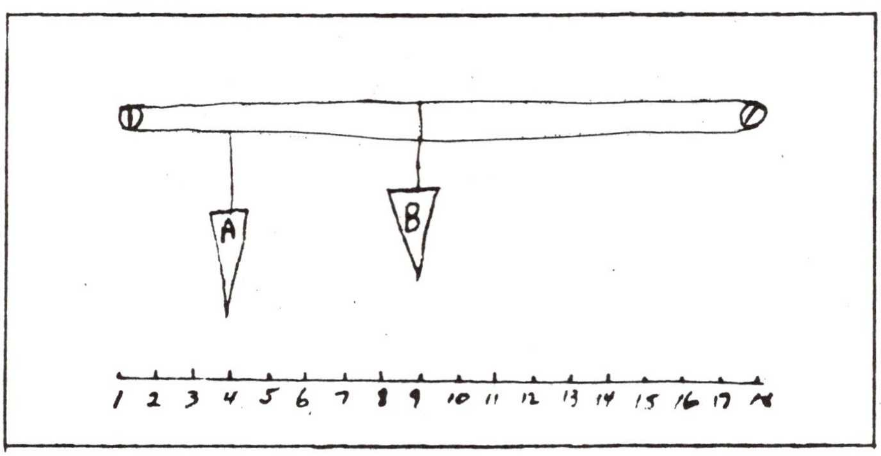
Take a rectangle of the size 3/4 by 2/3. What fraction of that rectangle have I shaded? I would hope that at the image-building stage I could relate that to 2/3 times 3/4.

Figure 7



Even the concept of a function needs some kind of imagery for it. Loosen 2 screws on the end of the blackboard. (See Figure 7) Draw a number line under it, with any kind of numbers you want. Run a piece of string around the screws. The tag moves as the spring moves. If I move tag A to 5, where does tag B move to? What kind of relation exists between the positions of A and B, a relationship that I could express in a function if I wanted to build an image for that function? B-A is 5 every time. The distance remains constant.

Figure 8



Put B on the top string. (See Figure 8) What kind of function do you have this time? You have a function that is a constant sum. If A increases by 1, B decreases by 1. Children could draw this kind of model at their desks using rulers which do exactly the same thing.

BUILDING GENERALIZATIONS

Physical things in various ways help to build images for some but not for all the things we want to teach in mathematics. The following is an example of something that cannot be taught well with physical imagery, and I don't think we ought to try to do that. Here is an organization of the basic facts for multiplication.

0 x 1	1 x 1	2 x 1	.	.	.
0 x 2	1 x 2	2 x 2	.	.	.
0 x 3	1 x 3	2 x 3	.	.	.
0 x 4	1 x 4	2 x 4	.	.	.
.	.	.			
.	.	.			
.	.	.			

I have reached the stage where I have imagery behind that and they mean something, but now I want to arrive at some generalizations about that list. I want children to recognize that they don't have to memorize the 0 facts or the 1 facts or the 2 facts. I do not know how to do that through a laboratory approach. We can focus in on the symbols only if we have imagery first. It's surprising that we do not let children into the secret of how many basic facts there are and what kind of organizational procedures we can use and what kind of generalizations we can build by making lists and by being honest with children. Often when I ask Grade IV children how many basic facts there are, they tell me that there are a million. It surprises a lot of children when they find out there are only 36. Let's be honest and let them in on this secret. They are not going to learn that with physical activity; concepts have to be organized.

Figure 9

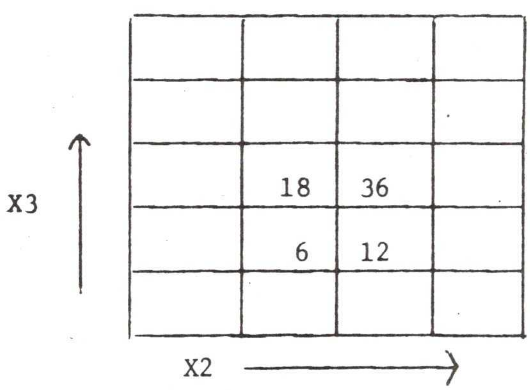


Figure 9 is a model for multiplication. If I go from 6 across to 12, I am multiplying by 2, and if I go from 6 up to 18, I am multiplying by 3. Completing the table in a physical model such as this will often build a generalization because students can often generalize when they see things organized. You can do exactly the same thing with multiplication of fractions.

PROVIDING PRACTICE

Practice in computation is important, and we make a big mistake if we ignore it. A student who enters Grade V without considerable multiplication skill is not likely to succeed. He just can't think mathematically unless he has the basic facts and can multiply with some facility. Not only does he lack confidence in what he is doing, but he also can't operate with area, ratio or fractions because all of these require fundamental computational skills. The reason for lack of skill on the part of many children is not because we have taught modern mathematics but because we as teachers have copped out. We haven't insisted that students compute. In my opinion, there will never be materials which can do that for us. We are responsible for our students being able to compute.

We can reduce the number of children with computation problems by providing our students with a variety of practice activities. Some mathematics educators have said that we should not use rote drill without pupils understanding what was going on before we used the drill. Many of us misinterpreted this to mean that we should use rote drill. There are many things in mathematics which we just have to memorize.

One simple way to provide immediate reinforcement in such things as basic facts is through an activity I call "Game 1".

Figure 10

BEAT THE TEACHER

Game 1

A	______	______
B	______	______
C	______	______
D	______	______
E	______	______
F	______	______
G	______	______
H	______	______

Copies of the above sheet can be used to provide practice in multiplication. I give, for example, orally a multiplication fact like 8 times 7. If a student writes down 56 before I say it, he writes it in the first blank. If he writes it down incorrectly or if he doesn't write it down before I say it, he writes in the second blank. When I say 6 times 8, 48, I will not say it as fast as I can. This provides practice with basic facts with immediate reinforcement. I can watch a student and put some pressure on him by adjusting the speed with which I say a fact. You can use this for fractions, decimals, multiplying by 10s and 100s and so on.

Another activity I have used involves the following two sheets.

Figure 11

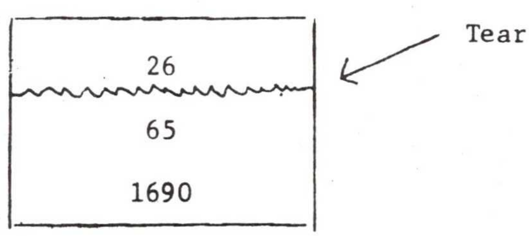
$\frac{648}{14}$	$\frac{483}{23}$
$\frac{851}{37}$	$\frac{910}{65}$
$\frac{544}{17}$	$\frac{4030}{62}$

21	32
65	47
14	23

In my class $\frac{648}{14}$ means 648 ÷ 14. The second sheet contains the answers. The students do the division and find out if their answer is on the second sheet. If it is, they get immediate reinforcement without me even being there, because they are quite sure that if they divided correctly, then the answer is there. Some say they can tell the answer without dividing. That is fine. Of course, the questions could be made harder.

Another practice activity with immediate reinforcement involves a calculator with a tape on it. I put into the calculator a number like 26 and then multiply it by 65 and get the product. (See Figure 12)

Figure 12



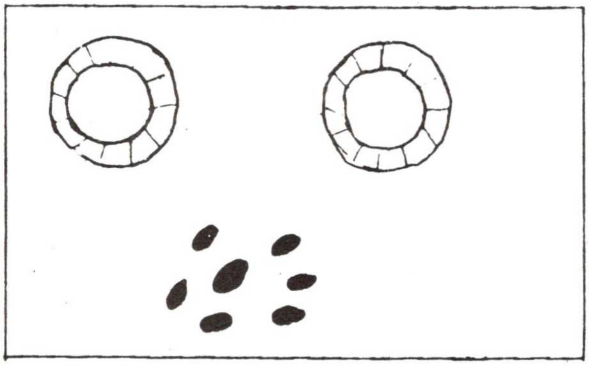
I tear the tape into two pieces as illustrated and give one piece to a student. He divides and then comes and asks for 26. What if 26 doesn't fit? He takes it, tries it, and if it doesn't fit, he goes back and divides again. If it does fit, he is immediately reinforced, goes back and does a different one. If I had given those children six or eight division problems, they wouldn't have done them. But if you give them one at a time with reinforcement, they'll do lots of them and enjoy it. Practice is important. We're not going to teach mathematics without it.

TEACHING APPLICATIONS

The things we use are also important in the application mode. There is a new mathematics revolution going on now which I call the "Application Revolution". One of the things we miss in the modern mathematics movement is the importance of problem-solving and application. That's what mathematics is all about.

Physical materials can provide application even at a very early level.

Figure 13

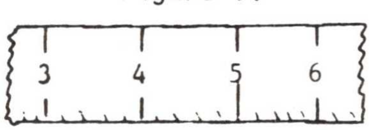


For example, Figure 13 shows two plates and seven beans. The pupil is to find out how many different ways he can put beans on each plate and to make a list of the ways. This is an application of the basic facts for addition. Things set it up well, better than listing all the basic facts for 7.

Another application involving area which is just the reverse of what we usually do makes use of a ditto sheet that looks like graph paper and a variety of rectangles. The pupils are to find the area of the rectangles. Is this a laboratory approach? It is to me because the pupils are manipulating things. Physical things *do* add a useful dimension.

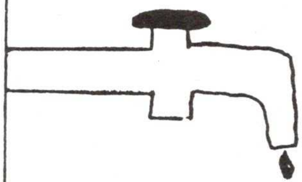
Other applications involve measurement. My students have difficulty using a ruler because the ends wear off. I want to teach the child to measure without using the ends of the ruler, so I break the ends of the rulers as illustrated in Figure 14. It is surprising how much that adds to a measurement experience. Now they have to do the subtraction. They have to use the ruler as I want them to use it.

Figure 14



If you have a sink in your room, you can use it to build a laboratory approach.

Figure 15



TIME (Minutes)	1	2	3	10	60
DROPS	15	30	45	150	

a. How many drops from the leaky faucet in 1 hour?

b. How many drops in 1 day? 1 week?

Let the faucet drip a little and prepare a set of problems about the faucet so that pupils have to measure the water from the faucet and then answer a series of questions related to the drippy faucet. They use a lot of mathematics in doing that. A table (see Figure 15) adds a great deal to the physical dimension in this kind of relationship because a table allows the student to get involved in the situation before he has to answer a lot of questions. We could put some entries in the table to encourage him to see the relationship. If I want him to be able to think for himself, I will not put entries in the table. Many of the slower students may not go beyond the use of the table to solve the problem. That's perfectly adequate.

The physical model suggested by the activity card in Figure 16 may be difficult to set up, that is, to actually have the pupils do the measurement.

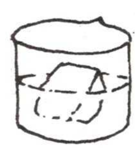
Figure 16

8% of a block of ice is out of water.

Height of ice out of water				
Height of ice				

1. If a block of ice is 10 inches thick, how much ice will float out of water?

2. If the ice is 15 inches thick, how much float out of water?



However, just having the ice cube there may be helpful. This activity is starting to set up the concept of percent. The day I proposed it, 8% of the block of ice was out of water. Again a table will help the child to translate the physical into what he is thinking about when he wants to solve the problem. Notice the table does not have any entries in it. I'm hoping the pupils will say that's 8 out of every 100 feet because that's what percent means. There are a lot of ways to describe that function. Children's entries are probably different from those of the teacher's. Four out of 50, 2 out of 25, and 16 out of 200 are examples of the idea of 8%.

It is important after a physical situation is set up to use a variety of questions about the same situation because it takes the focus off just the computation and places it on the situation. One of the difficulties we have had in building good problem-solving skills is that we have not provided enough experiences with the physical situations. It's not that children can't read the example, but they don't know enough about the situation on which the example has a bearing. Figure 17 illustrates what I mean.

Figure 17

Eggs	1	2	3	12	24
Time	10	10	10	10	10

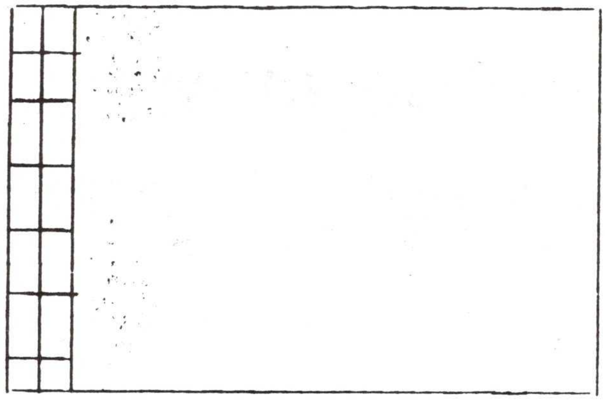
Equation: $T = 10$

If you can hard-boil one egg in 10 minutes, how long will it take to hard-boil a dozen eggs? Many pupils think of this as a ratio situation. You know enough about boiling eggs to know that it doesn't take any longer for one egg than it does for a dozen. How can you use the answer to that question? You might well express the relationship in the form of a table. You boil one egg in 10 minutes; two eggs in 10 minutes; three eggs in 10 minutes. You might be interested in writing an equation. It's not an important equation - as a matter of fact, it's one we often ignore.

If you put children into a physical setup, one of the first things they will do is to try, make trials and errors, make guesses, collect data, draw diagrams, and react to pictures. All these are problem-solving skills that the verbal problem completely ignores unless we happen to teach them. Most of those problem-solving processes require an imagery, something behind what we do when we solve problems. For example, providing a road map as a model on which children measure the distance, say from Calgary to Edmonton, provides an application setting which is better than verbal problems for which children don't have any particular meaning.

Another valuable kind of problem-solving skill develops when we place children in situations which require them to generate methods of solving the problem. For example, I asked a fifth-grade class how many fifth-graders we could get in the classroom. "Take out a piece of scratch paper and figure out what you would measure to find out." We had been working with area. I was impressed by the student who said he would line the students up as shown in Figure 18.

Figure 18



What impressed me was that he drew the students as a rectangle, not a square. Another student suggested that we get all the students in a corner. We had tile on the floor and we figured out that we could get 28 students on about 24 tiles. Then we found out how many tiles there were in the whole room and found that we could get 850 students into the room. Other students suggested other methods, but this is the one we chose to use.

Sometimes you have one student who has a unique idea. You should call on him because if you don't, you will turn off the rest of the class also. If they don't feel their question is good, then hardly anybody will ask. An example of a unique idea was given by one boy in my class who suggested that the 850 students would only be one layer.

When this same group was working with sugar cubes, I asked the children if they could place a million sugar cubes on the table. I left them there. Some pupils brought sugar cubes. Finally somebody laid them out and said, "Sugar cubes are about 1 cm, so it takes 100 sugar cubes long and 100 sugar cubes wide, and 100 sugar cubes tall. This would give you a million." Children don't have images for large numbers. They think a million sugar cubes would go out of the building.

CONCLUSION

Physical materials will provide two basic things: they will provide images for the things with which children think in mathematics, and they will provide applications which make mathematics real.