Transformations and School Mathematics

Transformations is one of the main themes in contemporary mathematics programs. It can form a basis for the study of geometry, functions such as mapping, trigonometry, and even the traditional topic of sections.

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INTRODUCTION

Transformations have played a very important part in our thinking over the past five years. Thinking about transformations has opened to us some of the true fascinations of mathematics and has showed us new directions for the development of student materials starting with five-year-old children. I am not alone in my enthusiam for this approach. My great teacher, Dr. Coxeter, perhaps the world's greatest living geometer, states the following:

> It is difficult to overestimate the importance of the notion of *mapping* in mathematics. In calculus it appears centrally in the concept of *a function*. In algebra we speak of a *correspondence*. In geometry we generally use the word *transformation*. Through the

concept of transformations we are able to characterize the geometry we are studying. It leads us, in fact, to a satisfactory answer to the question, "What is Geometry?" [unpublished manuscript]

I would like to go farther and say that transformations can lead a student to answer the question, "what is mathematics?"

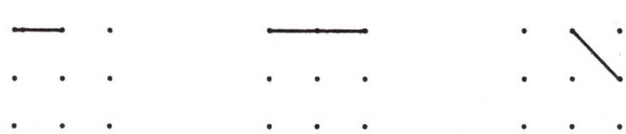
It has been clearly established that transformations are a vehicle for developing geometry. At an international conference on pre-college geometry held in southern Illinois, some of the world's foremost geometers spoke strongly in favor of transformations. Britain and many other European countries are now producing mathematical materials which rely on the idea of a mapping. In our work in Canada, we have found that the mapping idea has opened up new approaches to school mathematics. These approaches are proving both enjoyable and fascinating for teachers and children alike. I will try to outline briefly what has happened in the past few years.

TRANSFORMATIONS IN SCHOOL MATHEMATICS

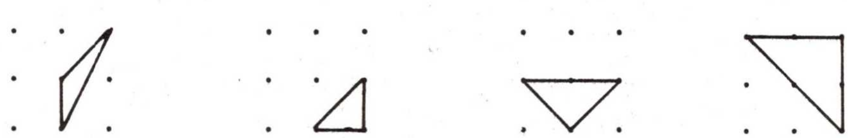
The most significant change in elementary mathematics is occurring through the introduction of transformations (Del Grande, 1972). Geoboards and colored elastics are used at a very early age, and children learn about shape and transformations through them. Special materials such as dot paper and plexiglass mirrors enable children to work with mathematical problems never before attempted.

Here are a few sample problems that children can try experimentally on the geoboard and then analyze.

- On a nine-nail geoboard, how many segments can you make the same size as the one given?



- How many triangles, the same size and shape, can you make?



From exercises such as those, children learn to talk about congruent figures they have made and describe how to move one figure onto another. These motions called *slides, flips* and *turns* form the basis for transformations in geometry.

Children act out slides, flips and turns with body motions. They move paper cutouts around a plane to illustrate the motions (Del Grande, 1972). They soon learn that pleasing patterns can be made by drawing the outline of a figure as it is moved according to a rule.

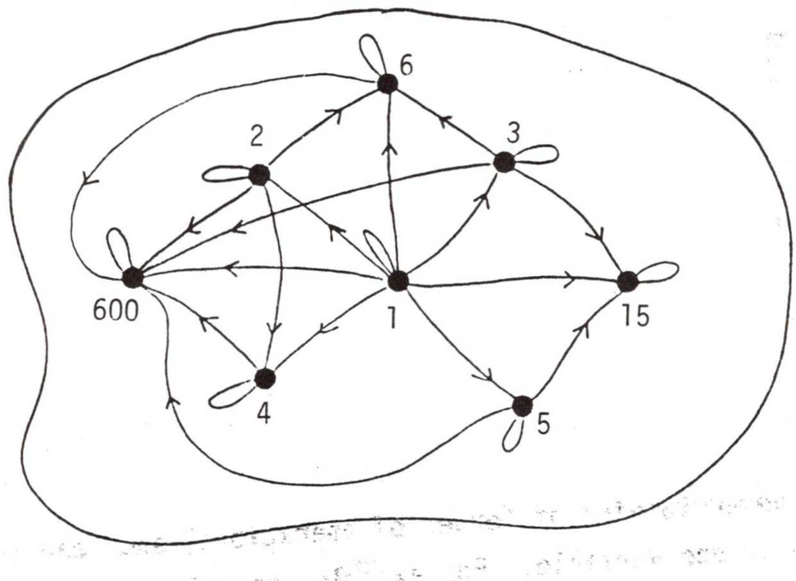
Reflection is first illustrated by using paper folding and flipping a figure about an edge. With the introduction of the semi-transparent mirror, actual mirror images can be drawn with ease. This special mirror has many applications to geometry and geometric constructions in later grades.

Our junior high program relies heavily on the idea of a mapping. The program starts with arrow diagrams first introduced by those brillant mathematics educators Georges Papy (1968) and his wife Frederique Papy (1971). Through these diagrams the student learns to pair things with things, number with number, and points with points.

If A is related to B, then A is joined to B with an arrow.

The following arrow graph or Papygram illustrates an interesting way in which Papy's approach can be used (Del Grande, Jones, Lowe and Morrow, 1971-72).

Draw an arrow diagram of the relation *is a factor of* in {1, 2, 3, 4, 5, 6, 15, 600}. (Since 2 is a factor of 4, we join 2 to 4 with an arrow, etc.)



IS A FACTOR OF

Notice that since 1 divides every whole number, 1 is joined to every other point.

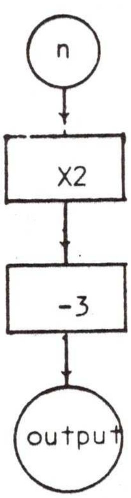
Why is there a loop at every point?

What kind of arrow diagram results if we have only prime numbers?

It is easy to design Papygram questions that involve drill and practice, and children really enjoy it.

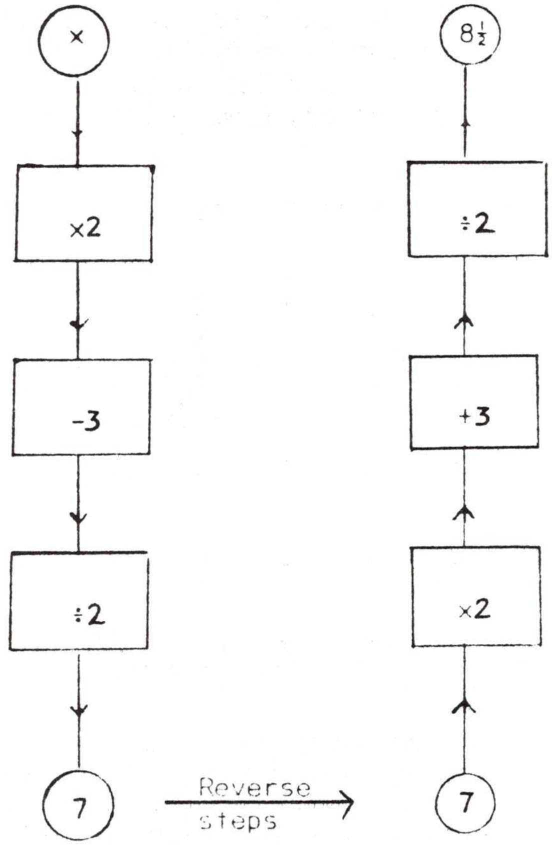
In pairing number with number, the student discovers pairing rules such as $n \to n + 3$, $x \to 2x - 5$. These pairings lead to graphing of the pairs of numbers with emphasis on those pairings that give a linear graph. Pairing of numbers have led us to introduce flow charts. For example

$$n \to 2n - 3,\ n \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$



Flow charts are used for evaluating algebraic expressions and result in considerable practice in computation. By pairing the "input" with the "output" we obtain ordered pairs that can be represented in a graph. These flow charts are like "function machines".

Flow charts require an understanding of "order of operations" and lead to the solving of linear questions in one variable. For example, to solve $\frac{2x - 3}{2} = 7$ the student produces the following flow chart and reverses the steps to solve for x.



Teachers in France are solving linear equations with Grade IV children using arrow diagrams and the renaming of numbers.

If $2x + 3 = 19$, then $2x + 3$ and 19 name the same number.

$2x + 3$

19

Find new numbers that have two names

$2x + 3 \rightarrow \boxed{+\ 2} \rightarrow 2x + 5$

$19 \rightarrow \boxed{+\ 2} \rightarrow 21$

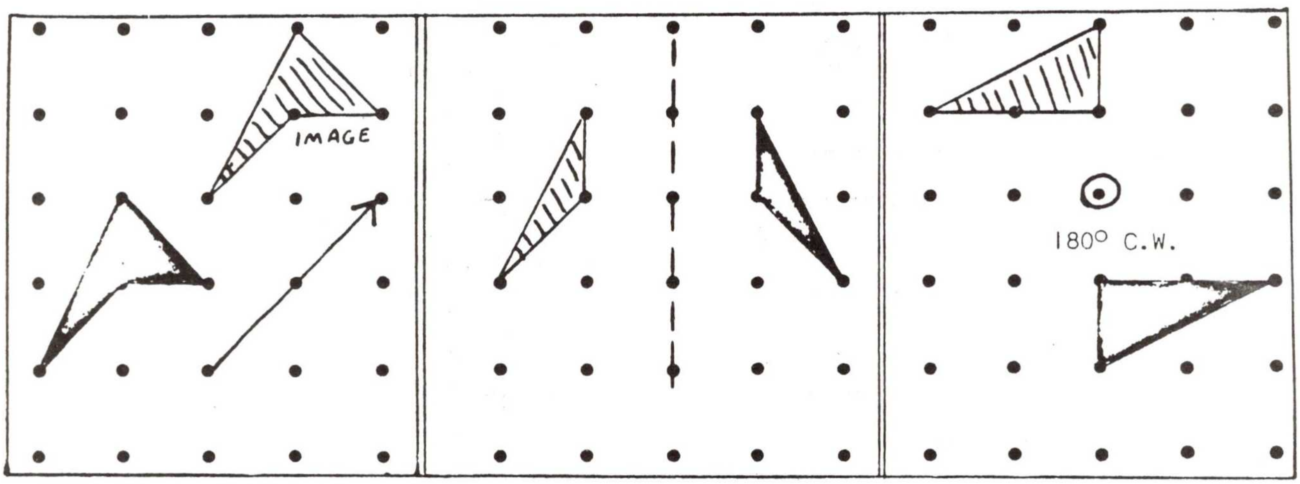
Find a number whose name is also x.

$2x + 3 \rightarrow \boxed{-3} \rightarrow 2x \rightarrow \boxed{\div\ 2} \rightarrow x$

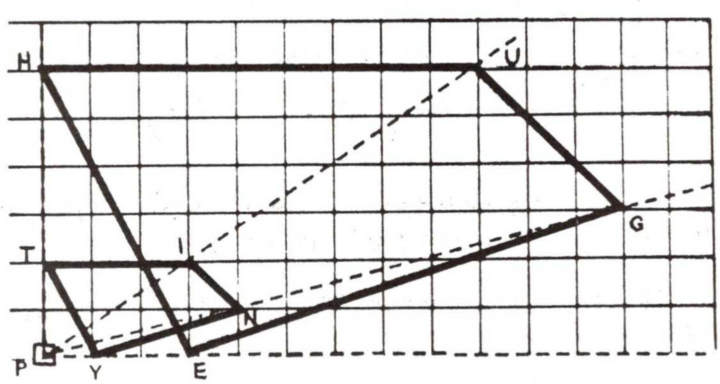
$19 \rightarrow \boxed{-3} \rightarrow 16 \rightarrow \boxed{\div\ 2} \rightarrow 8$

Thus, $x = 8$ is a solution of the equation $2x + 3 = 19$.

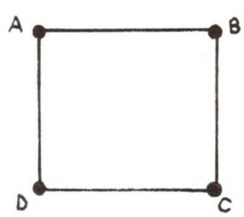
Papygrams lead naturally into the study of transformations through the pairing of points. A translation can be described using one point and its image, a reflection by the "mirror" line and a rotation by the center of rotation and the angle of rotation (Del Grande, Jones, Lowe and Morrow, 1971-72).

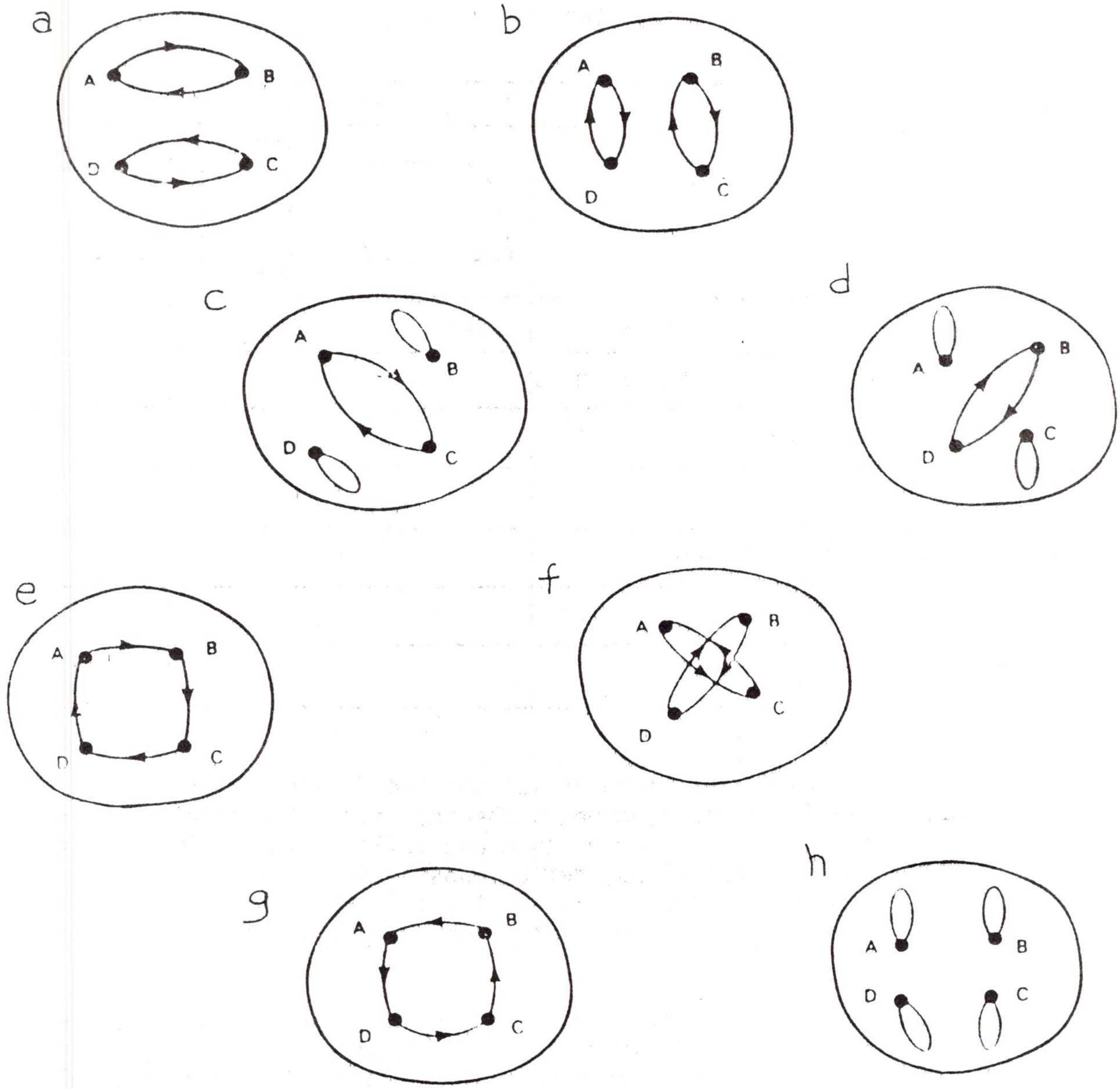


Dilatations and size transformations are studied and considerable work with similar figures, scale drawings and ratio is done (Coxford and Usiskin, 1971, Del Grande, Jones, Lowe and Morrow, 1971-72). The following diagram shows that HUGE is the image of TINY under a size transformation with center and scale factor 3:1.



Papygrams describe in a most unusal way some of the geometrical transformations (Del Grande, Jones, Lowe and Morrow, 1971). For example, given a square ABCD, which transformation does each Papygram describe?



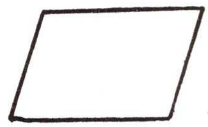
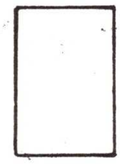
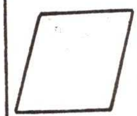
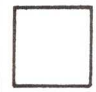


By applying the transformations of translation, reflection, rotation and glide reflection to geometric figures, children learn two important things:

(1) the properties of geometric figures,
(2) the properties of the transformations. (See Coxford and Usiskin, 1971, and Del Grande, Jones, Lowe and Morrow, 1971-72.)

The properties of geometric figures include the
- properties of isosceles triangles
- angle sum of a triangle
- properties of parallel lines
- properties of quadrilaterals.

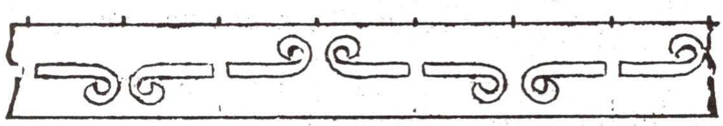
After activities involving quadrilaterals and transformations, Grade VIII children can fill in charts such as the one below.

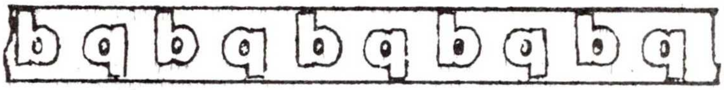
	Parallelogram	Rectangle	Rhombus	Square
				
a. The diagonals bisect each other.	yes			
b. The diagonals cross each other at 90^0.			yes	
c. All sides are congruent.				
d. All angles are congruent.		yes		
e. Opposite angles are congruent.				
f. Opposite sides are congruent.				
g. Opposite sides are parallel.				

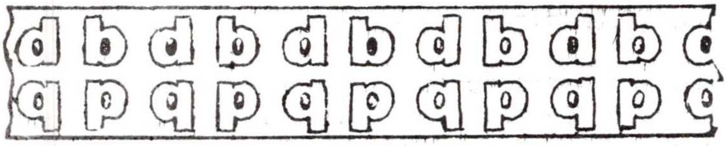
With this background of transformations and figure properties, it is an easy matter to prove the three congruency theorems side-side-side, side-angle-side, and angle-side-angle by showing that one triangle is the image of the other after a combination of translations, reflections or rotations. Thus, one can readily see that deductive geometry, in the traditional sense, can be attacked using the traditional tools along with the powerful yet natural weapons of transformations.

Dieudonné shocked the mathematical world when he said in his address at Cercle Culturel de Rogaumount in 1959, "Euclid must go". This modern approach to Euclid is what he wanted. His pleas have not gone unheard. We've accomplished what he predicted should happen; and it happened through transformations.

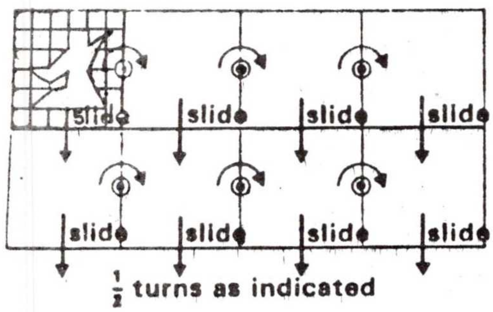
Transformations were never taught to young children in the past, nor were they usually outside the context of geometry. The study of the symmetries of a figure lead to the study of groups and all the fascinating aspects of strip patterns and wallpaper designs (Budden, 1972, Weyl, 1952). The following strip patterns are "generated" from a basic figure using transformations.

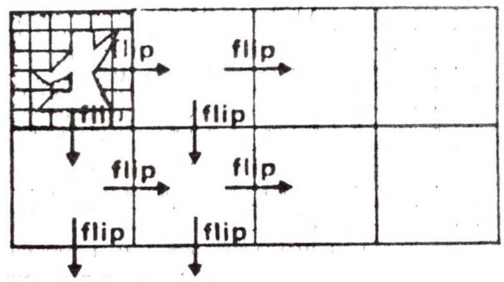




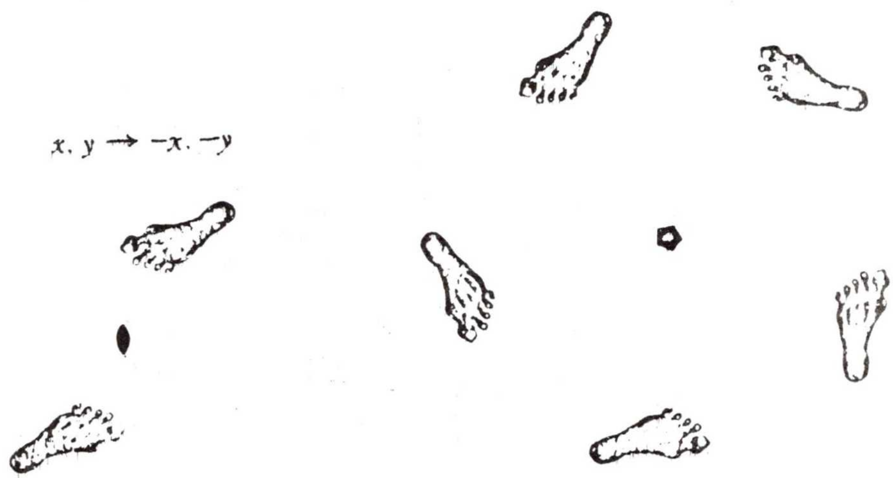


The following diagrams illustrate instructions which will generate two of the many possible wallpaper patterns.



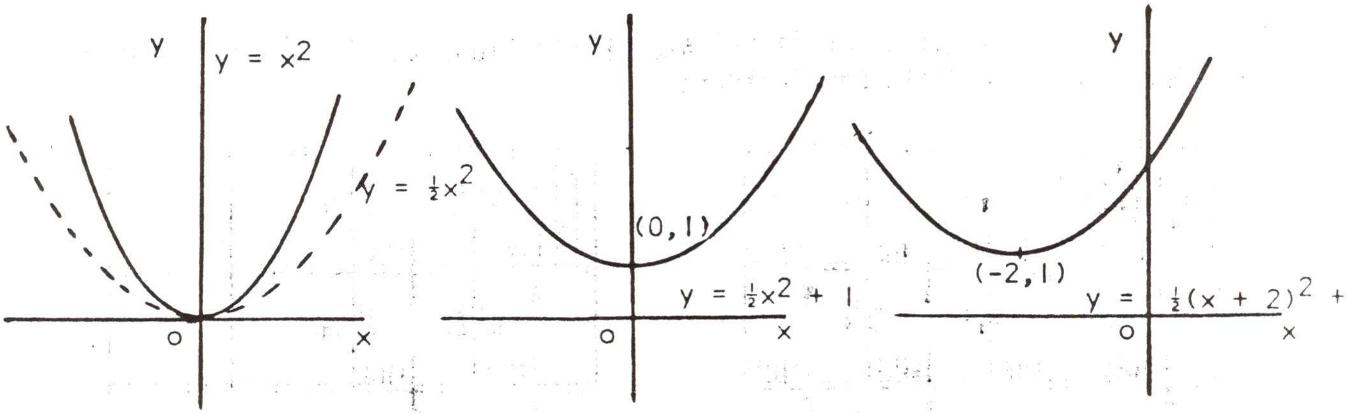


Symmetry leads to crystallography and nuclear structures. In fact, an excellent introduction to transformations is found in the book *Symmetry: a Stereoscopic Guide for Chemists* (Bernal, Hamilton and Rice, 1972). A few pictures from this interesting book follow.

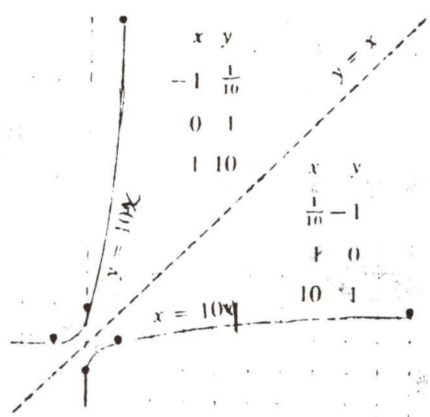


The work of the graphic artist M.C. Escher is primarily based on transformations. His pictures are famous and delight children of junior high school age who try to imitate his rules but with different designs.

The study of the linear and quadratic functions can be made dynamic and meaningful through transformations (Del Grande, Duff and Egsgard, 1970). For example, a parabola with equation $y = ax^2 + bx + c$ can be obtained from the parabola $y = x^2$ by a stretch and a translation. It is no great problem to develop a series of exercises through which a student can discover this fact for himself.



Functions and their inverses are shown to be mirror images of one another in the line $y = x$. The logarithmic function is defined as the inverse of a corresponding exponential function. The graphs of these two functions are shown below and are mirror images of each other in the line $y = x$.



The following gives a set of rules for relating functions to their graphs through transformations.

TRANSFORMATIONS AND CURVE TRACING

f is a function with defining equation $y = f(x)$.

Graph of f is symmetric about the y axis.

$$f(a) = f(-a) \quad (x,y) \to (-x,y)$$

Graph of f is symmetric about the origin.

$$f(a) = -f(-a) \quad (x,y) \to (-x,-y)$$

The function af

If $a = -1$, the graph of af is the mirror image of the graph of f in the x axis.

If $a > 1$, the graph of af is a "vertical stretch" of the graph of f.

If $0 < a < 1$, the graph of af is a "vertical compression" of the graph of f.

If $a < 0$, the graph of af is the mirror image of the graph of $|a|f$ in the x axis.

The function f^{-1} $\qquad (x,y) \to (y,x)$

The graph of f^{-1} is the mirror image of the graph of f in the line $y = x$.

$h:x \to f(x + a)$ $\qquad (x,y) \to (x - a,y)$

The graph of h is congruent to the graph of f.
If $a > 0$, the graph of h is a units to the left of the graph of f.

If $a < 0$, the graph of h is a units to the right of the graph of f.

$h:x \to f(x) + a$ $\qquad (x,y) \to (x,y + a)$

The graph of h is congruent to the graph of f and f is translated a units parallel to the y axis.

$h:x \to f(ax)$ $\qquad (x,y) \to (\frac{1}{a}x,y)$

If $a > 1$, the graph of h is a "horizontal compression" of the graph of f.
If $0 < a < 1$, the graph of h is a "horizontal stretch" of the graph of f.
If $a < 0$, the graph of h is the mirror image of the graph of $k:x \to f(|a|x)$ in the y axis.

$h: x \to f(ax + b),\ a > 1,\ b > 0$

$$f(ax + b) = f(a(x + \frac{b}{a}))$$

The graph of h is a "horizontal compression" of the graph of f and $\frac{b}{a}$ units to the left of the graph of $k: x \to f(ax)$.

Discuss the cases $a > 1,\ b < 0$

$0 < a < 1,\ b > 0$

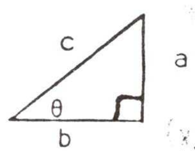
$0 < a < 1,\ b < 0$

$a < 0,\ b > 0$

$a < 0,\ b < 0$

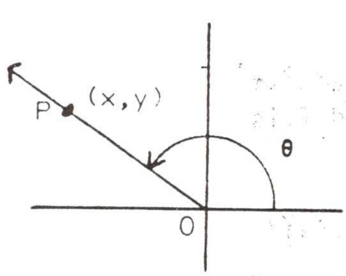
Although trigonometry can be approached using similar triangles and ratios in the early years, the trigonometric functions can be introduced in a meaningful way using a mapping.

The trigonometric ratios are first defined using a righ-angled triangle for $0^\circ < x < 90^\circ$



$$\sin\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c} \qquad \tan\theta = \frac{a}{b}$$

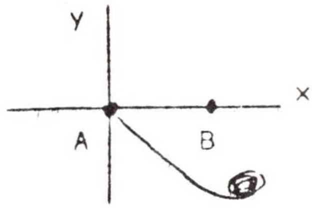
The definition may then be extended to angles of any measure by using the analytic approach

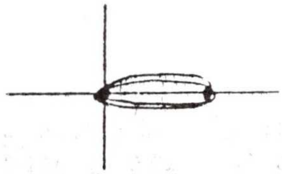


$OP = r$

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

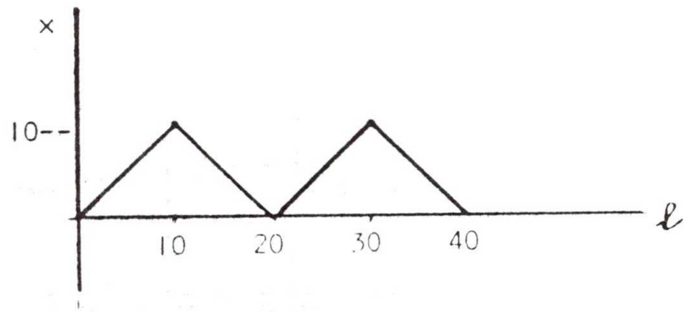
During these early years, we should develop some basic ideas for periodic functions through problems such as the following from Del Grande, Duff and Egsgard (1970) and Del Grande and Egsgard (1972).





- Two pegs are placed at A and B.
- Wrap a string around the pegs as shown.
- Each point on the string maps onto a point (x,y) on the plane.
- Relate the length of string ℓ to x and then ℓ to y.
- Graph the functions $\ell \to x$ and $\ell \to y$.

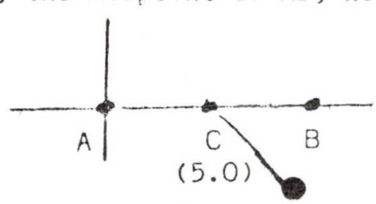
ℓ	0	1	2 ...	10	11	12 ...	20	21 ...
x	0	1	2 ...	10	9	8 ...	0	1 ...

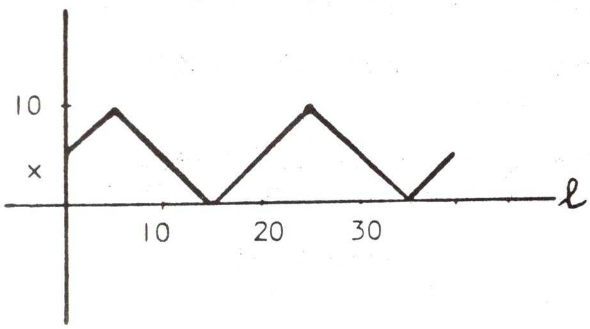


ℓ	0	1	2 ...	10 ...	20
y	0	0	0 ...	0 ...	0

The first graph allows us to study periodicity and amplitudes. Phase shift can easily be shown by using the same pegs but starting the string at some point between A and B. For example, if we start at C, the midpoint of $\overline{AB}$, we ge

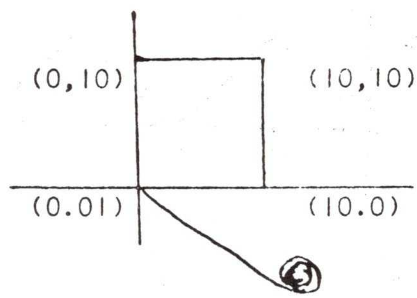
ℓ	0	1	2 ..	5	6	7 ..	10 ...	15 ...	20
x	5	6	7 ..	10	9	8 ..	5 ...	0 ...	5



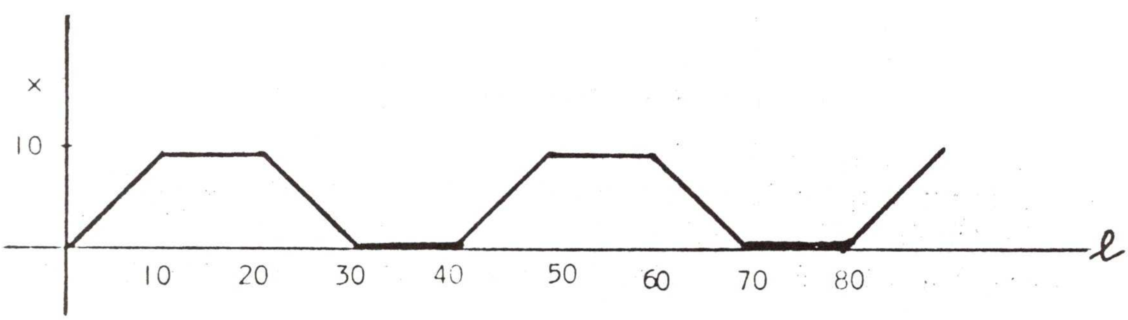


This new graph is a shift copy of the first with the exception of the starting portion. These graphs can be extended to the left by winding the string in the opposite direction and using negative values for ℓ. Periodicity amplitude and phase shift should be studied long before the graphs of the trigonometric functions are introduced.

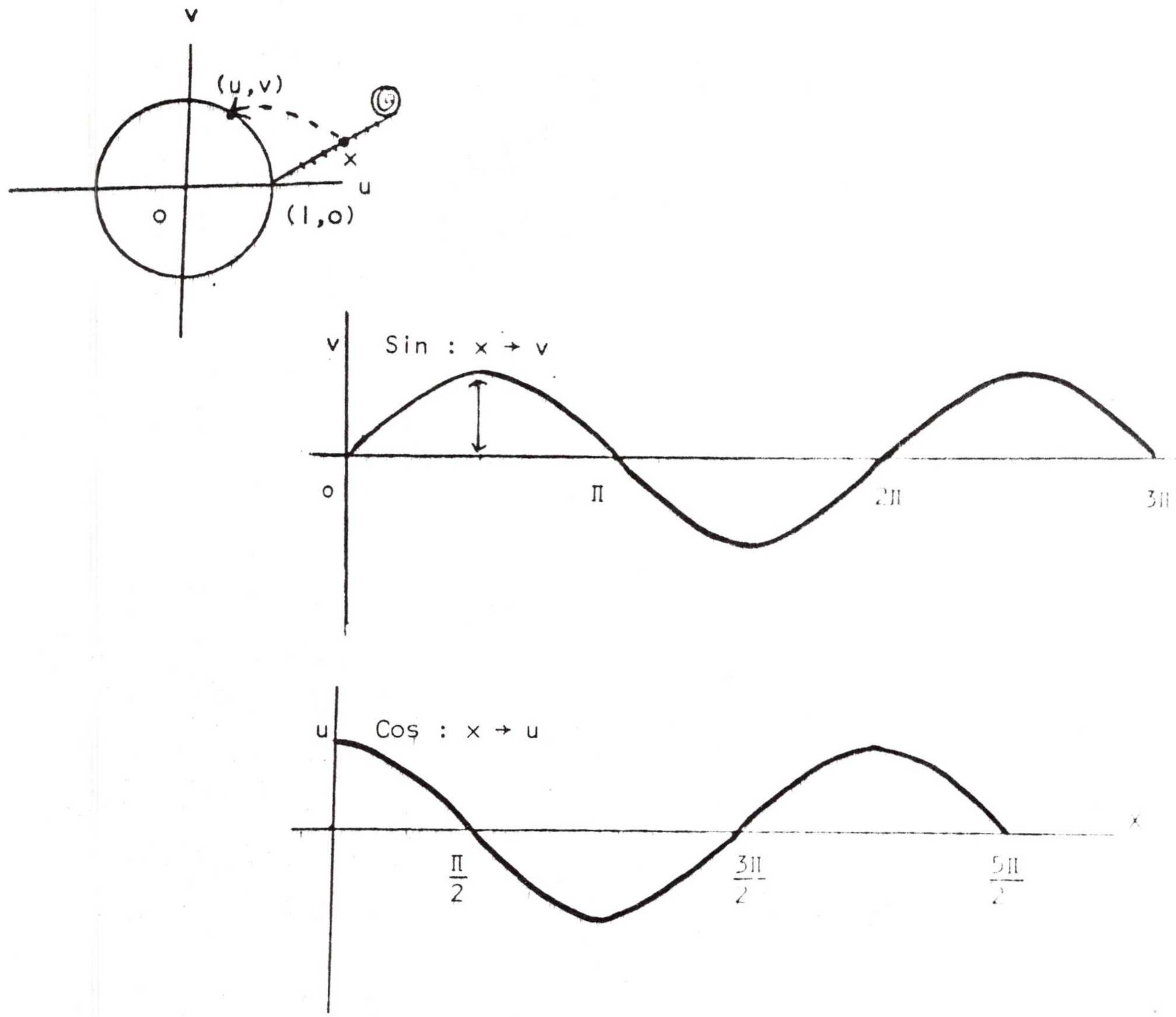
The next exercise might involve the winding of a string about a square.



The graph of $\ell \rightarrow x$ is as follows.



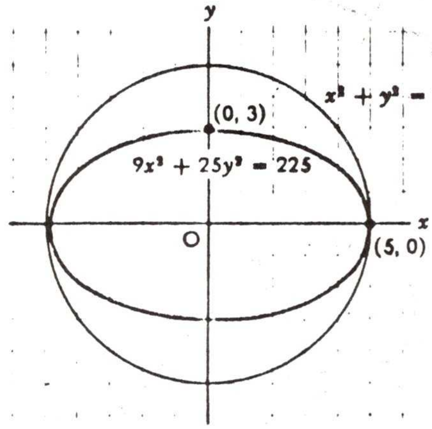
Finally, a string is wound around a unit circle and the familiar trigonometric functions emerge.



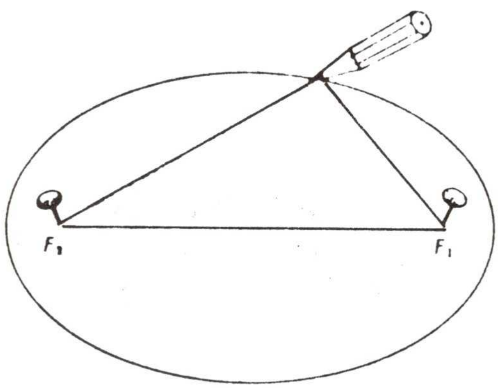
The change in symbols from (x,y) to (u,v) is necessary to arrive at the equations we desire, namely y = sin x and y = cos x. Notice that x is the length of string and is a *real number*.

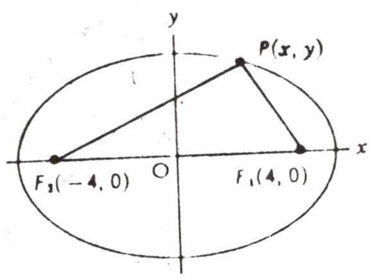
Graphs of functions such as y = a sin (bx + c) can now be related to transformation. The graph is a sine curve with amplitude $\underline{a}$, frequency $\frac{2\pi}{|b|}$ and phase shift $-\frac{c}{b}$.

Transformations enable us to approach the conic sections in a very interesting way (Del Grande and Egsgard, 1972). Starting with a circle with equation $x^2 + y^2 = 25$, a one-way stretch $(x,y) \to (x, \frac{3}{5}y)$ is applied. The resulting curve is $9x^2 + 25y^2 = 225$.



To show that the image curve is an ellipse, we define an ellipse using the constant sum of the focal radii.





By selecting suitable foci and a suitable sum we obtain the ellipse $9x^2 + 25y^2 = 225$.

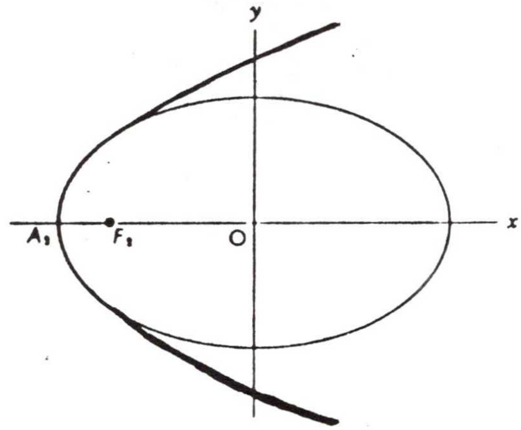
Thus, we show the image curve is in fact an ellipse.

These results can be generalized using the circle $x^2 + y^2 = a^2$ and the one way stretch $(x,y) \rightarrow (x, \frac{b}{a} y)$. To obtain an ellipse with foci on the y axis, we apply the stretch $(x,y) \rightarrow (\frac{b}{a}x, y)$.

Having stated that the graph of a quadratic function is a parabola (Del Grande, Duff and Egsgard, 1970), we show that parabolas have image parabolas under stretches. To show how to obtain a parabola from an ellipse, we start with the ellipse

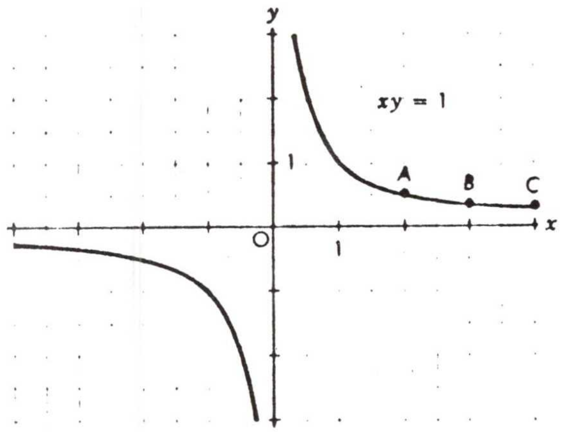
$$b^2x^2 + a^2y^2 = a^2b^2$$

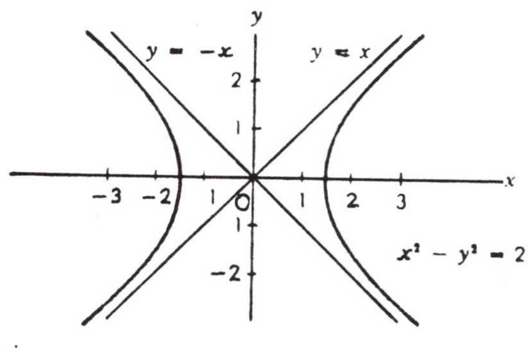
The ellipse is translated so that (-a,0) moves to the origin. We employ a special transformation that holds the vertex at (0,0) and the nearer focus fixed while the other focus moves to infinity along the positive x axis (Del Grande and Egsgard, 1972).



The final result is parabola!

For the hyperbola, we start be stating that xy = 1 is a hyperbola. This is easy to graph and the asymptotic properties of the hyperbola are apparent.



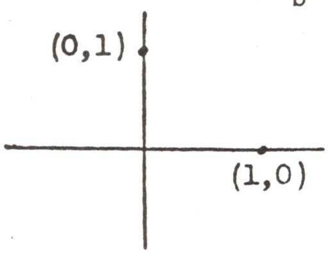


By a 45° rotation clockwise about the origin we obtain the image hyperbola $x^2 - y^2 = 2$. A two-way stretch, $(x,y) \to (\frac{a}{\sqrt{2}} x, \frac{b}{\sqrt{2}} y)$, of the image gives the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where the asymptotes are now $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$, which are the final images of the x and y axes.

It is interesting that starting with an hyperbola and holding one of its foci fixed, a suitable "stretch" will give an image that is a parabola. Thus the intuitive notion is established that when an ellipse is "stretched to infinity" it becomes a parabola, and when "stretched beyond infinity" it becomes an hyperbola.

Transformation gives us an excellent opportunity to introduce and apply matrices (Coxford and Usiskin, 1971). A 2 x 2 matrix can be used as an operator on the vertices of a figure to give the coordinates of the image points. Coordinates of points are written as a matrix

$$(a,b) \rightarrow \binom{a}{b}$$



If $(1,0) \rightarrow (x,y)$

$(0,1) \rightarrow (u,v)$

under a transformation, then

the matrix operator is $\begin{pmatrix} x & u \\ y & v \end{pmatrix}$

The matrix operator for reflection in the x axis is obtained as follows:

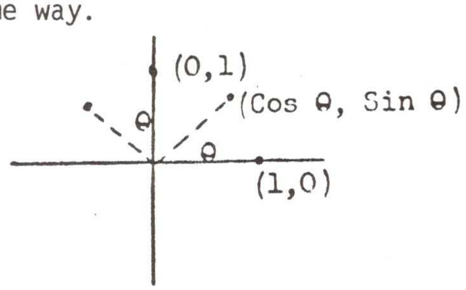
$$(1,0) \rightarrow (1,0)$$
$$(0,1) \rightarrow (0,-1) \qquad M_X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To find the image of ΔABC under a reflection in the x axis, where A(1,1), B(2,3) and C(-1,3), we perform a matrix multiplication as follows

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -3 & -3 \end{pmatrix}$$

↑	↑	↑
operator	coordinates of A,B,C	coordinates of image points

The matrix operator for a rotation of an angle θ about (0,0) is obtained in the same way.



$$(1,0) \rightarrow (\text{Cos } \theta, \text{ Sin } \theta)$$
$$(0,) \rightarrow (-\text{Sin } \theta, \text{ Cos } \theta)$$
$$R_\theta = \begin{pmatrix} \text{Cos } \theta & -\text{Sin } \theta \\ \text{Sin } \theta & \text{Cos } \theta \end{pmatrix}$$

Composition of transformation is obtained by multiplying matrix operators To illustrate, we use successive rotations of θ and α

$$R_{(\theta + \alpha)} = R_\theta . R_\alpha$$

$$\begin{pmatrix} \text{Cos}(\theta + \alpha) & -\text{Sin}(\theta + \alpha) \\ \text{Sin}(\theta \ \alpha) & \text{Cos}(\theta + \alpha) \end{pmatrix} = \begin{pmatrix} \text{Cos } \theta & -\text{Sin } \theta \\ \text{Sin } \theta & \text{Cos } \theta \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}\alpha & -\text{Sin}\alpha \\ \text{Sin}\alpha & \text{Cos}\alpha \end{pmatrix}$$

$$\begin{pmatrix} \text{Cos } \theta \text{ Cos}\alpha - \text{Sin } \theta \text{ Sin}\alpha \\ \_\ \ \_\ \ \_\ \ \_ \end{pmatrix}$$

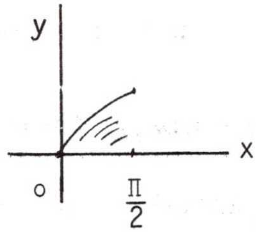
By comparing the two matrices we get the familiar result
$\text{Cos}\,(\theta + \alpha) = \text{Cos}\,\theta\ \text{Cos}\,\alpha - \text{Sin}\,\theta\ \text{Sin}\,\alpha$

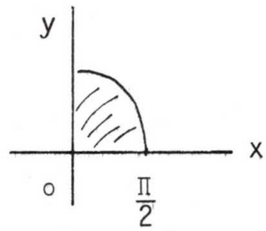
Can you complete the matrix equation to show that
$\text{Sin}\,(\theta + \alpha) = \text{Sin}\,\theta\ \text{Cos}\,\alpha + \text{Cos}\,\theta\ \text{Sin}\,\alpha$?

Throughout the work on transformations, symmetry appears time and again in most unusual ways. We use an example from calculus to illustrate.

To find $\int_0^{\frac{\pi}{2}} \text{Sin}^2 x \, dx$ we notice that the areas under the curves of

$y = \text{Sin}^2 x$ and $y = \cos^2 x$ in the interval 0 to $\frac{\pi}{2}$ are reflection images of each other.





$$\therefore \int_0^{\frac{\pi}{2}} \sin^2 x \ dx = \tfrac{1}{2}\left[\int_0^{\frac{\pi}{2}} \sin^2 x \ dx + \int_0^{\frac{\pi}{2}} \cos^2 x \ dx\right]$$

$$= \tfrac{1}{2} \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x) \ dx$$

$$= \tfrac{1}{2} \int_0^{\frac{\pi}{2}} 1 \, . \ dx$$

$$= \tfrac{1}{2} x \Big]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

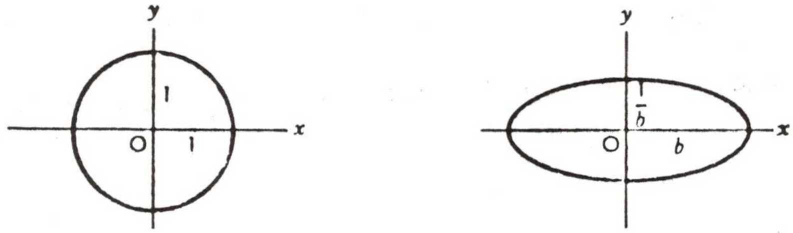
$$\therefore \int_0^{\frac{\pi}{2}} \sin^2 x \ dx = \frac{\pi}{4}$$

A second and interesting example from calculus shows the power and simplicity of transformations (Del Grande and Duff, 1972).

It is easily shown that the transformation

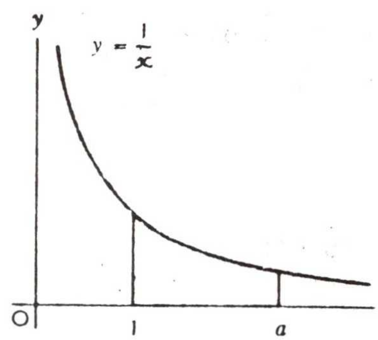
$$(x,y) \to (bx, \frac{y}{b})$$

preserves area. For example,



the circle has an ellipse as image but the areas are the same!

Under this two-way stretch $(x,y) \to (bx,\frac{y}{b})$ the curve $y = \frac{1}{x}$ maps onto itself. If the area in the interval $1 \leq x \leq a$ under the curve $y = \frac{1}{x}$ is defined as $\ell n\ a$



we can show that

$$\ell n\ a\ b = \ell n\ a + \ell n\ b$$

which is an important property of a logarithm.

By using areas, the result that

$$\frac{d\ \text{In}\ x}{d\ x} = \frac{1}{x} \text{ follows.}$$

Results such as these indicate that we can make mathematics more meaningful to our students to whom we have been entrusted to reveal the beauty, sense and fascination of mathematics.

CONCLUSION

The use of transformations produces a great quantity of mathematics suitable for young children and leads to a set of axioms in geometry on which a logical structure can be built. Transformations lead naturally into the ideas of relations, mappings and functions, especially for composition and inverses. Transformations clarify the addition of vectors, help to motivate matrix multiplication, provide better proofs of results in trigonometry, and make the study of curve tracing or graphical representation of functions both dynamic and lucid. Transformations is one of the main themes and is a unifying force throughout the whole of school mathematics.

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