Monograph no.2

September 1974

MATHEMATICS TEACHING: The State of the Art

Proceedings of the Edmonton Meeting of the NCTM

October 1973

EDITOR: W. George Cathcart

MCATA Publication of the Mathematics Council of The Alberta Teachers' Association

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Monograph No. 2 is an official publication of the Mathematics Council of The Alberta Teachers' Association. **Editor:** Dr. W. George Cathcart, Department of Elementary Education, University of Alberta, Edmonton T6G 2G5. **Publisher:** The Alberta Teachers' Association. Editorial and Production Service: Hilda M. Lindae. Address all correspondence about the contents of this publication to the Editor.

Members of the Mathematics Council receive this publication without charge. Non-members may obtain copies of this Monograph for $2.50 a copy by writing to The Alberta Teachers' Association, Barnett House, 11010 - 142 Street, Edmonton T5N 2R1.

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Editorial

The Time: October 4-6, 1973

The Place: Chateau Lacombe, Edmonton

The Event: Edmonton Meeting of the National Council of Teachers of Mathematics (MCTM)

Approximately 700 mathematics teachers from Canada and the United States attended the Edmonton meeting of the NCTM last October. The Mathematics Council of The Alberta Teachers' Association hosted the successful meeting.

This monograph contains the general session and general interest session speeches given at the convention.

PURPOSE OF THE MONOGRAPH

The purpose of this monograph is three-fold. First, to make available to MCATA members the thinking of some mathematics educators. Many of the speakers made recommendations regarding teaching strategies, program development, future trends and other important aspects of mathematics education. You should find many interesting, challenging and even controversial ideas in this monograph.

A second purpose is to make available the sessions you may have missed because you chose to attend another simultaneous session.

Finally, this publication provides you with a reference for the ideas you heard. Perhaps you have already been in the position where you have said, "Some-one at the convention was talking about that, what did he say?" Many of the ideas you heard are included in this monograph for your reference.

Those who were unable to attend the convention will find this monograph to be a valuable source of current ideas in mathematics education. Perhaps it will whet their appetite for the next convention sponsored by MCATA.

ORGANIZATION

The elementary general session speeches (Gibb, Immerzeel, Cathcart) are printed first. These are followed by the junior high (Bezuszka, Clary, Wells) and the senior high (Manning, Del Grande) general sessions. The last section contains the general interest session talks (Schaaf, Harrison, Cohen).

The addresses given by Dr. Eugene Smith and Dr. Eugene Nichols were not available for publication. We apologize to those readers who may have wanted to see one or both of these in print. Dr. Smith's keynote address is available on tape from the Department of Education.

THANKS

The MCATA executive extends thanks to Dick Daly (general chairman) and Stuart McCormick (program chairman) for organizing a successful and worthwhile convention. A word of thanks is also due the headquarters staff of the National Council of Teachers of Mathematics for their part in the organization and operation of the convention.

The editor would like to thank those speakers who provided him with a copy of their speech. This reduced the editor's workload and guaranteed accuracy of the ideas presented. He also wants to thank those speakers who revised the edited version of their speech which had been transcribed from tape.

Thanks to all of you who attended the Edmonton meeting of the NCTM - your attendance contributed to the success of the meeting.

The Editor

Through the Eyes of our Students

Observations of children in our mathematics classes provide basis for concern. Do we have the courage to get at the sources of evidence of frustration, resentment, "care-less" attitudes, and complacencies and then take actions on our findings?

E. GLENADINE GIBB

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University of Texas at Austin

Austin, Texas

INTRODUCTION

I've chosen the topic "Through the Eyes of our Students". It's one that has perplexed me, interested me, and has given me direction in guiding the mathematics learning on the part of those students who are taught by me. I realize that any experience which I share with you is my experience with the children whose eyes I have tried to see through. They have shared willingly with me as I I have guided their learning of mathematics. I hope that their experiences will provide an opportunity for you to think about the students to whom you are responsible for their mathematical understanding and skills.

Speaking of our students, the beginning of a school year seems to be the appropriate time for "sizing up". For us as teachers, another year often finds us with a different group for whom we have many responsibilities. One responsibility is to direct their learning in mathematics - their understanding, their skills, the fostering of capabilities in mathematical thinking. Teachers describe these new groups of students with varying responses. I have heard some say, "You know, I've just got a wonderful group this year. It's such a bright class. It's going to be a real joy." Others say, "It just seems like another year. There are always some that just don't have number sense and there are always a few who make life worth living." One said, "My whole class is weak, I just don't know where to start." Another said, "Well they are in their second year in school; they're not ready for the mathematics of that year, but I can't worry about it - here we go. There's so much that has to be covered this year."

While we are "sizing up" our students, we have to realize that they are "sizing us up". One of my own students recently shared with me the experiences that she had in her family with her younger brother who is in his fifth year in school. He had always enjoyed mathematics. Yet on the second day of school of this year he came home and announced to the family, "I'm not going to do very well in mathematics this year." "Oh, you're not, why?" asked her family. His response was, "The teacher doesn't like me."

Those of us who have read *I'm Okay, You're Okay* may find themselves as teachers saying, "I'm okay, but kid you're not." Some of us may learn to admit, "You're okay (student), but I'm not (teacher). I'm not the best teacher for you as hard as I try." There are many students, on the other hand, who get the same impression. "Okay, teach, you're fine, but I'm not sure." Some of them are more dramatic about it and say, "Well, I'm not and neither are you." There are others who say, "I'm alright but you're not." And then, there is that joyous feeling when we can all come together to feel "I'm okay, you're okay."

Thinking of bringing people together, let us not forget that when two people come together, there are really not two people - there are six: two people as seen by themselves, two as seen by the other, and two as they really are.

Thus, I want to share with you, first of all, some reflections I have had as I have observed and taught children in an effort to see learning experiences in mathematics through their eyes. Secondly, I wish to suggest some ways of preventing what seems to me symptoms of frustration, resentment, that "careless" attitude, and just plain complacency in these observations.

S Y M P T O M S

FRUSTRATION

I have observed many youngsters whom I think are frustrated. (You may want to give other labels to this symptom.) Some youngsters are frustrated because they are doing the same thing over again as they did last year. This also applies to students in college. Some of my friends in psychology tell me that as mathematics teachers we commonly commit the "sin" of not recognizing what students already know. We do come back on ideas in a spiral way of getting deeper and deeper and learning more and more about a particular mathematical concept. Although we know that we are doing things differently, or that we are helping the children to do more this year than they did last year, we don't share this with them. The youngsters do not see this themselves, that is, they do not experience the challenge that "This year we are going to learn more about this idea than we did last year." Since we don't share this, they become frustrated.

There are some who become frustrated because they don't get it the first time and here they are trying for a second time, and some children even a third and a fourth time. When I walk into a classroom, I usually can tell by the size of the children what year of school it is for them. I also check this out by observing what they're doing. I was surprised as I visited one classroom and saw a group just sitting there frustrated, not knowing what to do, and seemingly most unhappy. I wondered, "Now what year are they ?" "Oh, this is first grade, fourth chance", I am told, with the additional information that "They didn't learn the first year, so we put them through the second year, same thing all over again, and they still weren't ready to go on, so this is the fourth time." After any of us has experienced failure once, it is difficult to try again. To experience it twice, thrice, and still be expected to try again, takes much courage. I better understood that which I was observing.

In another classroom, I observed a sixth grade youngster just bang his book shut. I asked, "What's the trouble?" He replied, "Just about the time I think I understand something, the teacher goes on to something else. I just never quite make it." After a number of instances like this, a student finally gives up and frustration sets in.

Another source of frustration for pupils is the marking of their papers. We say, "You didn't do it right" or "There's something wrong here", and a child becomes frustrated because he doesn't understand why it is wrong. I recall one youngster who was sent to me because he was having trouble with subtraction, and here he was in his fourth year of school. I started out with a simple problem such as, "Suppose we have 7 apples and we eat 4 of them. How many are left?" As I spoke, I wrote the 7 down and under it, -4. I asked, "What would you put here?" He said 7. Well, so we often do, I thought we didn't communicate, so I repeated the problem. I got the same response. You don't go through the same thing three times. How am I going to get at this again? He thought faster than I and said, "Alright let me help you, that's a 7", as he pointed to the 7 on the paper. "Yes." "That's a 4, isn't it?", he said, pointing to the 4. "Take the 4 away (he covered up the 4), isn't there a 7 left?" He had gone on for three or four years with this thought and continually had all solutions to subtraction problems marked wrong.

In trying to abstract a mathematical notion, we may give illustrations, but the language we use is not sufficient to communicate. One student in his fifth year was having trouble with addition. He had not had trouble like this before. He was always able to add, for example, numbers in the thousands, the ten thousands, and even four or five numbers at a time. He had decided how the teacher made a decision about what to write down and what to carry. He said to me, "If you come out with 12, you write down 2 and carry the 1. If you come out with 21, you still write down the 2 and carry the 1, for you always write down the big number and carry the little one." If we think back on our teaching of addition computation, most sums in any column are not greater than 19, at least through the fourth year in school. His generalization had always worked until the sum became, say, 21.

I reflect on another youngster who was in his sixth year of school. He was frustrated because he had found a contradiction with previous learning. He had learned that in multiplication of whole numbers the product was always greater than either of the two numbers. Then, it became time to learn to find products of fractions and things didn't come out the same way. "This cannot be multiplication", he said. Until he could figure out or someone could help him to see why the product was smaller than the pair of factors, he wasn't going to do any more problems.

RESENTMENT

Why are some students bitter or resentful? One reason seems to be that they see through their eyes their teacher putting them down. When a supervisor or consultant visits the classroom, it doesn't bother the teacher to say, "That kid just can't learn. He'll never do anything. You might keep him here, but, we need something to entertain him with so he can at least be busy." Children don't like that just as we don't like it when a consultant says to another consultant, "That teacher just can't teach. Why we ever employed her, I'll never know."

Here are two other examples of student resentment. One little girl could not understand percentage problems. She was missing all of her work. She was visiting her girl friend who could do the work. When the time came to do their homework, her girl friend helped her. For the first time the girl understood percentage problems and could solve them. She got so excited over the problems she could do that she did the entire assignment. The teacher looked at them and tore her pages in pieces before her eyes. "You cheated, you don't know enough to do this." The teacher then gave a test, and the child wrote the test with no error - a perfect paper. Again, the teacher said, "You couldn't have made 100 percent - you flunked." That girl is now a teacher herself and claims that she cannot stand that teacher to this day. The second example is one shared with me by one of my own students. She, as a child, had a teacher who was fine to get along with in English and social studies but when it came to mathematics, there was conflict. One day as she was doing some problems, the teacher screamed at her, grabbed her pencil and broke it. The emotions and resentments that result from these kinds of experiences are unbelievable.

It has taken me awhile to realize the conflict and resentment that occurs for some children with ideas of subtraction and division. Prevelant among youngsters who are emotionally disturbed, from broken homes, or who have home turmoil in their lives, is an emotional conflict associated with these operations. According to my colleagues in special education, among the guidelines for teaching mathematics to emotionally disturbed children is "Do nothing with subtraction. Do nothing with division."

I-COULD-CARE-LESS

Another area which, I feel, reflects the thinking of some youngsters as I try to see what they see is the "I-could-care-less" attitude. "I'm just not turned on", said one youngster as he refused to do work with fractions. "Why don't you want to work with fractions?", I asked. He replied, "I don't need them." "Why don't you need them?", I asked. "Well, he said, "because they're girl problems. Fractions are for girls, they are not for boys." I then turned to the textbook being used in his school. Yes, every problem on that page was a "girl problem".

In discussing a similar disinterest in problems with another boy, I said, "How about making that party problem a baseball problem?" He looked up with a smile on his face and said, "Could we? I'd like that." Not until we begin to see what kind of response our students have to that which we expect of them will we be able to change an existing attitude or foster a change in attitude.

COMPLACENCY

I characterize complacency as an attitude which reflects, "This is something that I have got to cope with - it's here. I have got to do it so I'll try to make the best of it." One example occurred in a classroom where the children were all sitting around a table for instruction and each child with his own set of sticks. When I visit such sessions, I usually pull up a chair and become one of the youngsters. I had my sticks, and when the teacher told us to put our hands in our laps, I responded as did the children. When we were instructed to bundle our sticks in groups of 10, I grouped my sticks. As a member of the group, I observed that of the 21 of us, the teacher seemed to talk to only four. One child assured me that I was not alone in my observation, for he said to his teacher, "Mrs. Smith, you never did look at Dr. Gibb's sticks", to which I replied, "I think there are several of us that she did'nt see today." In other words, "I'm here, but I don't think you know that I am here, teacher."

Some of you may be involved with computer-assisted instruction. In working with children in these experiences, I recall an eight-year-old whose teacher didn't know what to do with him, so he was sent to the laboratory to see what he could do on the computer terminal. After he had been working for a period of time, I asked him, "How's it going?" "Oh", he said, "this is good. It's so nice to sit down to when you are lonely." Another student had also come to work on a learning sequence. Somehow he became involved in a lesson that was not necessarily appropriate for him but nevertheless he insisted on doing it. You would not believe what change came over the youngster upon successfully completing the lesson. He returned to his classroom, he did his mathematics, he completed his assignment in social studies, he completed other tasks that had been assigned. He had taken a new lease on life - a new motivation for learning. In response to one of my colleagues about the effectiveness of the computer in mathematics instruction, my reply was, "I seem to be doing mental therapy rather than mathematics."

Sometimes our students do not see a purpose in that which we expect them to do. Many of us have been involved in the spirit of mathematics laboratories - materials, activities, games, and so on. Sometimes learning mathematics is perceived by children as one game after another. "Why don't we do some mathematics instead of just playing games", they ask. When a parent asked his child what he did in school, the reply was, "Just played games again." We don't share with our students how these experiences can be expected to help them for a particular kind of learning, practice, or review.

Other children are complacent about the judgments and evaluations made by us. A teacher handed me a paper of one of her students and remarked, "I think this child got 40 percent of his work correct. How can I help him." I took the paper, examined it, and replied, "According to my calculations, I find that he has 92 percent of his work done." The teacher had responded only to the answer - the solution. It was either wrong or right. In the stages of learning a process, we lose a lot of information if we do not examine both the process and the product. This child had been right up to the very end of most problems and yet was given no credit for his accomplishments.

What we see, or what we think we see, as we teach mathematics and what our students see us doing either in relating to them or in our judgments of them and their work are not always the same. To be effective, we just see through their eyes.

C U R E S

What can we do? Some suggestions follow.

Be positive in your attitudes about the ability of each child to learn, and demonstrate that you are positive.

One teacher told me that she really had a slow class, but she led them to believe that they were the best class she had ever had and they just worked, worked, worked. We, as teachers, are human too. It is difficult for us to be positive in our attitude toward each of our students, but we must strive to do so. There is some good in what each can do. We must spend less time talking about what our students can't do and focus on what they really can do.

Start at a child's entry to the road of learning, not where you would like him to be or where you expect him to be.

We need to listen to our students and to hear what they say. Also, we may have some diagnostic measures in the way of tests, interviews, performance tasks - asking, on the one hand, "Are they ready to move ahead?" and on the other hand, "Have they already moved ahead?" Often, however, our questions are knowledge questions. They involve what I call the "spit back", recall response. From these kinds of questions, it is difficult to tell whether or not a child can think on a higher level. Questions to assess high-level thinking are more difficult to formulate. Although we need knowledge, a higher level of thought involving comprehension and application are needed even more. Listening to what students say can help us to determine what they really understand.

One little girl could hardly wait to get into "two-digit" multiplication. She got a book off the shelf and decided to teach herself. Her teacher was disturbed because she was using a different method than that used or taught by the teacher. The teacher asked, "What do we do now?" In response, I said, "It is an interesting method. I wish I had thought of it myself." Children get the feeling that there's only one way to do something, and if dad or mother or big brother or big sister or anybody else suggests doing it another way, they are quick to respond, "Oh, no, it's got to be done this way." I cannot believe that, particularly in the level of skill. We all, once we understand what the job is, have unique skills in accomplishing that task. The best way for me to do it is not necessarily the way you find best for yourself.

Respect a child's value system although it may be different from yours.

This suggestion has been very difficult for me to follow. I seem not to be as sensitive as I should be to the value systems of others. During my childhood, I learned to respect the judgment of adults, to think for myself, to defend my thoughts, and to take the consequences of my decisions. In teaching, after I feel that my students have the idea being developed, I am inclined to make an intentional (sometimes unintentional) error with the expectation that my students will correct me. Some children, however, have been taught from early childhood that adults are always correct and even if they do make mistakes and you know that they have made a mistake, you do not correct them. Thus, for these children a correction would be improper to make.

Other children learn to bow their heads in respect to someone speaking to them. I need to remember this instead of saying, "Look at me, you're not even paying attention", while they are trying to give me their deepest respect. Granted, it is not the eye-to-eye contact that I might otherwise expect.

Be patient.

Learning mathematics takes time. It may take one day, one week, one year, or even longer. I feel that one of the things which has confused and perplexed children is that we tend to symbolize ideas before they have an idea to symbolize. My position is that children need to have an idea, have a thought, and then learn to symbolize it using mathematical language. There are those who disagree with me and say, "Here are the symbols. Now let us get some meaning for them." Symbols, the language of mathematics, should bring some kind of mental image. Furthermore, we need to communicate to children using a language that makes sense to them. I recall working with a group of seven-year-olds and problems such as, "You have 6 objects. You need 15 objects. How many more will you need? ($6 + n = 15$)." I thought all was going well until I observed 25 pairs of eyes staring at me as blank as they could be. Fortunately, one boy said, "Miss Gibb, this may be easy for you, but it sure is hard for us." Too often we forget or are not sensitive to the level of thought children have attained. We need to provide many opportunities for communication on the part of children. What they say and what they do can give us insight into what they understand which, in turn, can better enable us to pace learning.

Be proficient in using several modes of instruction.

We need to continually seek ways to help children learn. A few years ago, we were "gunho" on what we called discovery (or guided discovery) teaching. Evidence from research supports our own experiences that all children do not learn in the same way. Two modes of learning that seem to categorize differences in learning are field sensitive learners and field independent learners. Field sensitive learners need an adult demonstration of that which is to be learned. Field independent learners, on the other hand, seem to learn best in discovery-type modes.

One of the ways in which we can attune to children's learning needs is by developing different styles of teaching. Just as we need to be multilinguists in order to communicate with many children with whom we work, we also need more than one style of teaching. We may have our preferred style, but we should have other styles which we can utilize in order to enhance opportunities for learning on the part of all children.

Move slowly enough to allow time for mastery.

Sometimes we get geared up to do so many pages or so many problems, thinking that such experience is sufficient for students to learn. There is no assurance that by simply covering the ground we are enabling children to learn and that they indeed learn. Yet, one can spend too much time on one idea or skill and likewise turn off learning. I recall one teacher who stayed on one page from October to January because every youngster in her class did not know the basic facts on that page. We must use judgment, but we do need to make it possible for children to feel that they have learned and to feel good about that which they have learned. It can certainly lessen their frustrations. Many people say that they have so much to cover; but wouldn't it be better to go slower so that children can learn? Think again of those children who were trying to learn for the fourth time.

Provide children with individual records of their progress.

What motivation when a child can say, "Well, I know this but look, see how much better I did!" There is nothing that is more wonderful than success - a feeling of accomplishment - to motivate further learning.

Understand mathematics yourself.

Experiences in the text materials we use are not necessarily appropriate for the children to whom we are responsible. We need to be able to create new experiences that will really get at the essence of the idea and be more meaningful to those children we are teaching. Sometimes we do not use the materials we have to build a program or to build a course. Rather, we become slaves to a set of printed materials. Instead of letting materials help us, we let them dictate to us.

SUMMARY

The intent of my remarks has been to provide time to reflect on how the students in our classes see us and what we expect of them. What are they taking home? As a way of looking at this problem, I have suggested looking through the eyes of those students who show symptoms of frustration, resentment, as well as those who just seem to go along. (Sometimes we like the latter the best because they do not bother us.) I have suggested a few guidelines for our consideration in overcoming these interferences and, what is more important, to prevent them. Many of us are already doing this. But whatever we have done, I think we would agree that we have to work awhile, we have to endure much, and certainly we have to believe in what we are doing. With our ability to help students perceive the ideas of mathematics by seeing through their eyes, we can help to become competent in that which they do, to think sharply and to be creative. Our effectiveness as teachers of mathematics is measured by those children whose future lies in the world of tomorrow.

When is the Thing the Thing?

It is important to recognize what things can do and what things can't do. Things can build images, but they can't teach the students to compute. They can provide experiences, but they can't make generalizations. Examples from the elementary classroom will illustrate when the thing's the thing.

GEORGE IMMERZEEL

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INTRODUCTION

I want to relate to the idea of using things to teach mathematics. I also want to clarify a few things related to these things. There is, for example, the idea of a mathematics laboratory which has two meanings, one of which is a physical place. I believe in the other meaning. Any classroom can be a mathematics laboratory if we provide the proper atmosphere, the feeling of doing things rather than talking about them. It is an attitude, an approach, but it is not an approach which will solve everybody's problems. It is an approach that can do certain things for us and do those things very well. We seem to try to find one solution to all of our dilemmas, but that solution does not exist. It is very important to look at what we are doing and try to point out those things which the laboratory approach can do well and those things which it cannot do well at all.

Another matter I want to clear up is the belief that a mathematics laboratory necessitates some kind of individualized instruction program. While you could operate an individualized program and use a laboratory approach to teaching some of the concepts, they are not dependent. The laboratory approach does not necessitate an individual approach, nor does an individual approach mean that you have to use a mathematics laboratory. They are just different kinds of things.

It is useful to establish some stages in developing concepts with children. The stages which help me design a program or organize curriculum are image, symbols, organization, generalization, practice, and application.

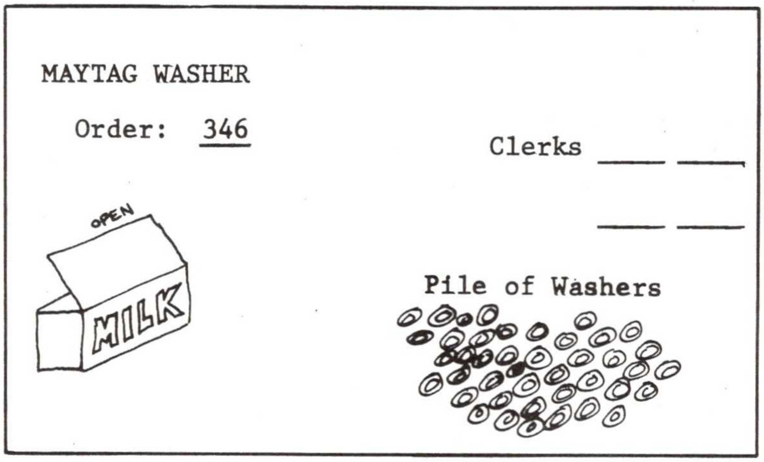
We need to be sure that all children, whether they are in Grade VIII, XII, or I, will think with images rather than just symbols. When a child sees 2 x 5, he must have some way to relate to that. When he sees a probability, he should have had some kind of experience to which he can attach that probability. When I am talking about congruent triangles, I want the child to have done things with congruent triangles so that mathematics becomes more than just maneuvers with symbols. However, I must get more sophisticated than that if I am going to develop learning very efficiently. So as I build a concept, I want to move into some kind of development which provides the student with an opportunity to arrive at some kind of generalization. Practice using the developed ideas is essential for fixing learning, and applications are of great significance to the student. If mathematics doesn't have a real meaning to him, then we are in trouble. We have to do a lot more with application.

BUILDING IMAGES

When I write down table, chair, apple, and so on, the word brings an image to your mind. It may not be the same image that I have, but you do think in terms of the images they present. In teaching reading, those words which do not have images are particularly difficult for the student. There is nothing to tie it to except the symbols. We have had difficulty with those things in mathematics also for which we do not have imagery. Everything we do in mathematics should have some kind of image. Most images can be built through physical experiences in a laboratory setting. Pictures just are not as good as the objects themselves.

Many image-building activities happen at the primary level. If they don't happen there, you will find out fairly quickly.

Figure 1



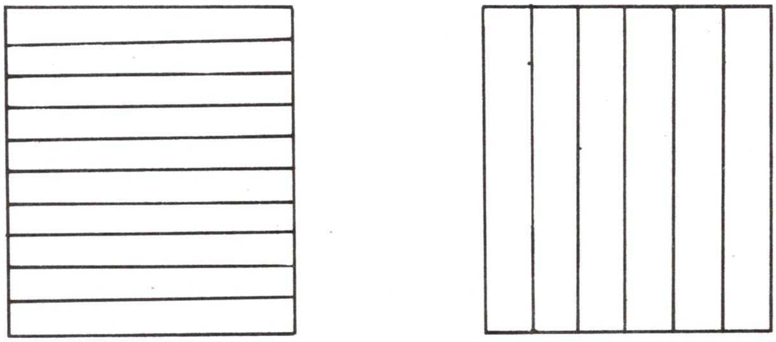
If we want to build some concepts of numeration, we can use the activity in Figure 1. We do a fairly good job talking about small numbers with children, but we "cop out" when it comes to talking about 346, for instance. Many children in Grades V and VI could not count 346 and be sure that they had 346. The Grade III activity in Figure 1 is very useful. The class is divided into two sections. One section consists of clerks - they will be in small groups of four or five students, and their job is to put 346 washers in the boxes. One student may have 45, another 52, another 85, and so on. They usually conclude that they don't know how much that is. Often some student will say, "Let's put them in groups of 10", so everybody puts them in groups of 10; and somebody else says, "let's put 10 tens together and put them in the box. That's 100." By the time they have filled that order and maybe two or three other orders, they're beginning to build an image of the importance of the numeration system in terms of 10s and 100s. It is much different than telling the students that this means three hundreds, four tens, and six. If I ask one of my students to tell me how many tens there are, I want him to tell me that there are 34, because I want him to relate to the number involved.

Students in the other half of the class are inspectors who check to see if the first students have correctly counted the order. I get the students started ahead of time by filling a few boxes with handfuls of washers. This usually results in the order not being correct. Not being correct provokes more arguments than being correct, because the students are not sure that they counted correctly. Very often they go through the first order two or three times. They can't imagine that the teacher would give them a thing entitled 346 washers without there being 346 washers in it. Both of these groups are developing images for numeration. You are going to have to do things to build imagery for fairly large numbers if you expect children to operate with them.

Some of the things I'm going to show do not require a lot of laboratory equipment. For example, a book is a good image-building device, and it is available in every classroom. I'm on page 37, I turn ahead 10 pages, what page am I on? I'm on page 52, I turn back 5 pages, what page am I on? This device may even be better than the number line.

Another example involves finding the number of intersections when these transparencies are put one over the other. There are nine lines on one and five lines on the other.

Figure 2



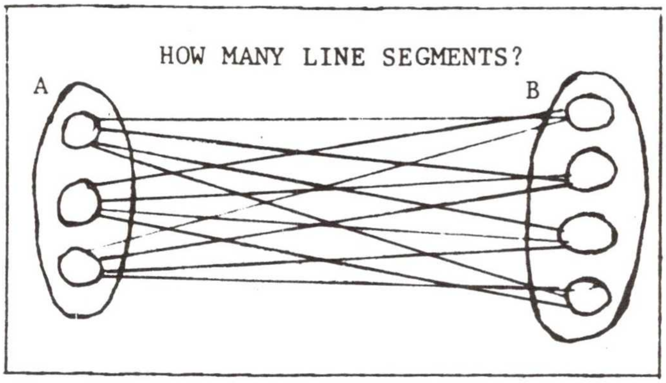
This can also be used to build an image for multiplication by 0, which is essential. Simply use a blank transparency for one of the factors. How many points of intersection will I get after I have established the model? That's far better than the teacher I had in the fifth grade who carried fruit jars from the basement as the image for multiplication. Going to the basement three times and not bringing up any fruit jars each time didn't make any sense to me. But it made better sense than the time I didn't go to the basement at all and brought up three jars each time I didn't go.

Another way to build images for multiplication is through the use of an array. A picture of an array in a textbook isn't as important as some physical objects the child can arrange. At the second or third grade level you could give students some objects and ask them to arrange them so you can see how many you have. Many students will automatically arrange them in an array. Asking them to describe the array helps translate that particular physical experience into symbols. They will tell you that there are six in each row and four rows. They will use their language, but they will have a better image than if you gave them the picture of the thing because you can't maneuver and manipulate the picture.

One important characteristic is that physical things have built into them a "forgiveness" factor. You can maneuver them, you can correct your mistakes, you can do things with them without being brought to a fatal conclusion which often happens on paper.

Not all the things that build images have to be physical. Figure 3 illustrates a perfectly good way to build images for multiplication.

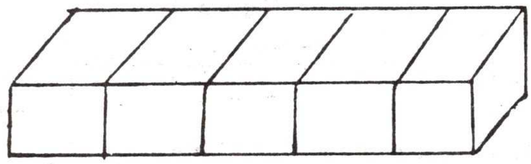
Figure 3



Using the two sets of points, Set A and Set B, how many line segments can I draw which have one end point in A and the other end point in B. The students will draw them, rather than look at a picture. You can explain that for each of the points in A there will be four segments (4 times 3). Multiplication has to take on a variety of images because it is used in a variety of ways.

Figure 4 illustrates a repeated addition model. There is the same number of objects in each of the boxes. Have the students complete the table.

Figure 4



TABLE

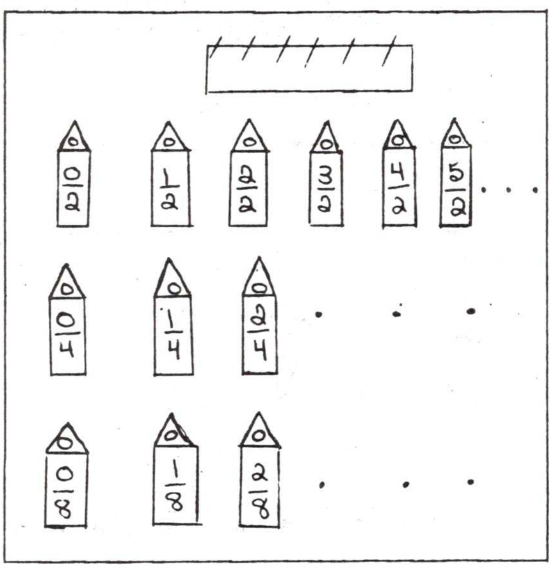
BOXES	1	2	3	4	5
OBJECTS					

Another multiplication model is just a ditto sheet with a lot of dots on it. Ask the children to circle, say, 23 times 45. How many do you have? This can be used to provide an image for the multiplication algorithm, or, if the children have had some experience with the algorithm, they can relate the algorithm to what they are going to draw.

You should follow up the image you have built with some kind of paper-and-pencil-activity so that students can translate what you do in the physical world into something that makes sense in the symbol world.

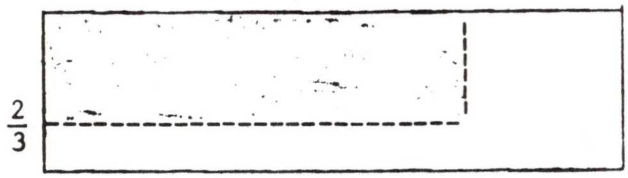
Each time we attach a new concept with children, we should have in mind some good image-building activities. Equivalent fractions is one concept for which we often ignore image-building. One model for equivalent fractions consists of some nails and tags as shown in Figure 5. What other tags go on the same nail? Seeing tags tacked up seems to have a different effect than simply talking about equivalent fractions.

Figure 5



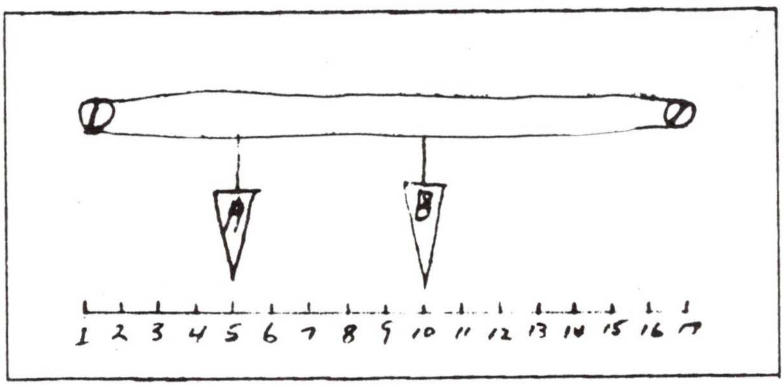
An image you might use for multiplication of fractions is shown in Figure 6.

Figure 6



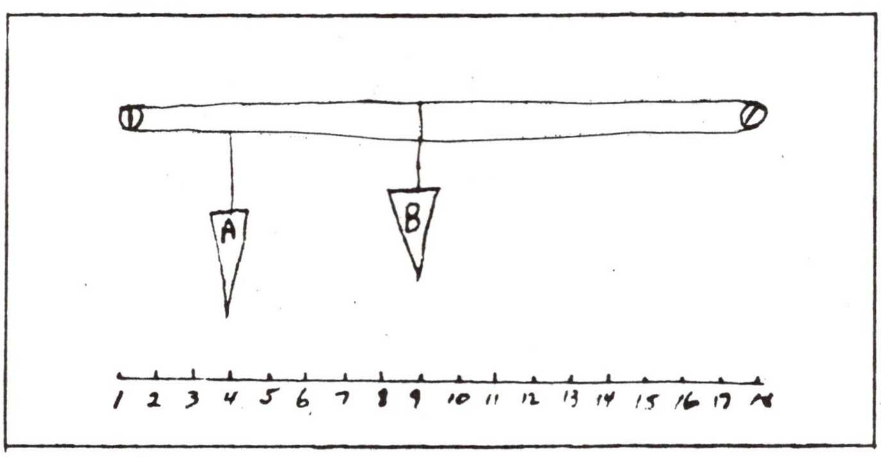
Take a rectangle of the size 3/4 by 2/3. What fraction of that rectangle have I shaded? I would hope that at the image-building stage I could relate that to 2/3 times 3/4.

Figure 7



Even the concept of a function needs some kind of imagery for it. Loosen 2 screws on the end of the blackboard. (See Figure 7) Draw a number line under it, with any kind of numbers you want. Run a piece of string around the screws. The tag moves as the spring moves. If I move tag A to 5, where does tag B move to? What kind of relation exists between the positions of A and B, a relationship that I could express in a function if I wanted to build an image for that function? B-A is 5 every time. The distance remains constant.

Figure 8



Put B on the top string. (See Figure 8) What kind of function do you have this time? You have a function that is a constant sum. If A increases by 1, B decreases by 1. Children could draw this kind of model at their desks using rulers which do exactly the same thing.

BUILDING GENERALIZATIONS

Physical things in various ways help to build images for some but not for all the things we want to teach in mathematics. The following is an example of something that cannot be taught well with physical imagery, and I don't think we ought to try to do that. Here is an organization of the basic facts for multiplication.

0 x 1	1 x 1	2 x 1	.	.	.
0 x 2	1 x 2	2 x 2	.	.	.
0 x 3	1 x 3	2 x 3	.	.	.
0 x 4	1 x 4	2 x 4	.	.	.
.	.	.			
.	.	.			
.	.	.			

I have reached the stage where I have imagery behind that and they mean something, but now I want to arrive at some generalizations about that list. I want children to recognize that they don't have to memorize the 0 facts or the 1 facts or the 2 facts. I do not know how to do that through a laboratory approach. We can focus in on the symbols only if we have imagery first. It's surprising that we do not let children into the secret of how many basic facts there are and what kind of organizational procedures we can use and what kind of generalizations we can build by making lists and by being honest with children. Often when I ask Grade IV children how many basic facts there are, they tell me that there are a million. It surprises a lot of children when they find out there are only 36. Let's be honest and let them in on this secret. They are not going to learn that with physical activity; concepts have to be organized.

Figure 9

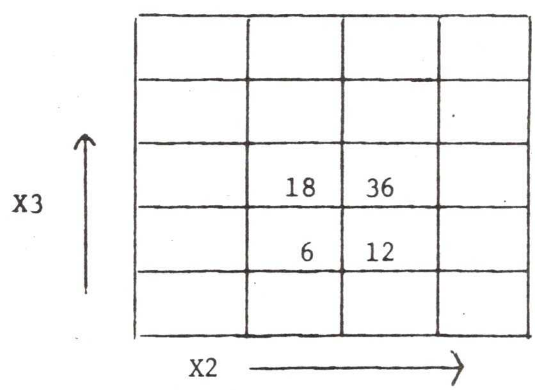


Figure 9 is a model for multiplication. If I go from 6 across to 12, I am multiplying by 2, and if I go from 6 up to 18, I am multiplying by 3. Completing the table in a physical model such as this will often build a generalization because students can often generalize when they see things organized. You can do exactly the same thing with multiplication of fractions.

PROVIDING PRACTICE

Practice in computation is important, and we make a big mistake if we ignore it. A student who enters Grade V without considerable multiplication skill is not likely to succeed. He just can't think mathematically unless he has the basic facts and can multiply with some facility. Not only does he lack confidence in what he is doing, but he also can't operate with area, ratio or fractions because all of these require fundamental computational skills. The reason for lack of skill on the part of many children is not because we have taught modern mathematics but because we as teachers have copped out. We haven't insisted that students compute. In my opinion, there will never be materials which can do that for us. We are responsible for our students being able to compute.

We can reduce the number of children with computation problems by providing our students with a variety of practice activities. Some mathematics educators have said that we should not use rote drill without pupils understanding what was going on before we used the drill. Many of us misinterpreted this to mean that we should use rote drill. There are many things in mathematics which we just have to memorize.

One simple way to provide immediate reinforcement in such things as basic facts is through an activity I call "Game 1".

Figure 10

BEAT THE TEACHER

Game 1

A ________ ________

B ________ ________

C ________ ________

D ________ ________

E ________ ________

F ________ ________

G ________ ________

H ________ ________

Copies of the above sheet can be used to provide practice in multiplication. I give, for example, orally a multiplication fact like 8 times 7. If a student writes down 56 before I say it, he writes it in the first blank. If he writes it down incorrectly or if he doesn't write it down before I say it, he writes in the second blank. When I say 6 times 8, 48, I will not say it as fast as I can. This provides practice with basic facts with immediate reinforcement. I can watch a student and put some pressure on him by adjusting the speed with which I say a fact. You can use this for fractions, decimals, multiplying by 10s and 100s and so on.

Another activity I have used involves the following two sheets.

Figure 11

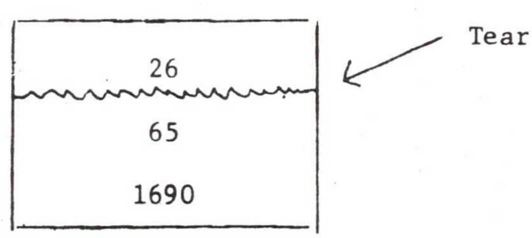
$\frac{648}{14}$	$\frac{483}{23}$
$\frac{851}{37}$	$\frac{910}{65}$
$\frac{544}{17}$	$\frac{4030}{62}$

21	32
65	47
14	23

In my class $\frac{648}{14}$ means 648 ÷ 14. The second sheet contains the answers. The students do the division and find out if their answer is on the second sheet. If it is, they get immediate reinforcement without me even being there, because they are quite sure that if they divided correctly, then the answer is there. Some say they can tell the answer without dividing. That is fine. Of course, the questions could be made harder.

Another practice activity with immediate reinforcement involves a calculator with a tape on it. I put into the calculator a number like 26 and then multiply it by 65 and get the product. (See Figure 12)

Figure 12



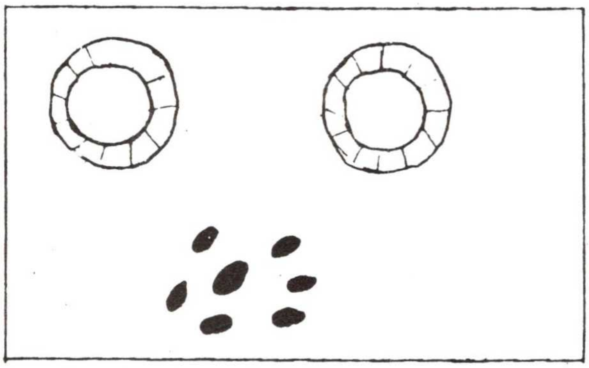
I tear the tape into two pieces as illustrated and give one piece to a student. He divides and then comes and asks for 26. What if 26 doesn't fit? He takes it, tries it, and if it doesn't fit, he goes back and divides again. If it does fit, he is immediately reinforced, goes back and does a different one. If I had given those children six or eight division problems, they wouldn't have done them. But if you give them one at a time with reinforcement, they'll do lots of them and enjoy it. Practice is important. We're not going to teach mathematics without it.

TEACHING APPLICATIONS

The things we use are also important in the application mode. There is a new mathematics revolution going on now which I call the "Application Revolution". One of the things we miss in the modern mathematics movement is the importance of problem-solving and application. That's what mathematics is all about.

Physical materials can provide application even at a very early level.

Figure 13

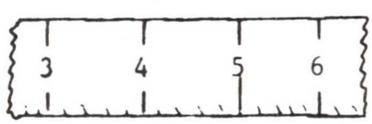


For example, Figure 13 shows two plates and seven beans. The pupil is to find out how many different ways he can put beans on each plate and to make a list of the ways. This is an application of the basic facts for addition. Things set it up well, better than listing all the basic facts for 7.

Another application involving area which is just the reverse of what we usually do makes use of a ditto sheet that looks like graph paper and a variety of rectangles. The pupils are to find the area of the rectangles. Is this a laboratory approach? It is to me because the pupils are manipulating things. Physical things *do* add a useful dimension.

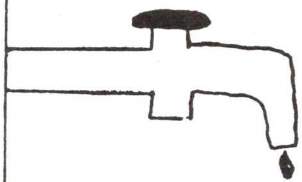
Other applications involve measurement. My students have difficulty using a ruler because the ends wear off. I want to teach the child to measure without using the ends of the ruler, so I break the ends of the rulers as illustrated in Figure 14. It is surprising how much that adds to a measurement experience. Now they have to do the subtraction. They have to use the ruler as I want them to use it.

Figure 14



If you have a sink in your room, you can use it to build a laboratory approach.

Figure 15



TIME (Minutes)	1	2	3	10	60
DROPS	15	30	45	150	

a. How many drops from the leaky faucet in 1 hour?

b. How many drops in 1 day? 1 week?

Let the faucet drip a little and prepare a set of problems about the faucet so that pupils have to measure the water from the faucet and then answer a series of questions related to the drippy faucet. They use a lot of mathematics in doing that. A table (see Figure 15) adds a great deal to the physical dimension in this kind of relationship because a table allows the student to get involved in the situation before he has to answer a lot of questions. We could put some entries in the table to encourage him to see the relationship. If I want him to be able to think for himself, I will not put entries in the table. Many of the slower students may not go beyond the use of the table to solve the problem. That's perfectly adequate.

The physical model suggested by the activity card in Figure 16 may be difficult to set up, that is, to actually have the pupils do the measurement.

Figure 16

8% of a block of ice is out of water.

Height of ice out of water				
Height of ice				

1. If a block of ice is 10 inches thick, how much ice will float out of water?

2. If the ice is 15 inches thick, how much float out of water?

However, just having the ice cube there may be helpful. This activity is starting to set up the concept of percent. The day I proposed it, 8% of the block of ice was out of water. Again a table will help the child to translate the physical into what he is thinking about when he wants to solve the problem. Notice the table does not have any entries in it. I'm hoping the pupils will say that's 8 out of every 100 feet because that's what percent means. There are a lot of ways to describe that function. Children's entries are probably different from those of the teacher's. Four out of 50, 2 out of 25, and 16 out of 200 are examples of the idea of 8%.

It is important after a physical situation is set up to use a variety of questions about the same situation because it takes the focus off just the computation and places it on the situation. One of the difficulties we have had in building good problem-solving skills is that we have not provided enough experiences with the physical situations. It's not that children can't read the example, but they don't know enough about the situation on which the example has a bearing. Figure 17 illustrates what I mean.

Figure 17

Eggs	1	2	3	12	24
Time	10	10	10	10	10

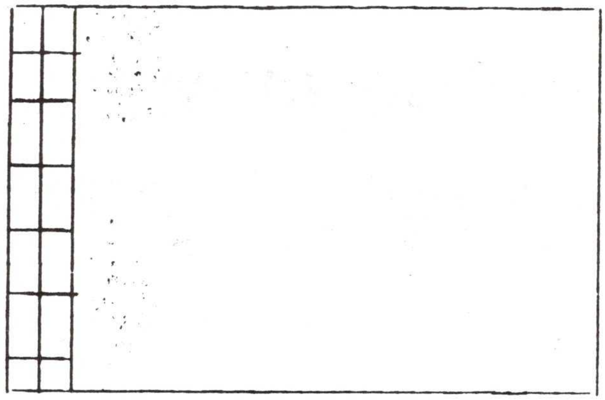
Equation: $T = 10$

If you can hard-boil one egg in 10 minutes, how long will it take to hard-boil a dozen eggs? Many pupils think of this as a ratio situation. You know enough about boiling eggs to know that it doesn't take any longer for one egg than it does for a dozen. How can you use the answer to that question? You might well express the relationship in the form of a table. You boil one egg in 10 minutes; two eggs in 10 minutes; three eggs in 10 minutes. You might be interested in writing an equation. It's not an important equation - as a matter of fact, it's one we often ignore.

If you put children into a physical setup, one of the first things they will do is to try, make trials and errors, make guesses, collect data, draw diagrams, and react to pictures. All these are problem-solving skills that the verbal problem completely ignores unless we happen to teach them. Most of those problem-solving processes require an imagery, something behind what we do when we solve problems. For example, providing a road map as a model on which children measure the distance, say from Calgary to Edmonton, provides an application setting which is better than verbal problems for which children don't have any particular meaning.

Another valuable kind of problem-solving skill develops when we place children in situations which require them to generate methods of solving the problem. For example, I asked a fifth-grade class how many fifth-graders we could get in the classroom. "Take out a piece of scratch paper and figure out what you would measure to find out." We had been working with area. I was impressed by the student who said he would line the students up as shown in Figure 18.

Figure 18



What impressed me was that he drew the students as a rectangle, not a square. Another student suggested that we get all the students in a corner. We had tile on the floor and we figured out that we could get 28 students on about 24 tiles. Then we found out how many tiles there were in the whole room and found that we could get 850 students into the room. Other students suggested other methods, but this is the one we chose to use.

Sometimes you have one student who has a unique idea. You should call on him because if you don't, you will turn off the rest of the class also. If they don't feel their question is good, then hardly anybody will ask. An example of a unique idea was given by one boy in my class who suggested that the 850 students would only be one layer.

When this same group was working with sugar cubes, I asked the children if they could place a million sugar cubes on the table. I left them there. Some pupils brought sugar cubes. Finally somebody laid them out and said, "Sugar cubes are about 1 cm, so it takes 100 sugar cubes long and 100 sugar cubes wide, and 100 sugar cubes tall. This would give you a million." Children don't have images for large numbers. They think a million sugar cubes would go out of the building.

CONCLUSION

Physical materials will provide two basic things: they will provide images for the things with which children think in mathematics, and they will provide applications which make mathematics real.

The Nonsense in My Little Girl's Geometry Program

The current elementary school geometry program will be reviewed, some nonsense pointed out, and suggestions made for more 3-D geometry in the early grades.

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Mathematics educators have devoted considerable energy to justifying the inclusion of geometry as a legitimate content area in the elementary school mathematics program. (Vigilante, 1965; Robinson, 1966; Inskeep, 1968; and Vance, 1973 are some examples.) "Children greatly enjoy working with this aspect of mathematics", "geometry is encountered in everyday life", "geometry can extend and enrich the study of arithmetic", "geometry should be taught ... because of its inherent beauty and vast utility in everyday life" are examples of the kind of arguments used by these writers.

While we must be ready to justify everything we include in the curriculum, geometry has become so generally accepted that our efforts should change from justifying to evaluating. It is time we took a careful look at the geometry program in the elementary school.

TABLE 1

GEOMETRY CONTENT OF FIVE ELEMENTARY MATHEMATICS TEXTBOOK SERIES

Grade \ Textbook	T1	T2	T3	T4	T5
1	Open and closed curves. Interior and exterior of curves. Betweenness for points	Incidental	Simple closed curves, segments. Plane figures	Identification of plane figures. Symmetry.	3-D shapes, lines, segments, curves, 2-D shapes. Similarity, congruence.
2	Points, lines, segments, intersecting and parallel lines. Polygons.	Incidental	Circles. Congruence by tracing. Diagonals. Symmetry by folding and tracing. Square corners.	Open and closed figures. Segments, betweenness. Congruence of segments and figures.	More 3-D and 2-D shapes. Open and closed figures. Transformations. Patterns. Intersecting shapes. Sorting shapes.
3	Sets of points. Rays, angles. Congruence for segments and angles. Triangles, parallelograms, rectangles, squares, circles.	Angles, triangles, quadrilaterals, polygons. Parallel lines right angles. Circles. Coordinate system.	Rigidity. Correspondence in congruence. Solid shapes (cube, pyramid, sphere).	Interior, exterior. Angles, right angle. Parallel lines. Similarity. Corresponding parts.	Characteristics of solids. Symmetry. 2-D derived from 3-D. Intersecting and parallel lines. Angles.
4	Planes. Perpendicular. Right angles. Classification of polygons. Congruence of polygons.	Simple closed curves. Cubes, triangular pyramids, cylinders and cones.	2-D Figures (extended). Rotations. Scale. Ray, angle, plane, right angles. Perpendicular and parallel lines.	Triangles and other 2-D figures. Perpendicular lines. Space figures.	Compass constructions. Tiling. Congruence - Extension.
5	Open and closed surfaces. Plane and curved surfaces. Space figures. Similarity. Congruence by sliding.	Sets of points. Planes. Congruent segments. Perpendicular lines. Congruent triangles. Space figures.	Classification of triangles. Transversals. Regular polyhedron. Compass constructions. Curve stitching.	Classification of angles and triangles. Circle. Compass constructions. Scale. Coordinate system.	Cross section of solids. Planes of symmetry. Scale. Construction of solid shapes. Planes.
6	Classification of angles and triangles. Congruence by reflecting. Intersection of planes and surfaces. Space figures.	Congruent angles and triangles. Compass constructions. Pythagorean theorem. Cross sections. Solid shapes.	Regular polygons, interior and exterior angles. Planes of symmetry.	Extension of previous ideas.	Rotations of regular polygons. Figures as sets of points. Bisection Classification of triangles. Angle sums.

WHAT ARE WE DOING?

Before I level some criticism against the geometry program, let me review the program outlined in a number of textbooks. I do this on the assumption that the textbook is the basic curriculum guide in most classrooms.

Table 1 is an abbreviated sequence chart for five modern textbook series for Grades I through VI. T_1 and T_2 were published between 1965 and 1969, while T_3 , T_4, T_5 are 1970 or newer publications.

With the exception of T_5, the geometry program outlined in Table 1 can be generalized into a sequence something like this: Points, lines, shapes in the plane (including classification, similarity, and congruence), angles, shapes in space. While programs differ in degree of integration of these ideas and with respect to other topics such as curve stitching, compass constructions, transformations, and other concepts, the sequence stated above seems to be a common one.

There are programs which deviate from the above sequence - T_5 for example, which integrates solid and plane geometry from Grade I through to Grade VI - and many teachers add to and subtract from the program in a textbook. However, the sequence represented by textbooks T_1 through T_4 in Table 1 represents the most common type of geometry program presented to elementary school pupils today.

WHAT IS WRONG WITH WHAT WE ARE DOING?

At least three major criticisms can be levelled against our current elementary school geometry program. These will be discussed in order of increasing severity and seriousness.

First, some of the concepts being taught are not really very important. Perhaps the only justification for including them is that they are prerequisite to more advanced work in geometry. For example, I question the value of the point-set approach in the primary grades. While the concept of a point is basic to much of our advanced geometry and the idea of a set is certainly a unifying factor in mathematics, the meaningfulness of these ideas to a young child is questionable.

We place far too much emphasis on definitions based on the point-set approach. For example, defining polygons as the union of certain kinds of line segments is not satisfactory to many pupils in the primary grades. Another example of a concept of little importance is the distinction between open and closed curves.

Of more serious concern is that we start our geometry program with the abstract rather than with the concrete. All but one of the textbook series outlined in Table 1 introduce geometry "logically" with lines and various polygons defined as sets of points. Points, lines, polygons are all abstract concepts.

Beginning with the abstract contradicts the best theories of learning which we have. Bruner (1966), for example, has found that children first code and represent the world around them in an enactive way, later in an iconic or visual way, and finally, only after sufficient experience, they can make use of a symbolic coding system. According to Piaget, most children are not able to operate at this symbolic or abstract level until they reach the stage of formal operations near the *end* of the elementary school years, not at the beginning as we assume in our geometry program.

Dienes (1967) maintains that we need to embody mathematical concepts in many different physical materials before we can expect children to abstract the mathematical concept.

Beginning with the abstract as we do in geometry ignores the child's conception of space. Piaget and Inhelder (1963) and others who have replicated their work have found, for example, that young children do not believe that a line segment can be a set of points. If asked what the smallest line segment is, they insist that it is still a line segment until they are approximately 11 years of age.

Thus, as Copeland (1972) says:

> to begin in first grade with the notion of the basic element in geometry as the "point" and that lines, squares and so forth are "sets of points" ignores the child as a prelogical rather than a logical person, assuming instead that he has the logical apparatus of an adult mind [p. 23].

The third and most serious criticism is that most of the geometry taught in the primary grades does not build on the child's previous knowledge and experience. We start off teaching points, lines and polygons without asking what kinds of geometric experiences children have had when they come to school. What kinds of experiences have children had with geometric concepts before we see them in school? While they have *not* played with points, lines, triangles, squares and so on, almost every child has built castles out of blocks or cubes, thrown balls or spheres aound the house, licked ice-cream cones, and helped his mother by taking a tin or cylinder of soup out of the cupboard. The child lives in a three-dimensional world. He is frequently manipulating solid shapes. In fact, a very high proportion of a child's out-of-school geometry experiences revolves around solid shapes.

In failing to build on the three-dimensional experiences which a child has had prior to coming to school, we are violating a basic principle of good learning theory which says that new knowledge should be built upon previously learned knowledge.

Delaying the teaching of 3-D geometry to the fourth or fifth years of school, as most of the textbooks in Table 1 do, is pure nonsense. Thus, the nonsense in my little girl's geometry program lies in the poor sequencing of the topics of geometry.

WHAT SHOULD WE BE DOING?

What can be done to improve the geometry program in our elementary schools? If the major criticism that the sequence is backwards, or, at best, all screwed up can be overcome, the other criticisms may also disappear.

What we should be doing is starting in the early grades with three-dimensional geometry and graduate to two- and one-dimensional geometry in the upper elementary grades. This is the reverse of our present order. The remedy, however, is not simply reversing the order. That is too easy and naive. We can't simply take the 3-D ideas and materials presented in the fourth and fifty years of school and put them into the first. The whole approach has to be changed.

Since children's experience with three-dimensional geometry has been at the active and experiential level, we need to continue this approach and build on that experience. "Action on objects precedes perception and, of course, conception." (Skypek, 1965, p.443)

Activities in the primary grades involving three-dimensional geometry might fall into the following broad categories.

Getting a feel for solids

Young children should have opportunity to simply play with various solids. A useful activity is for one student to hide a shape in a cloth and another student to feel the shape and try to guess what it is. His description can be general rather than involving technical terms.

Relating solids to familiar objects in the environment

Let the child find objects in the classroom, his home, the grocery store, or other places, which look like some of the regular solids we want him to become familiar with.

Figure 1, on the following page, is an example of a possible assignment card for such an activity.

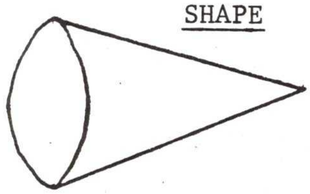
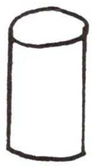
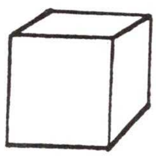
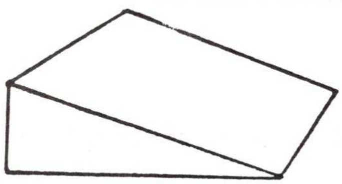
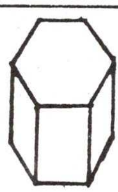
Examine properties

Another useful activity for primary pupils is to have them examine some of the properties of the various solids. For example, they could count the number of edges, faces, and corners (vertices) of the solids. This is not a trivial exercise because children must devise some kind of a scheme for keeping a record of which edges, for example, have been counted. They frequently lose track of which ones they have counted. Pupils can also examine the kinds of faces (curved or flat) and the kinds of edges (curved or straight) which various solids possess.

Figure 1

FAMILIAR OBJECTS

Find some familiar objects at home, school or in the grocery store which remind you of each of the solid shapes on the left.

	SHAPE		REMINDS ME OF
1)		1)	
2)		2)	
3)		3)	
4)		4)	
5)		5)	

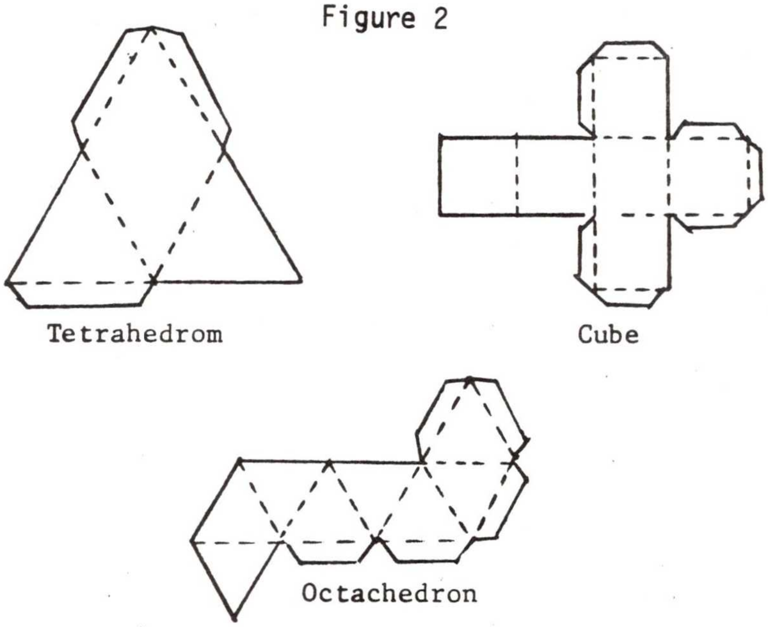
Classification

There are many criteria which young children could use to sort and classify a set of solid shapes. For example, they could sort them on the basis of the number of faces, number of edges, number of corners, kinds of faces (flat or curved), whether the objects roll easily or not, and so on. They should be encouraged to sort the same set of objects several times using a different criterion each time.

Build

Children should be given an opportunity to construct various solid shapes. They can do this in two ways. First, they can use a pattern and fold it into a solid. Some examples of patterns are given in Figure 2.

Figure 2



A second activity involving construction makes use of small sticks and modelling clay or marshmallows. Pupils can use these to make skeleton models, a few of which are illustrated in Figure 3. Sticks and rubber bands also work well, especially if larger models are desired. (See Scott, 1970.)

Figure 3

Having had these kinds of experiences, it should be easy for children to make a transition to two- and one-dimensional geometry in the upper elementary grades. For example, the faces of the solids are two-dimensional shapes. Lines can be demonstrated by extending the edges of solids, including parallel and perpendicular lines. The corners of the solids serve as illustrations of points.

SUMMARY

Geometry has been a very valuable addition to the elementary school mathematics program. However, sequencing of topics within the geometry section should be improved through some reorganization. We begin with the abstract and irrelevant when we should begin with the concrete and relevant. We should begin our program with activities revolving around three-dimensional shapes because these are the things with which children are familiar and, in fact, have already had experience. Such a reorganization is one way of contributing toward "excellence in mathematics education".

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Motivating Number Fumblers

The acquisition of arithmetical skills often depends on the students' interest and motivation. To get the students interested, tasks must be appealing. To motivate students, the problems must be relevant to the students' environment. A collection of tasks, projects, and problems to interest and motivate the student will be discussed.

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The low achievers, underachievers and reluctant learners need motivation more than anything else in mathematics. Frequently, these students are mentally capable of performing the required tasks but are not interested and, consequently, neither concentrate nor apply themselves.

For the past three years, we have been gathering materials which, we hope, will help the number fumblers to concentrate, to acquire some of the manipulative skills, and be sufficiently motivated to take part in the class activity. These materials are to be given to the students after the regular classroom presentation of the corresponding topic and the accompanying minimal drill. In place of the extended and dull pages of unmotivated problems, may we suggest the following problems and others similar to these.

PROBLEMS FOR WHICH CONCENTRATION IS REQUIRED

MAXI-COLUMNS

Place the numbers 1, 2, 3, 4, ..., and so on in column 1 or in column 2. There is only one rule: no number may be put in a column if it is the sum of any 2 other numbers already in that column.

Example

Column 1	Column 2
1	2
3	4
5	

We placed 1 in column 1, 2 in column 2, 3 in column 1. Notice that we had to put 4 in column 2, since we could not put it in column 1 (1 and 3 are already in column 1 and 1 + 3 = 4). We placed 5 in column 1. Now we cannot place 6 in either column 1 (since 1 + 5 = 6) or column 2 (since 2 + 4 = 6). With the above arrangement of numbers, we have gone from 1 to 5. Can you go higher by arranging the numbers differently? (Answer: yes, you can go to 8.)

Problem 1: How high can you go with 3 columns? (Answer: 23)

Problem 2: How high can you go with 4 columns? (Answer: 65)

In solving these problems, the student will see a pattern developing as he goes from two columns to three and four columns.

PROBLEMS FOR WHICH CONCENTRATION IS REQUIRED AND OPPORTUNITY FOR DRILL IS PROVIDED

HAPPY NUMBERS

Take a number, say 13.

Square each digit and add. $1^2 + 3^2 = 10$

Repeat the above with 10. $1^2 + 0^2 = 1$

A number for which this pattern yields finally a 1 is a happy number.

The number 13 is a happy number.

Take the number 2.

Square it and add. $2^2 = 4$

Repeat the above with 4. $4^2 = 16$

Square each digit in 16 and add. $1^2 + 6^2 = 1 + 36 = 37$

Repeat.

$3^2 + 7^2 = 9 + 49 = 58$

$5^2 + 8^2 = 25 + 64 = 89$

$8^2 + 9^2 = 64 + 81 = 145$

$1^2 + 4^2 + 5^2 = 1 + 16 + 25 = 42$

$4^2 + 2^2 = 16 + 4 = 20$

$2^2 + 0^2 = 4$

Notice that we are back to the number we started with. The pattern will cycle forever.

The number 4 is not a happy number.

Problem 1: Find all the happy numbers less than 100.

Problem 2: Let A = 1, B = 2, C = 3, D = 4, ... and so on through the alphabet. Take a name: ALLEN. Add the numbers corresponding to each letter. Thus,
ALLEN 1 + 12 + 12 + 5 + 14 = 44
Test whether or not 44 is a happy number. If 44 is a happy number, then ALLEN is a happy number name.

Problem 3: Problem 2 can be adapted to the days of the week, to the months of the year and to the years themselves.

Thus, take 1973. $1^2 + 9^2 + 7^2 + 3^2 = 1 + 81 + 49 + 9 = 140$.

Continue with 140. Is 1973 a happy number year?

PROBLEMS FOR WHICH CONCENTRATION IS REQUIRED, DRILL IS PROVIDED, AND WHICH MOTIVATE SOLUTION STRATEGIES

PATTERNS FROM THE ADDITION AND MULTIPLICATION TABLES

Using the addition table, solve the following problems:

+	0	1	2	3	4 ....
0	0	1	2	3	4 ....
1	1	2	3	4	5 ....
2	2	3	4	5	6 ....
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

1. Take 3 consecutive numbers in a row (column) - for example

2	3	4

What is the sum? Is it 3 times the middle number? (yes. 3 x 3 = 9)

2. Take 5 consecutive numbers in a row (or column)

2	3	4	5	6

What is the sum? Is it 5 times the middle number? (yes. 5 x 4 = 20)

3. Take 7, then 9 consecutive numbers in a row (or column). What is the sum? Did you use the same pattern as in (1) and (2)?

4. Take 4 consecutive numbers in a row (or column) - for example

2	3	4	5

What is the sum? Is it twice the sum of the first and last number? (yes. 2(2 + 5) = 14)

5. Take 6 consecutive numbers in a row (or column)

2	3	4	5	6	7

What is the sum? Is it 3 times the sum of the first and last number? (yes. 3(2 + 7) = 27)

6. Try the above for 8 and 10 consecutive numbers in a row (or column). What is the sum? Did you use the same pattern as in (4) and (5)?

7. Discover the rule for finding the sum of the numbers in the addition tables that appear in the following patterns.

(a)

2	3
3	4

Square 2 x 2

Rule: Number of numbers in the square (4) times the number on the dotted diagonal.

(b)

1	2	3
2	3	4
3	4	5

Square 3 x 3

Does the rule in 7(a) hold here?

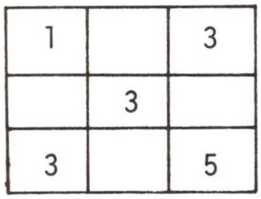
(c)

+	0	1	2	3 . . . 10
0				
1				
2				
.				
.				
.				
10				

Challenge: What is the sum of all the numbers in this addition table (do not consider the numbers here ↗ or here ↙)?

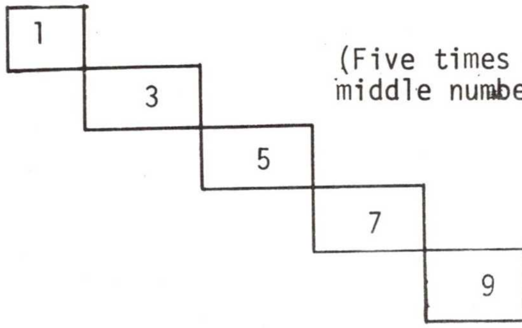
8. Find a rule for the sum of the numbers in the addition table that appear in the following patterns.

(a)



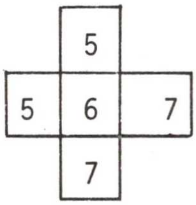
(Five times the number in the center. $5 \times 3 = 15$)

(b)



(Five times the middle number

(c)



(Five times the middle number)

9. Challenge: Find other patterns in the
   (a) addition table, and
   (b) in the multiplication table.

PROBLEMS WHICH INTER-RELATE TOPICS

HOW DO YOU FIND THE GREATEST COMMON DIVISOR OF TWO NUMBERS?

A. Intersection of sets of divisors

We illustrate this method by an example.

Find the GCD of 6 and 4.

The set of divisors of 6: $D(6) = \{ 1, 2, 3, 6 \}$

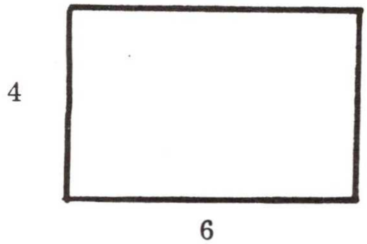
and the set of divisors of 4: $D(4) = \{ 1, 2, 4 \}$

Now $GCD(6,4) = D(6) \cap D(4) = \{ 2 \}$

B. Geometric method

Take the numbers 6 and 4.

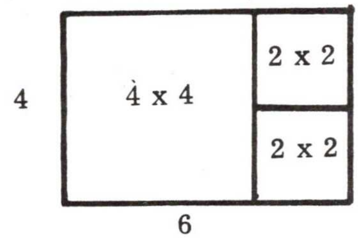
Construct a rectangle with sides 6 and 4 units.



Partition the rectangle according to the following procedure:

1. Find the largest square in the rectangle (there may be more than one).
2. Repeat step 1 for what is left in the rectangle after each partition.

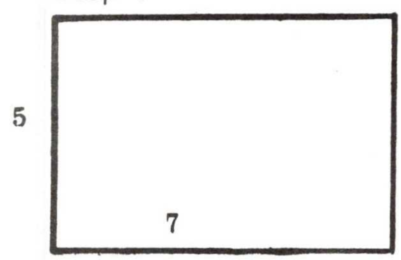
Thus,



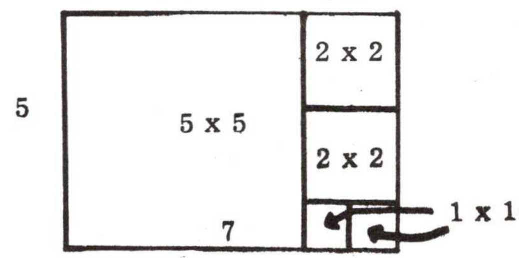
The dimension of the smallest square(s) is the greatest common divisor of 6 and 4. In the above, GCD(6,4) is 2 since the dimension of the smallest square is 2.

Example: Find the GCD(7,5)

Step 1



Partition into squares



GCD(7,5) is 1 since the dimension of the smallest square is 1.

C. Division method

Example: Find the GCD(7,5)

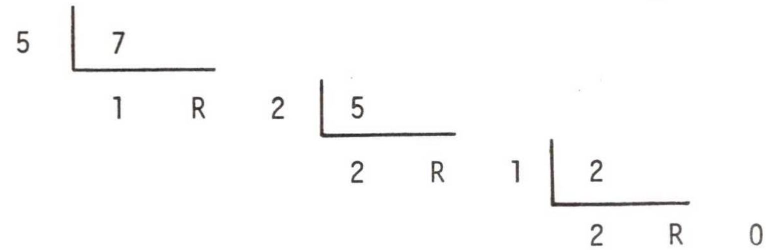
1. Divide the smaller number into the larger number according to the following procedure

   5 | 7 — 1 R 2 — 5 goes into 7 with a remainder of 2

2. Now divide the remainder (2) into the previous divisor (5):

   5 | 7 — 1 R 2 | 5 — 2 R 1

3. Repeat the step (2)



The divisor (1) which leaves a remainder zero (0) is the GCD of the two numbers 5 and 7.

Problem 1: Use the above procedure to find the GCD of

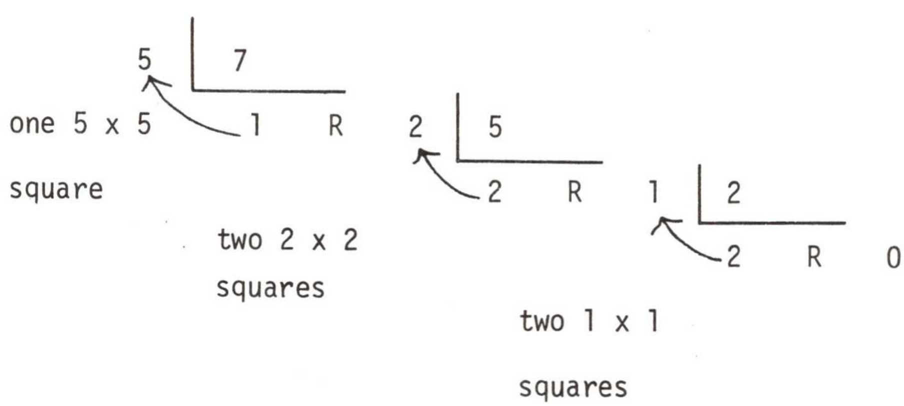
(a) 8 and 13

(b) 21 and 34

(c) 35 and 55

Note: The above numbers are consecutive numbers in the Fibonacci sequence. These numbers give the largest chain of steps in the division algorithm above.

D. Compare the division algorithm in (c) with the rectangle partition procedure in (b).



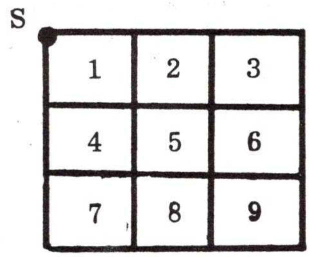
The division algorithm gives the partition into squares of the rectangle consisting of sides 5 and 7.

PROBLEMS FOR MODULAR UNITS AND PROJECTS

A student can enjoy working on a project if it is challenging and interesting. The following project is best done by using inch squares rather than trying to show the squares on graph paper.

PROJECT

1. Make a 3 x 3 square out of 1 x 1 unit squares.

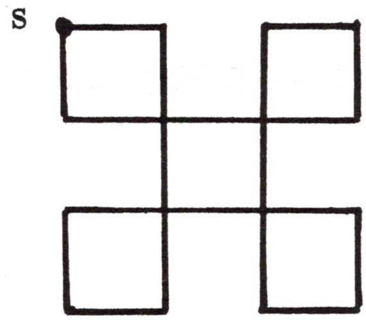


The perimeter is 12 units.

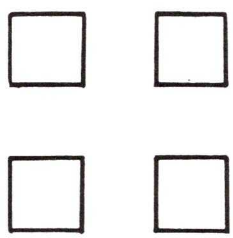
Now remove squares to make the perimeter larger. The rule is:

you must remove squares in such a way that starting at any point S you can trace a path back to S.

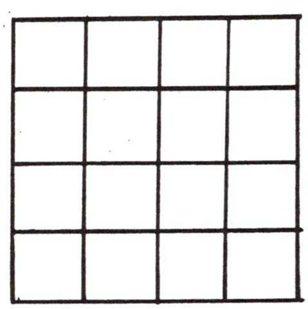
The maximum perimeter for a 3 x 3 square is made by removing squares 2, 4, 6, and 8 to get



The perimeter here is 20 units. Notice that starting at any point S of the above arrangement, we can trace the perimeter and return to S. Figures such as the following are excluded.

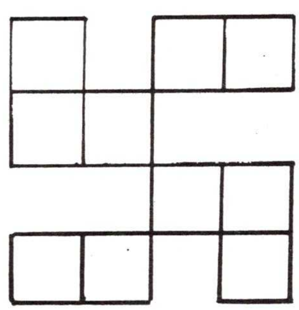


2.



4 x 4 square

Perimeter 16 units



Maximum perimeter

28 units

Problem 1: Take 5 x 5, 6 x 6, 7 x 7, 8 x 8 squares.
Find maximum perimeter.

Problem 2: Can you find formula(s) that will give you the answer for the maximum perimeter for an n x n square where n is an element of 1, 2, 3, 4, ...

Challenge: Apply the above to unit cubes and find arrangement of n x n x n unit cubes with the maximum surface area.

What About Drill?

The difference between developmental teaching and drill will be discussed, along with the necessity for drill, when and how drill should take place, and principles to follow in administering drill.

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Before I begin to talk about drill, we should look at the purpose of junior high school mathematics. For too long it has been remedial mathematics, getting ready for the next step. It should be an extension of mathematics already learned and a preliminary exploration in the broad area of mathematics. Students of this age are rebellious and do not like mathematics presented as a finished product. Their energy should be directed toward exploration and questioning in mathematics. They should be encouraged and challenged to think about such things as why we divide fractions the way we do, why a negative times a negative is a positive, why we sometimes multiply exponents and at other times add them, and so on. After skills, concepts or principles have been developed, it is essential to carefully develop and plan a program of recurring experiences which will assure that these skills, concepts and principles are not forgotten. I refer to this as drill.

Arlington County, Virginia, has one of the highest levels of adult education achievement of any county in the United States. Teachers there say that their major problem is pupils coming into the ninth and tenth grade not understanding or knowing how to do the basic operations. That is why drill is important.

WHAT ABOUT RESEARCH?

What does research have to say about drill? I would like to review three areas of this research. The first one is the amount of time to be devoted to drill and practice. This research does not differ much from 1914 until now. Researchers in the early part of this century found that periods of about 20 minutes were most effective. Meddleton (1956) cited strong evidence to show that systematic short review work produces higher achievement. In a more recent study (Shipp & Deer, 1960) it was revealed that less than 50 percent of class time should be spent on practice activities since increased achievement resulted when up to 75 percent of the time was spent on developmental activities. This finding has been supported by three or four different pieces of research since that time. So roughly about 20-25 percent of our time, according to the research, ought to be spent on drill practices.

What type of drill produces most effective results? Drill should be constructed to fit a particular purpose. Functional experience is important. Distributive practice is more helpful than concentrated practice. Children should use practice materials on their own difficulty level and progress at their own rate. Varying the type of drill and the use of "frames" were found to be effective by Sandefur (1966).

Where in the sequence of learning mathematics is drill most effective? According to Brownell and Chazal (1935), the time for drill is after effective teaching. This has been generally supported and accepted.

WHAT ABOUT DEVELOPMENTAL TEACHING?

Many teachers use only one instructional strategy, namely the one suggested in the textbook. This is not good. A teacher should acquire alternate instructional strategies. How? - by generating them himself or by seeking procedures already devised by someone else. One way in which this can be done is by looking at other textbooks. Rather than using one textbook for 32 students, use five copies of seven different texts for children. You could work out activity sheets for the various levels of pupils in the class referring them to textbooks, tapes, and so on. Tapes are useful for children who have difficulty in reading the texts.

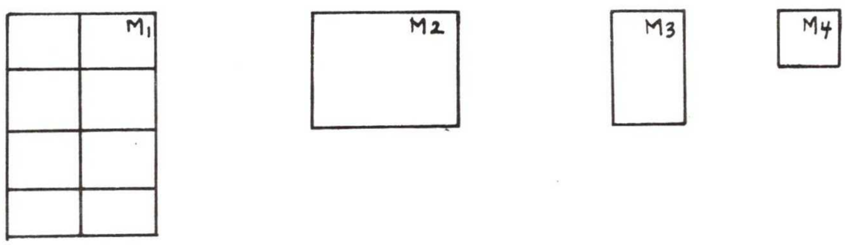
Individualized instruction can take place with 30 students, or with eight, or only one. We must not necessarily have a one-to-one ratio in all individualized instruction, but we should attempt it.

We also have to look at the prerequisites for learning the concepts we try to teach at the junior high level. Often we do not do this. We need to find out if our pupils have the proper background; if they do not, we need to find some way of providing for it in our strategy.

One strategy that has helped me is making copies of students' worksheets and test papers, particularly of those in which many mistakes were made. Last year I placed copy machines in several classrooms. Teachers made copies of the pupils' work sheets so that I could look at their error patterns. If you keep a record of the error patterns your pupils make, you might be able to devise better developmental teaching. Some research findings indicate that in Grade VII basic properties of addition were not clearly understood, the distributive property apparently being the most difficult. If there is one property that we should get across in pre-high school mathematics, it is the distributive property. It can be of tremendous value. It has been found that only three of ten fraction principles were known in Grade V, while four were known in Grade VI. This means that we must still do a lot of developmental teaching and drill in the basic operations in whole and rational numbers at the junior high school level.

The type of developmental learning I would like to see take place with all students is that in which the student has his own package of materials such as the package partially illustrated in Figure 1. In this package are colored

Figure 1



acetate sheets labeled M_2, M_3 and M_4 (M_2 is ½ the size of M_1, M_3 is ½ the size of M_2, M_4 is ½ the size of M_3) and a plain sheet of paper labeled M_1 that has been divided into eight equal sections. The activity sheet will ask the students to use these sheets to find answers to the following problems: ½ times 1; ½ times ½; ½ times ¼. Just let them play around with these for a while. The next thing I ask them to do is to find the answer to these three problems. Soon many of the students come up with the correct answers by proper overlapping of the sheets. In drill work it is important that we reinforce learning immediately. (I prefer giving back papers to students immediately to giving them a lot of things to do.)

In my activity sheets I like to move from the complex to the simple so that the good student will not waste time reading all of the other questions which I put down for the weaker student. What about the bright student? Every time I make an activity sheet, I develop three. The first one is a basic activity for all students who are going to be functional citizens, the second is for the average students, and the third is for the accelerated students in the class. The wording on these is quite different.

If teaching takes place so that concepts are meaningful to pupils, then the amount of drill needed might be reduced.

WHAT ABOUT DRILL?

Drill should be relevant and well motivated. For example, I have used some cardboard index paper that simulates a sleeve and piston of a gasoline engine. The student is told that all he has to do is roll the piece of index paper until the four holes of the paper match. He thinks of this as a sleeve for the piston of an engine. We have a piston that can be made to go into the sleeve. Then we proceed to develop exercises related to this. Some of the sleeves have a radius of one inch so that most of the pupils can work with it. I give the better students actually the specifications of an automobile and have them work with the real thing. With this, they can see the need for fractions and decimals. We can give them drill this way without their actually thinking they are being drilled.

Figure 2 illustrates a gimmick I like which camouflages drill. This is necessary for some students. It can be used at the grade three level or at the high school level. Figure 3 shows how it can be used at the high school level.

Figure 2

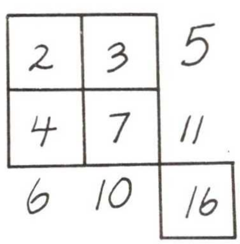
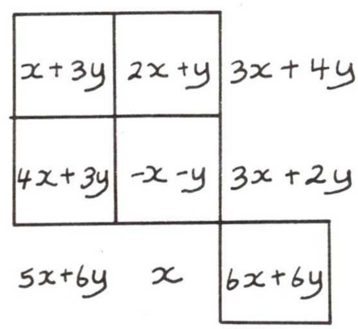


Figure 3



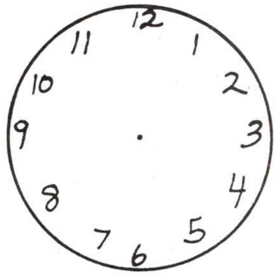
In junior high school, we can use fractions or decimals instead. We can also leave out some of the factors, put in the answer, and let the student find the other factors. You can also use this with multiplication. Your better students might want to prove that this does work.

Another source of good exercises is a world book of records. There are many highly motivational and relevant facts in such a book. By using such things, you can do a tremendous amount of drill work which is not a repetition of things students have had in the past.

Similarly, *Popular Science* is a source of an amazing number of exercises at various interest levels. Let the students make up their own exercises and pass them around. I set up four or five stations, order the material according to difficulty, and have the students go to these tables according to their ability.

I have also made good use of catalogues. I have a catalogue of 100,000 car parts which is highly motivational and extremely relevant, particularly to boys who are interested in automobiles. You might also collect mathematical puzzles out of the magazines to which you have access. One of the best I have seen relates to the watch face.

Figure 4



Can you divide the watch face with 2 **straight** lines so that the sums of the numbers in each part are equal?

Can you divide the watch face into 6 parts so that each part contains 2 numbers and the 6 sums of 2 numbers are equal?

This gives drill in logical thinking and in addition. Sometimes the child who, in your opinion, cannot add very well will come with the answer first.

Research has shown that games and puzzles are great devices to motivate students. You are treading on thin ice when you try to make them competitive. This discourages some of your weaker students and also some of your better students very quickly.

You should let the students know the reason you are doing things. You should tell them in clear and concise language what you want them to learn. The student ought to have some way of evaluating himself so that he knows whether or not he has achieved the things you wanted him to achieve.

How much drill? I say enough. The amount of drill will vary according to students' difficulty with mathematics. Do not give them more than they need. Do not give them problems too difficult for them. One of the worst things we can do is simply assign problems on a certain page to do for homework to all students in the class. At best this will hit a small percentage of the class. Some will be beyond this, others will not be ready. Give them enough drill to satisfy their needs but not too much. This can be determined by giving them self-evaluation sheets, letting them decide when they can do certain examples.

What should be emphasized? We should never forget the aesthetic value of mathematics, but to many pupils the division of fractions is never going to be very useful. For that matter, neither is working with 11's, 17's or 43's. Table 1 contains the results of a survey conducted in the business world.

TABLE 1

FREQUENCY OF USE OF SOME FRACTIONS IN THE BUSINESS WORLD*

Fraction	Percent frequency of use
Halves	31
Fourths	36
Eighths	20
Thirty-seconds	9
Sixteenths	1
All others	3

**Mathematics Newsletter*, Department of Public Instruction, Raleigh, N.C., Fall 1973.

A survey of Grade VII mathematics books revealed that the fractions ½, ¼, 1/8, 1/16, and 1/32 represented about 15 percent of the fractions considered in these books, all others about 85 percent. We ought to take a look at what we are teaching our students.

We ought to look at the practical uses of mathematics. The other night, I got a check at a restaurant and knew that it must have been added incorrectly because I had estimated how much my bill was going to be. We need to teach pupils to estimate because it is a real help as far as drill is concerned. How much of us actually figure out if our banker has charged us the correct interest when we borrow money? Yet a fair amount of our time is spent around Grade VIII trying to get students to do complicated problems involving interest. This is something they will probably not need again with the invention of the pocket electronic calculator which is going to make a difference in the amount of mathematics needed.

Emphasis should be placed on correctness of response rather than speed. We give drill so that a correct pattern of behavior can be developed. A pupil who practises incorrect behavior, even if he gets his work done quickly, is in trouble. We ought to give students immediate feedback by giving them certain check points. For example, every third problem could have the answer worked out or a way provided for checking.

As much as possible, drill should be self-diagnostic. As the pupil works on the assigned exercises, he should become aware of the behavior which he has not learned. This would imply that he can go back and find answers quickly.

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The Last Twenty-Five Years -- What Have We Learned?

During the past 25 years, much significant work has been done to improve the mathematics programs of junior high schools. This discussion will describe how some of the outcomes of this work can give direction to the present and future efforts to continue improving programs.

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It was about 25 years ago that the movement to weigh the values in mathematics seriously began in this country. It was then that the works of promise for the teaching of the subject began to appear. It was only a little later that the textbooks first attacked successfully the unreal topics and problems of the past and set a standard for the real ones for the present and the future. Since the movement began, there has been a veritable revolution in the subject matter, in its arrangement, in the spirit with which it is presented and in the textbooks in which the work is set forth.

To give you some feeling for the way things go in mathematics, the above statement was made in 1923 by Eugene Smith who was reporting on the progress of teaching mathematics in the 25 years prior to 1923.

Many good things have been happening in mathematics, and our modern programs are better because of them, but we ought to look at how they were done so that we can learn something for the future.

WHAT HAS HAPPENED?

In the late 1940s, there was considerable conflict about what really ought to be done in mathematics. People like Brownell and others were talking about teaching mathematics for meaning so that children would understand it and have a rationale for what they did. Then came the war, which proved conclusively that America was a mathematically illiterate nation. For example, over 1,600 out of 2,000 officer candidates from colleges across America failed a mathematics examination which covered material through Grade VIII or the early part of Grade IX. These people came through a mathematics program in which the skills were the dominant thrust. They performed well in order to get to college, but after they had been away from the skills for a while, they forgot much of what they had learned. When candidates for electronics training in the navy were writing a test which included problems dealing with division of fractions, you could depend on someone to say, "If you tell me which one I turn upside down, I can do this problem." They learned the skills, but they never thought very much about their meaning. So the conflict in the late 1940s was between understanding and being able to perform certain competencies such as the 29 competencies set forth by the National Council of Teachers of Mathematics (1945) which every intellectual citizen ought to know or be able to perform.

In the 1950s and early 1960s, new programs came along to replace the old ones. They were not designed to enhance the old or replace part of the old, but to completely take out the old ones and put in the new ones in one swoop.

These programs got their impetus in America when Sputnik flew because America became frightened by the notion that it was not as technically competent as the Russians. This situation was viewed as a national emergency, and the federal congress allocated all kinds of funds for the design of these new programs.

One consequence of the new programs was massive teacher training programs. The National Science Foundation paid people to come in and learn about the new mathematics programs. Unfortunately, teachers think that children ought to know exactly the same about mathematics as they do. If teachers know it, everybody ought to know it; if teachers don't know it, it probably shouldn't be learned by anybody. The teachers who came to these training courses learned the new mathematics but went back and put it into the schools the way they had learned it. This created problems. The teacher training was only moderately successful at best.

In the 1960s and early 1970s we began to move away from an emphasis on technology and science. There is now discontent with the programs which were designed to meet the national emergency of the late 1950s. Evidence of discontent is widespread. Journals such as *Wall Street Journal*, *Time* and other similar magazines have had articles in the last year about the failure of mathematics programs.

Many promises were made with the new mathematics programs. For example, we were promised that if you really stressed the concepts, the skills would take care of themselves. So the emphasis was on content. The discontent started when the test results began to come in. The old programs were not sufficient, because results showed that children could not do certain kinds of things. So new programs came into being, and then the old tests which measured only skills were administered again. Children went down on computation, and the argument was that the tests do not measure the things we are teaching.

One of the problems was that the people who are paying for the programs were not told what the goals of the new mathematics programs were. The new mathematics programs were never designed to make children better computers but to help them become better at understanding mathematics and hopefully better at problem-solving. Many people think that if children cannot compute, the program must be a failure. Consequently, new mathematics is in disrepute in the United States today. This illustrates the point that if you are doing something,you better know why you are doing it and what you can expect from it.

Some discontent is caused by the fact that new programs make it possible for students who learn mathematics well to learn more while the kind of students who were marginal students in the early 1950s are learning less. New mathematics programs are more appropriate for the brighter students than for others. So the enrollment in high school mathematics classes in the United States is decreasing substantially. Why? If we have a good mathematics program, one that accommodates all pupils, the number of pupils who enroll will increase. A corollary of that is an apparent lack of concern for the mathematical literacy of the nation. I have very little regard for the mathematics programs which makes it possible for fewer and fewer children to learn more and more because the function of the school with respect to mathematics should be to make as many students mathematically literate as it possibly can. This should be the major criterion in evaluating a mathematics program.

Discontent has also arisen out of lack of attention to application and problem-solving. The idea that if the students get the concepts, they will be able to solve the problems, has proven false. The National Council of Teachers of Mathematics and the Mathematical Association of America have a joint proposal before the National Science Foundation to prepare a source book for junior and senior high schools on applications in mathematics. Its content is to be compatible with all programs so that every teacher can supplement his program with the source book.

Another reason for discontent is that mathematics educators have not provided the people, who are paying the bill, the kind of information they need in order to make judgments about whether the program is successful in achieving its objectives. Beckmann (1970) was one of the fortunate people who took the 29 competencies when they first came out and designed a test to see how well students in Nebraska performed. In 1965-66 he gave the test again. He found that students of the modern mathematics programs were learning the 29 competencies better than those of mathematics programs 15 years earlier. In addition, the modern math students learned more mathematical ideas. There are many studies such as this, but we don't hear much about them because we haven't made a very good effort of disseminating them.

WHAT HAVE WE LEARNED?

One of the things we have learned from the events of the last 25 years (I think we knew most of them in the first place) is that it is almost always impossible to completely take out one program and put in a completely new one with a new set of objectives and with new emphasis. A program has to be sold. Selling a new mathematics program is just like selling a refrigerator. The first thing you do is to establish a need for it. The people who buy it - *not* the people who make - have to believe that need exists. Once they recognize the need, they have to be shown that the product will satisfy their need for it. They also have to be shown that it is economically feasible for them to place it into their house, that they can afford to buy it and cannot afford *not* to buy it. The same applies to selling a mathematics program.

Very few teachers had any input into the new programs, very few teachers were ever consulted, very few teachers ever really saw the need for new programs. Regardless of what textbooks people were given, everything went on very much as usual when they closed the door to the classroom. We talked about how important it was for children to understand and get a feel for the spirit of mathematics, but when you collected the tests that were given, you found that those ideas were not tested. In fact, tested was how well they could get the answer or compute. Students soon learned that they didn't have to understand it in order to get good grades, they just had to deliver the solution to the equation or perform the computation.

Another thing we learned is that any national emergency will pass, one way or another, but a product designed to meet some national emergency will outlive the emergency. This results in discontent which puts pressure on teachers to report to the recommendations of incompetents. What has happened is that all kinds of panaceas are now coming across our educational threshold which are supposed to eliminate all the difficulties and make children mathematically literate. Every one of these proposals has enough truth in it to make it salable to some people.

Individualized instruction is an example of one of these panaceas. There are many definitions of individualized instruction, but it often degenerates into something where students are reacting to materials only, to materials on a machine, or some other way on a one-to-one basis. It is impractical to select a myth such as individualized instruction and make it a blanket which covers the whole mathematics program. The best strategy is to decide first what it is that we ought to be teaching, then decide which of those topics can best be taught by individualized instruction, which of those are best handled by group instruction and which ones can best be taught with just a text and workbook approach. There is no single method I would like to recommend as a blanket to impose over the whole mathematics program. To me, individualized instruction means making arrangements for every child to get his best shot at learning whether that is prescribed by the teacher or chosen by the student himself.

The next time someone wants you to do the whole program by individualized instruction, pose the following hypothetical situation to him. The Province of Alberta made a ruling that every junior high school teacher, in order to maintain his credentials beyond the end of the current school year, must pass an examination on the undergraduate mathematics program. The government also said that there are two ways you can prepare to pass that examination. One is that you can come to a class with a competent instructor and discuss the content on which you were to be examined. You can work with the instructor, do the problems in the usual way, discuss, and work by yourself when it is appropriate. The other way is to pick up a package of materials, go home, study it on your own, and take the examination. Which method would you choose? There are probably some who would learn individually, but that is an art. Many people need the interchange with an instructor. The place where you hone ideas is in a good discussion with somebody who understands them well. If individualized instruction means that discussion is at a minimum and workbook work is at a maximum, then I doubt if many students will be successful learners. The function of the school is to make students successful learners.

Very little research is being done on the characteristics of topics which can best be learned individually. Some skills can be learned individually, but there are many things about proof, for example, that students can learn better with some help.

Another panacea is accountability. Usually accountability means that the teacher is going to be held accountable for the achievement of children. I believe in accountability, but it has to apply to the whole system, not just one segment of it. Accountability ought to be with parents to send children to school who are willing and ready to do the work required to achieve the objectives. The superintendent and principals ought to be held accountable for providing all the necessary materials to achieve the objectives in the most efficient way.

Out of accountability came behavioral objectives which were intended to lay out the whole thing so that people would know what they had achieved. I have no objections to behavioral objectives. I was writing them in 1942, but I wasn't writing them about ideas - I was writing them for skills. In America, we have people next to each other writing behavioral objectives for the same topic. Everybody is inventing the wheel.

Teachers should not be required to write behavioral objectives because this does not make the most productive use of their time. However, every teacher ought to know specifically what objectives he is trying to achieve in any lesson. In Michigan, we asked the teachers not to write the objectives (because so much time is wasted in arguing over the correct verb) but to write down what students should learn in terms of exercises, problems, attitudes, and so on. These exercises were used as the criterion for objectives, and we hired a technician to write objectives to fit the exercises. You have to have a criterion in mind, and teachers can write the criterion well. This procedure takes less time.

There should be at least two lists of objectives. One list related to skills and certain elgorithms could be stated behaviorally because they can be measured that way. Another list would contain goals related to such things as proof and problem-solving. We might be able to write objectives in these areas, but when I have them all checked off I still cannot certify that a student is a problem-solver or that he can do proof. Teachers ought to make it clear what, in their opinion, it is that can be stated and measured behaviorally and what it is that cannot. The whole program need not be defined by a set of behavioral objectives.

We need more data on children's problem-solving abilities. It seems that children who can solve problems in Grade III can still solve them in Grade VII and students who cannot solve them in Grade III still cannot solve them in Grade VII. We need to collect more observations over time which will enable us to make statements such as, for example, "John is better in problem-solving now than he was last year for these reasons". But we do not assign a number to it. Teachers who go to school four or five years to learn how to do business with children should be able to make subjective judgments about their achievement. To do this, they have to know the goals, but they ought to know what they can state behaviorally and demonstrate achievement in that area and what it is that they can only *say* students are making progress in.

Another subject falling into the category of panaceas is laboratories. I have regard for laboratories in mathematics, and yet, I have been in places where students were very good in laboratories (setting up experiments and following directions) but didn't learn much mathematics. Certain things in mathematics can be learned in a laboratory setting, but I know schools in which the entire program is built around the laboratory. Many things cannot be taught that way.

What we also see happening today is that publishers, as a result of pressure partially from curriculum committees, are producing textbooks with much less reading in them than found in earlier books. The new texts are much more skill-oriented. When you take reading out of the textbook, you do two things: (1) you destroy the students' chance to hear the ideas discussed on the page by somebody who ought to understand the ideas; (2) and more important, you deny the student the opportunity to learn to read things technically. Adults have a hard time reading technical things because it is not taught in the schools. Most mathematics teachers use the textbook as a problem source. The developmental work is usually done at the board. The students read only the exercises and not the material itself. It is imperative that students learn to read something technical, because one of the major goals of mathematics should be to make students independent learners. They will not always have someone laying it out for them on a chalkboard.

We still don't know at the grass roots what the impact of the new mathematics programs has been. We ought to pressure organizations like the Mathematics Council, The Alberta Teachers' Association, and the National Council of Teachers of Mathematics to begin to generate ways of determining the impact of new programs. They need to make the objectives and goals of the new programs known so that we can determine what we have achieved in terms of the predetermined goals.

We should establish a large item bank from which people could choose, at random, items to determine the achievement of certain objectives. If a single test is to be given at the end of the year, I teach that test because my reputation depends on it. But if some items are chosen at random from a bank of, say, 1000 items, I am a little more general in my teaching.

After a student has completed a topic or a program, we should be able to say that he can demonstrate performance of the skills required in mathematics, that he is making progress in problem-solving according to some criteria set up by us. To say that a student is at the 98th percentile doesn't give me any useful information. In other words, we must answer the question "how many can" rather than establishing percentile ranks.

Performance-based teacher education is being tried in America today. There are some things which teachers ought to know, but we may not agree on what these things are. One skill which teachers ought to be able to perform is to conduct a class in the discovery mode for some competencies and to do a good exposition of other topics because this is how they can be handled best. Many teachers could certify that they have accomplished all of the performance tasks I have seen and still be poor teachers. Others may find it impossible to do some of those things, and yet they may be good teachers.

Many universities in the United States have been slow in developing performance-based teacher education. In some of these situations, the state is doing it for them. The lesson is that we should be aggressive and exert our influence before the time comes when we have no influence to exert.

IMPLICATIONS FOR MATHEMATICS TEACHERS

We need to be firm on some matters about any program or product to be implemented in our district. First, we should request that goals be reasonably well specified. We should also request that evaluation be available and that the evaluation and the goals have a good match with what we think is important in our school system. If we were firm on these things, we would avoid the situations I have seen where the same textbook is sold in one district for the good students and in another district for the slow learners.

We have a responsibility to experiment with new programs, but we don't involve the whole school system. No experimental program should be carried out unless it has first been tested with the smallest population possible to apply the results to a larger population. It is a catastrophe to introduce the whole school to a new program without testing it first to see if it is going to work.

Another matter I would insist on is that in any new program, equal attention to method and content is given. We were caught in a trap in 1950s and 1960s. with too much emphasis on content. A knowledge of content is necessary but not sufficient for a competent teacher. A new program should make it possible for the teacher to use the material in a variety of ways.

Teachers ought to be responsible for long- and short-range planning. Whenever they don't carry out their responsibilities in this area, somebody else will do it for them. It will be done at the state or provincial level, or the university people will do it for the teachers.

An organization such as the Mathematics Council ought to project what the needs and major goals in mathematics are. This should be done in realistic terms, not to meet some national emergency or to react to some criticism that is in the forefront.

Teachers should insist on mathematical literacy for the greatest number of people. Every year a student ought to have a mathematics program available to him in which he has at least a 0.8 probability of success if he is willing to do a reasonable amount of work. Teachers who like to teach the mathematics instead of working with children to make them achievers bother me. The school is measured by how many literate people it turns out and by how good they are. The teachers' organization should require that all colleges, universities, and other organizations help plan and carry out that mission.

The amount of a mathematics teacher's knowledge has little impact on the achievement of students. The attitude of the teacher toward mathematics has little impact on the attitude of students. In spite of all this, it turns out that the teacher is still the most important element.

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Student Needs and Subject Requirements -- Can One be Met Without Sacrificing the Other?

The real issue in the '70s is to improve teaching strategies to effectively use the materials presently available - and adapt them to meet the needs and interests of the students. Are we equal to the challenge?

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When I accepted the invitation to speak at this session of your conference, I deliberated on what I could discuss with you that would be worth your time, because I feel a deep sense of responsibility for providing a few ideas which will repay you for your attendance at this meeting. I thought about an open letter from Dan Valentine, a columnist for a Salt Lake City newspaper, concerning his son starting school. In the letter he lists some basic and important matters to be considered by every teacher at every level. The letter reads something like this:

Dear World:

My young son starts to school today. ... It's all going to be sort of strange and new to him for a while, and I wish you would sort of treat him gently.

You see, up to now he's been king of the roost... He's been boss of the backyard. His mother has always been near to soothe his wounds and repair his feelings.

But now things are going to be different.

This morning, he's going to walk down the front steps, wave his hand, and start out on the great adventure... It is an adventure that might take him across continents, across oceans, ... It's an adventure that will probably include wars and tragedy and sorrow... To live his life in the world in which he has to live will require faith and love and courage.

So, World, I wish you would sort of look after him... Take him by the hand and teach him things he will have to know.

But do it gently, if you can.

He will have to learn, I know, that all men are not just, that all men are not true.

But teach him also that for every scoundrel there is a hero, that for every crooked politician there is a great and dedicated leader. Teach him that for every enemy, there is a friend.

It will take time, World, I know. But teach him, if you can, that a nickel earned is of far more value than a dollar found. Teach him to learn to lose so he'll enjoy winning that much more...

Steer him away from envy if you can ... and teach him the secret of quiet laughter.

Let him learn early that bullies are the easiest people to whip in the schoolyard... Teach him, if you can, the wonders of books. But also let him ponder the eternal mystery of birds on the wing, and bees in the sun, and flowers on a green hill.

In school, World, teach him it is far more honorable to fail than to cheat. Teach him to have faith in his own ideas, even if everyone says they are wrong...

Teach him to be gentle with gentle people and tough with tough people.

Try to give my son the strength not to follow the crowd when everyone is getting on the bandwagon. ... Teach him to listen to all men - but teach him also to filter all he hears on a screen of truth and take just the good that siphons through.

Teach him, if you can, how to laugh when he's sad. Teach him there is no shame in tears... Teach him there can be glory in failure and despair in success.

Teach him to scoff at cynics and to beware of too much sweetness... Teach him to sell his brains and brawn to the highest bidder but never to put a price tag on his heart and soul.

Teach him how to close his ears to a howling mob ... and to stand and fight if he thinks he's right.

Teach him there are times when a man must gamble... And there are times when a man must pass the dice.

Treat him gently, World, if you can. But don't coddle him... Because only the test of fire makes fine steel. Let him have the courage to be impatient. Let him have the patience to be brave. . .

Let him be no man's man... Teach him always to have sublime faith in himself.

Because then he will always have sublime faith in mankind.

This is quite an order, World, but see what you can do... He's such a nice little fellow, my son!

Dan Valentine

As you reflect on that letter, you find any number of challenges which we could discuss in the next few minutes, but there is one that I would like to review with you: "Take him by the hand and teach him things he will have to know." There is a challenge!

What mathematics will your students have to know in the next year - five years - ten years? What will the world be like in 1985? To try to get some feeling for the magnitude of the problem, let's take a short look at what has happened in the last 15 years.

FIFTEEN YEARS AGO:

1. The U.S.A. placed its first satellite in orbit. (Russia had startled the world with its Sputnik one year earlier.)
2. Jets were beginning to cross the oceans.
3. There were no supersonic commercial jet liners, communications satellites, and space agency.
4. The so-called "New Math" was about to make its appearance.

In this 15-year period, 12 men have trod the moon; we have just witnessed the conclusion of the second extended sky lab experiment where three men spent 59 days in an orbiting space laboratory. All in this world were able to watch the living color transmission from China by satellite as President Nixon made his recent trip to that country. The supersonic jet transport - the Concord - has made its first flight to the American continent.

During this 15-year period, we have gone from a situation of surplus energy - especially electricity - to a point where we face a very serious energy crisis.

FIFTEEN YEARS FROM NOW:

What dramatic changes will we see in the next 15 years? What will the world be like? It is really quite difficult to predict. However, here are a few suggestions that will have a significant impact on all of us.

1. Space communications will play an increasingly important role in education. In 1974, an experiment will be conducted in the Rocky Mountain States using the ATS-F satellite. This project will bring some unique educational opportunities into remote areas of the states. There will be direct two-way communications between the remote areas and the project center via the satellite as well as TV broadcasts.

   (a) We are moving into the computer age. Schools need access to large-scale digital computers - either on a local or regional basis. Space communications can play an important role in making such a computer system available.

   (b) Libraries and universities could be tied together through a communications network. With the "knowledge explosion" and great mass of printed materials, it will not be feasible for each library to maintain a comprehensive collection of pertinent materials in all areas. With the use of micrographics, most of the books in the world could be made available to schools and universities.

2. Weather satellities are in operation now, and it is likely that we will have a global weather network for long-range forecasts as well as disasters and emergencies, and even weather management is a possibility.

3. The development of the resources on earth and global crop management and prediction have far-reaching implications to us all. Orbiting satellites offer real hope in achieving major advancements in these areas.

4. Space sensors may have an impact on the diagnostic problems of the sick or could be a valuable resource in emergency medical care. For example, it is estimated that 50 percent of those who die of a coronary could be saved if they had been transported to an intensive care unit in time and the necessary information and procedures received.

5. Skylab experiments are extremely interesting and offer the potential for a variety of breakthroughs such as improved vaccine production techniques due to zero gravity effects.

* * *

This brings us to three questions:

1. *WHAT EFFECT WILL THIS HAVE ON OUR MATHEMATICS EDUCATION PROGRAM?*
2. *WHAT MATHEMATICS SHOULD WE TEACH?*
3. *WHAT IS THE BEST PROGRAM FOR TEACHING THE NECESSARY MATHEMATICS?*

We can be sure that mathematics will continue to play a very important role in our society; therefore, we should provide a program that will help every student to develop the necessary mathematical skills for whatever occupation he chooses.

The question of what mathematics should be taught is much more difficult, and I'm not sure that the answer for the schools in Edmonton is the same as the answer for those in Lethbridge or in Whitefish or Idaho Falls.

OBJECTIVES

Just as each new building requires a set of architectural plans reflecting the local physical and terrain conditions, each district should determine the goals and objectives important for their students. Should your maximum efforts be directed toward a program of college preparatory mathematics classes, or does the large majority of your students obtain employment on farms, in stores or in jobs where the basic skills and shortcuts in computing, such as the number of board feet in a stack or the number of acres in a plot of ground, are the most important? It is essential that the teachers in your schools develop the objectives for their classes, but they should use available resources so that they don't have to "plow the same ground" as other groups have already completed. There are several sources available. It is much easier to adapt a list of objectives to your own needs than to start from the beginning.

The School Mathematics Study Group Newsletter No. 38, dated August 1972, contains some very pertinent information and references concerning educational objectives for mathematics. The S.M.S.G. Advisory Board has listed seven principles which they feel are important for those who are developing objectives:

1. Statements of objectives should be hortatory. They should be taken seriously by teachers, curriculum workers, and textbook writers as important and realistic guidelines. They should *not* be expressions of wishful thinking.

2. On the other hand, statements of objectives should be taken as floors, not ceilings. If a teacher or a school can go beyond stated objectives, so much the better.

3. If the statement of a particular objective is to be taken seriously, then the purpose of the objective has to be made clear. Furthermore, a serious, relevant objective must be so clearly characterized as such as to be easily distinguishable from a personal whim.

4. If statement objectives are to be taken seriously, then the objectives must be clearly verifiable and feasible. It is not enough to know that an objective has not been shown to be infeasible. Before it should be advocated, it should have been positively shown to be feasible (and verifiable).

5. To be consonant with the above, we believe that all statements of mathematics educational objectives should be put in terms of student behavior. (The one exception is that we advocate a particular pedagogical objective: *Teach understanding of a mathematical process before developing skill in the process.* We believe that there is enough empirical evidence in favor of this to make it a realistic objective.)

6. Also, to be in conformance with point 4., we advocate, at present, no affective objectives. There is no evidence available to show that attitudes toward mathematics can be manipulated; consequently, such objectives are not, at present, feasible.

7. None of the above should be taken as suggesting that we ignore goals which are, at the moment, not feasible or not verifiable. Indeed, such goals indicate the most important areas in which to concentrate our future research efforts.

MATHEMATICS PROGRAMS

Once the objectives have been developed, the appropriate instructional materials and media can be selected. There is a variety of good mathematics programs available. There is also the technology and media available to teach practically any concept we desire. At a recent educational media conference I observed a demonstration involving six slide projectors and two 16 mm movie projectors operating in a synchronated display. That was very effective. Another demonstration involved the use of 12 slide projectors operating simultaneously. It is not feasible to have systems like that in too many schools, but it is reasonable to expect the teachers to be proficient with the available equipment and materials and to use them effectively to accomplish the desired goals and objectives.

There are other problems related to materials and hardware which help to prevent the fullest blossoming of educational technology. These were pointed out by Congressman Orval Hansen at the Annual Meeting of the Association for Educational Communications and Technology in Las Vegas:

1. Educators must learn to describe their needs with greater precision.

2. Educators face a serious information gap when making purchases because of inadequate product assessment and consumer information.

3. Rapid technological changes make equipment obsolete too quickly.

These problems are closely related to the need for development of local goals and objectives and assessing the needs of the students. When that has been accomplished, it is much easier to evaluate the materials and media available. The fact that equipment becomes obsolete so rapidly emphasizes the importance of the judicious selection of appropriate materials to help teachers teach the stated objectives. Finances are not adequate to be able to afford the luxury of purchasing some media or materials solely because they are attractive or have limited applications.

After the educational specifications and objectives have been identified, programs can be developed to accomplish those objectives. However, this can still be a difficult task because of the different views and backgrounds of people. It has been said that we see things and perceive them not as they are, but rather as we interpret them to be. This was emphasized at a recent Conference on Contemporary Issues and Problems in Mathematics held at Temple University. Participants at the conference were unable to generate an acceptable definition for the target population. Terms such as "slow learner", "low achiever", "under-achiever", and "general mathematics student" were used synonymously. Apparently the majority of the conferees "lumped" all students who were not in the college preparatory programs into the "same bag".

Participants of the afore-mentioned conference were in agreement that the non-college-bound student is programmatically short-changed, and that all too often the offerings become a "take what's left over" program composed of "bits and pieces" from other courses centered around the computational skills. They felt that a lot of effort had been devoted to the development of programs for the gifted and slow learners while the middle track had been neglected.

If an accurate needs assessment has been completed and the goals and objectives developed are appropriate for your target population, it is anticipated that gaps such as the one mentioned above would not be present. Hopefully the needs of all the students would be met.

The selection and development of the best program to accomplish the objectives is a real challenge. There are many different types of programs involving "individualized instruction, mathematics laboratories, teaching by discovery, the inquiry approach, computer-assisted instruction, and programmed learning". Different things are meant by different people as they use these terms. There is also wide variation in the degree of involvement in the programs. Certainly the interest and training of your staff and the physical resources available play key roles in determining the most appropriate program.

PROGRAM AT IDAHO FALLS

I would like to review with you the program which we developed for our district. Perhaps you will find one or two ideas useful to you. Idaho Falls is a district with about 10,000 students and has two high schools. There are approximately 2,300 students in Grades X through XII. One school is very traditional, while the other operates on a modular schedule.

As we considered ways in which to improve the mathematics curriculum, it was decided that the program should be consistent with the following:

1. Responsibilities of the schools to the community.
2. Responsibilities of the community to the schools.
3. What are the common concerns and interests of the students?
4. How do the students differ in interests and abilities? Will the program provide for these differences?
5. Will the program prepare students for the changes occurring in society?

The major goals were:

- to provide facilities for more individualized instruction in mathematics by the use of a mathematics resource center;
- to increase the motivation and problem-solving ability of low-achieving students by the use of a variety of media and instructional materials.

It was desired to develop a program that would provide for individualized instruction and independent study without resorting to the preparation of a large number of packets. We also wanted to retain some of the traditional classroom instruction because some students just do not have the self-discipline or ability to do well on independent study. They need the constant reinforcement of daily discussions and assignments.

It was decided to establish a mathematics resource center at one of the high schools and staff the center with a full-time master teacher. This was accomplished through the assistance of a Title III ESEA grant. The students who wish to do so can take their mathematics classes from the resource center teacher on an independent study basis. The teacher is available at any time for consultation, help and encouragement. The other students take their mathematics classes in the traditional manner. It is possible for students to transfer at any time from one program to the other so that it has the added advantage of providing a choice of programs to the students. It has not been uncommon for a few students to transfer from one program to the other two or three times before they feel they are in the program best suited for their needs and interests. It has been found that even highly motivated students on independent study benefit from discussing certain concepts. For that reason, small group seminars are held periodically where students can discuss important points and thus obtain a broader understanding of the concepts.

In order to make the independent study program as effective as possible, learning objectives are written for the courses. Supplementary references and suggested assignments keyed to the textbooks are included in the objectives, allowing students to proceed with a minimum amount of direction from the instructor. The teacher is thus in a position to devote most of his time helping students who need assistance.

Individual conferences are held with each student at the beginning of each grading period. The student lists the number of units he will try to complete and the average test scores during that grading period. At the end of the period, the teacher reviews the student's achievement with him and makes any appropriate suggestions or comments. It is not uncommon for students to complete two years of work in one year and score outstanding grades on the chapter and unit tests. These students have also made outstanding scores on standardized tests, national mathematics contests, and have been very successful in college mathematics courses.

Included in the resources center is a variety of mathematical games, laboratory activities, computing devices, reference materials and a small computer. The student spends whatever available time he desires in the resource center, but is not required to be there unless he has a conference scheduled with the teacher or is not maintaining a satisfactory achievement record.

Most of the general mathematics students do not work on independent study. However, the teacher uses the materials in the resource center to provide a variety of learning experiences for these students. Short units designed to help them improve their computational skills and problem-solving ability are used rather than a single textbook.

The program has had a dramatic impact on the mathematics program in our district. In 1967-68 (the year just prior to starting the program), the following courses were available at Skyline High School:

General Math	Algebra I
Geometry	Algebra II
Trigonometry*	Solid Geometry*
Math Analysis*	Introduction to Calculus*

The 1968-69 school year was the first year in which the resource room and independent study program were available. The mathematics curriculum was expanded to include the following:

Basic & Business Math	Survey of Math
Algebra I	Algebra II
Geometry	Algebra III*
Trigonometry*	Math Analysis*
Introduction to Calculus*	Computer-oriented Math
Calculus I*	Calculus II*
Machine Language Programing	

In 1970-71, the curriculum was further expanded, without an increase in staff, and included the following courses:

*semester courses

Basic & Business Math	Survey of Math
Algebra I	Algebra II
Geometry	Algebra III*
Trigonometry*	Functional Analysis*
Analytic Geometry*	Probability and Statistics*
Linear Algebra*	Algebraic Systems*
Calculus I*	Calculus II*
Calculus III*	Computer-oriented Math
Computer Systems and Machine Language	

All of the courses, except calculus, are available on independent study. The other courses are taught in the regular classroom or independent study. It is felt that students in calculus need the additional discussion time periodically.

The number of students enrolled in mathematics classes has increased significantly. Following is a summary of the increase at Skyline High School:

	1967-68	1969-70	1970-71	% Increase
Total number of students at Skyline	1371	1391	1306	4 (decrease)
Number of mathematics classes available	23	27	33	43
Number of students enrolled	550	750	780	42
Number of students on independent study	0	36	120	
Number of basic mathematics classes	1	2	2	100
Number of students in basic mathematics classes	20	30	32	56
Number of students in advanced classes (above Algebra II)	100	147	160	60
Number of students enrolled in two or more mathematics classes	0	10	27	
Number of students enrolled in three mathematics classes	0	0	10	
Number of students involved in computer programming	0	22	35	

*semester courses

The number of classes and enrollment remained approximately the same in 1971-72. There was a slight reduction last year and again this year. However, there are some unique reasons for this.

The program has had a significant impact on the mathematics curriculum at Idaho Falls High School also. During the second year of the program, many Idaho Falls High School patrons wanted to know why their students could not have similar opportunities. There were not sufficient funds in the Title III Project to provide the necessary materials, so the patrons contacted a local company and requested their help. The company agreed to buy a computer for the center if the district would provide the remaining items. This was done and I assumed the position of the resource center teacher. It was the most interesting and enjoyable year I have spent in education.

The Title III Project terminated at the end of the 1970-71 school year, but the program has continued. In fact, there is presently more activity in the traditional high school program than at Skyline. The facilities have been expanded and more teachers are involved in the program.

It has been clearly demonstrated in the development and maintenance of this program that the personnel is the key to success. Except in very unusual situations, the following is very true: *IT'S NOT HOW MUCH YOU KNOW, IT'S HOW MUCH YOU CARE!*

Another very beneficial spin-off has occurred as a result of the project. In 1968, Skyline High School had a small (PDP 8/S) computer. In 1970, Idaho Falls High School obtained a small (PDP 8/I) computer. The district has now secured a PDP II which is capable of doing the district accounting functions and performing instruction activities. As a result, each high school has a small computer and two remote terminals available for use by the students. This has been accomplished for approximately the same cost as was involved when a local service bureau performed the accounting activities.

The students have benefitted in many ways from the program. The benefits include:

1. a feeling on the part of the students that the mathematics teachers really care about them and are looking for better ways to meet their needs (this was indicated by the results of a survey conducted by the students at Skyline);
2. the strengthening of the mathematics curriculum;
3. provision of a center for interesting activities and opportunities for motivation and success activities;
4. greater involvement of the mathematics staff - an extensive in-service program has been conducted to prepare the teachers to meet the challenges of the new project;
5. improved attitude on the students' part toward mathematics;
6. independent study provides a greater opportunity for "one-to-one" discussion and direction between student and teacher;
7. provision of opportunity for a student *to choose* - he can *set his own goals;*

8. students learn to "read" a mathematics textbook and to use the resources available and necessary to solve the problem (one of the real keys to helping students prepare to take their place in a rapidly changing society);

9. increased patron and community involvement.

* * *

TO SUMMARIZE, let's go back to the opening letter. Is it possible for us to teach one of our students that it is more honorable to fail than to cheat, to have faith in his own ideas, to filter all that he hears on a screen of truth, and to have faith in himself? Can we do all of this and struggle with the last few chapters in the text for which we never have time? What a challenge! But it can be met if we will take the time to know and understand our students, find out what their interests and needs are, maybe even visit them and their parents in their homes before a problem develops - just to let them know that we care.

I am sure that most of us are here today because somewhere during our school days there was a teacher who cared enough to touch our hearts and gave us the vision of the great responsibilities and intangible rewards that come from shaping the minds of the young and helping them to discover the truths of the universe.

We can meet the challenge of the '70s, but we have to understand and care about our students, know what our goals are, where we are going, and how we are going to get there.

Transformations and School Mathematics

Transformations is one of the main themes in contemporary mathematics programs. It can form a basis for the study of geometry, functions such as mapping, trigonometry, and even the traditional topic of sections.

JOHN DEL GRANDE

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INTRODUCTION

Transformations have played a very important part in our thinking over the past five years. Thinking about transformations has opened to us some of the true fascinations of mathematics and has showed us new directions for the development of student materials starting with five-year-old children. I am not alone in my enthusiam for this approach. My great teacher, Dr. Coxeter, perhaps the world's greatest living geometer, states the following:

> It is difficult to overestimate the importance of the notion of *mapping* in mathematics. In calculus it appears centrally in the concept of *a function*. In algebra we speak of a *correspondence*. In geometry we generally use the word *transformation*. Through the

concept of transformations we are able to characterize the geometry we are studying. It leads us, in fact, to a satisfactory answer to the question, "What is Geometry?" [unpublished manuscript]

I would like to go farther and say that transformations can lead a student to answer the question, "what is mathematics?"

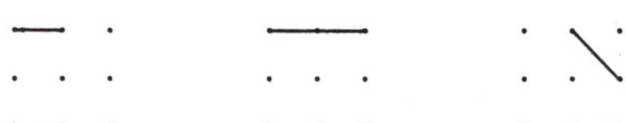
It has been clearly established that transformations are a vehicle for developing geometry. At an international conference on pre-college geometry held in southern Illinois, some of the world's foremost geometers spoke strongly in favor of transformations. Britain and many other European countries are now producing mathematical materials which rely on the idea of a mapping. In our work in Canada, we have found that the mapping idea has opened up new approaches to school mathematics. These approaches are proving both enjoyable and fascinating for teachers and children alike. I will try to outline briefly what has happened in the past few years.

TRANSFORMATIONS IN SCHOOL MATHEMATICS

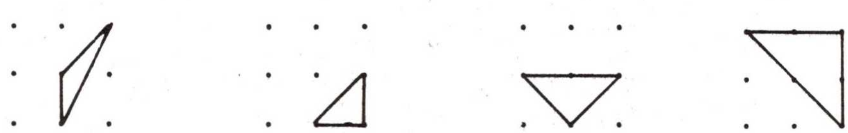
The most significant change in elementary mathematics is occurring through the introduction of transformations (Del Grande, 1972). Geoboards and colored elastics are used at a very early age, and children learn about shape and transformations through them. Special materials such as dot paper and plexiglass mirrors enable children to work with mathematical problems never before attempted.

Here are a few sample problems that children can try experimentally on the geoboard and then analyze.

- On a nine-nail geoboard, how many segments can you make the same size as the one given?



- How many triangles, the same size and shape, can you make?



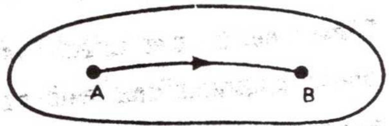
From exercises such as those, children learn to talk about congruent figures they have made and describe how to move one figure onto another. These motions called *slides, flips* and *turns* form the basis for transformations in geometry.

Children act out slides, flips and turns with body motions. They move paper cutouts around a plane to illustrate the motions (Del Grande, 1972). They soon learn that pleasing patterns can be made by drawing the outline of a figure as it is moved according to a rule.

Reflection is first illustrated by using paper folding and flipping a figure about an edge. With the introduction of the semi-transparent mirror, actual mirror images can be drawn with ease. This special mirror has many applications to geometry and geometric constructions in later grades.

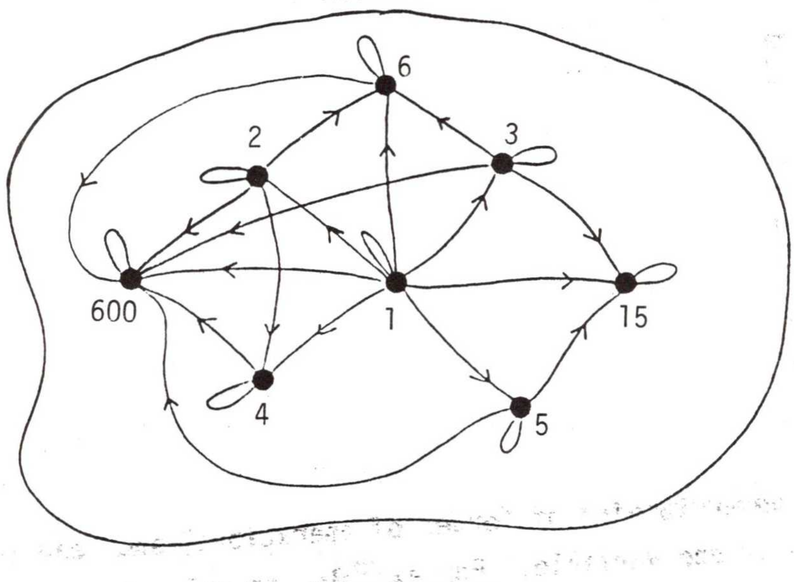
Our junior high program relies heavily on the idea of a mapping. The program starts with arrow diagrams first introduced by those brillant mathematics educators Georges Papy (1968) and his wife Frederique Papy (1971). Through these diagrams the student learns to pair things with things, number with number, and points with points.

If A is related to B, then A is joined to B with an arrow.



The following arrow graph or Papygram illustrates an interesting way in which Papy's approach can be used (Del Grande, Jones, Lowe and Morrow, 1971-72).

Draw an arrow diagram of the relation *is a factor of* in {1, 2, 3, 4, 5, 6, 15, 600}. (Since 2 is a factor of 4, we join 2 to 4 with an arrow, etc.)



IS A FACTOR OF

Notice that since 1 divides every whole number, 1 is joined to every other point.

Why is there a loop at every point?

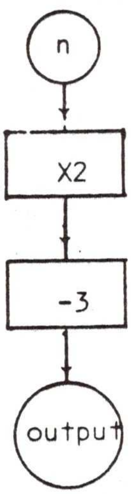
What kind of arrow diagram results if we have only prime numbers?

It is easy to design Papygram questions that involve drill and practice, and children really enjoy it.

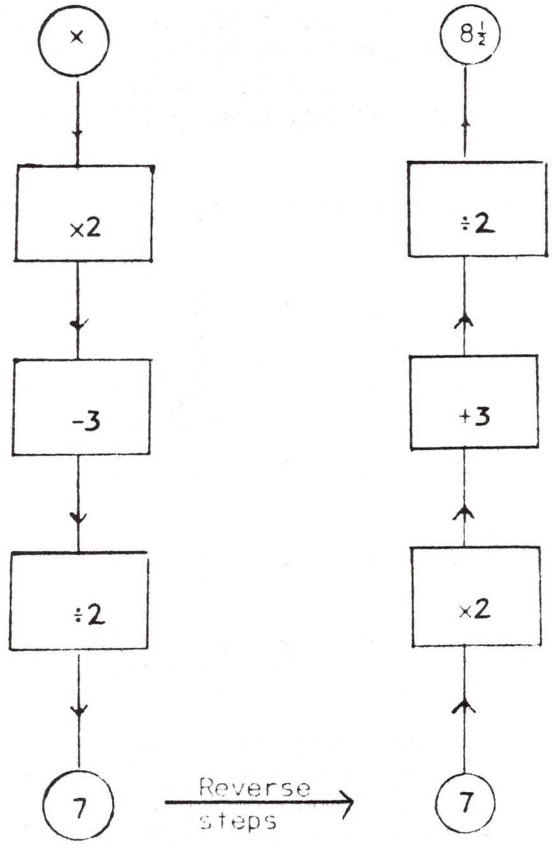
In pairing number with number, the student discovers pairing rules such as $n \to n + 3$, $x \to 2x - 5$. These pairings lead to graphing of the pairs of numbers with emphasis on those pairings that give a linear graph. Pairing of numbers have led us to introduce flow charts. For example

$$n \to 2n - 3, n \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

Flow charts are used for evaluating algebraic expressions and result in considerable practice in computation. By pairing the "input" with the "output" we obtain ordered pairs that can be represented in a graph. These flow charts are like "function machines".



Flow charts require an understanding of "order of operations" and lead to the solving of linear questions in one variable. For example, to solve $\frac{2x - 3}{2} = 7$ the student produces the following flow chart and reverses the steps to solve for x.



Teachers in France are solving linear equations with Grade IV children using arrow diagrams and the renaming of numbers.

If 2x + 3 = 19, then 2x + 3 and 19 name the same number.

2x + 3

●

19

Find new numbers that have two names

2x + 3 → [+ 2] → 2x + 5

●————→————●

19 → [+ 2] → 21

Find a number whose name is also x.

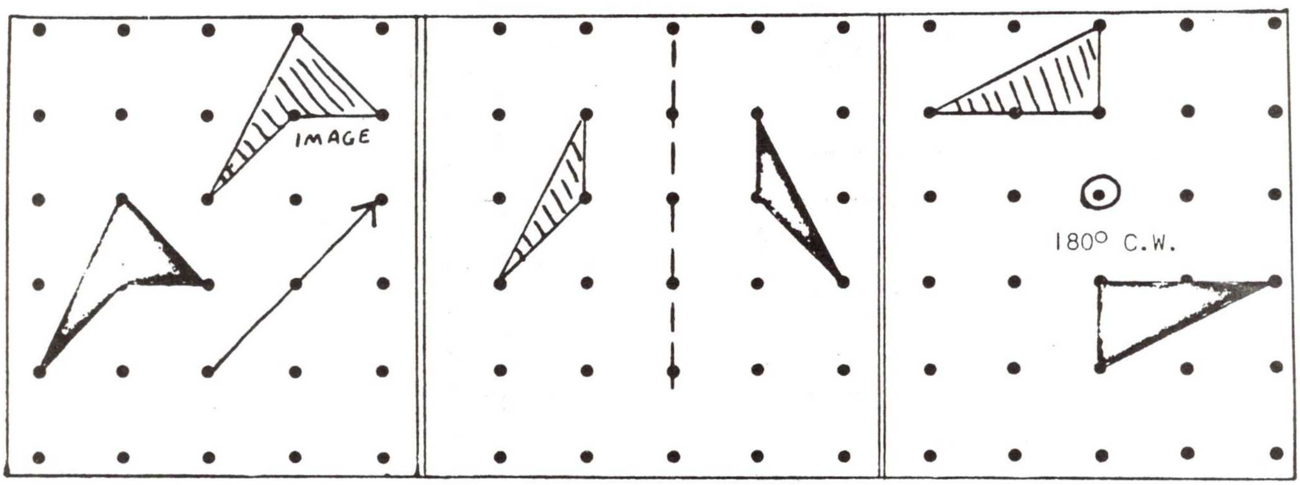
2x + 3 → [-3] → 2x → [÷ 2] → x

●————→————●————→————●

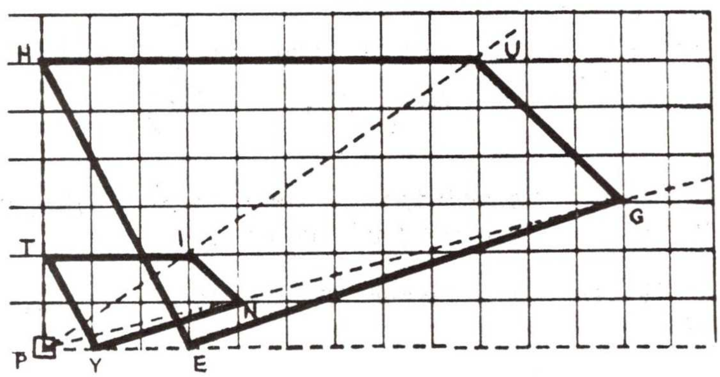
19 → [-3] → 16 → [÷ 2] → 8

Thus, x = 8 is a solution of the equation 2x + 3 = 19.

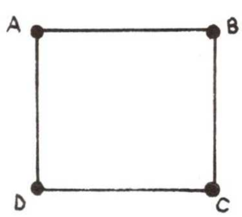
Papygrams lead naturally into the study of transformations through the pairing of points. A translation can be described using one point and its image, a reflection by the "mirror" line and a rotation by the center of rotation and the angle of rotation (Del Grande, Jones, Lowe and Morrow, 1971-72).

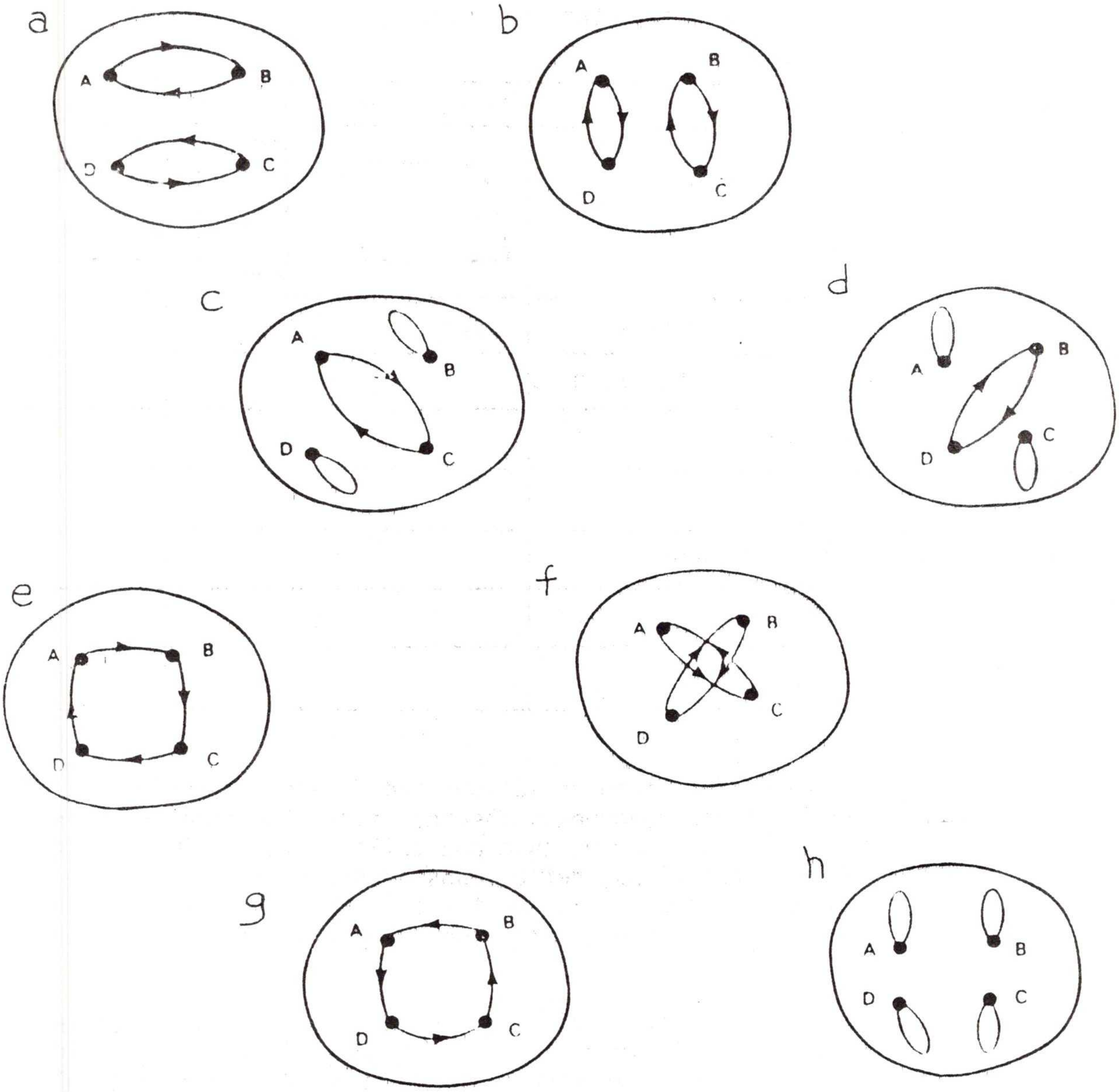


Dilatations and size transformations are studied and considerable work with similar figures, scale drawings and ratio is done (Coxford and Usiskin, 1971, Del Grande, Jones, Lowe and Morrow, 1971-72). The following diagram shows that HUGE is the image of TINY under a size transformation with center and scale factor 3:1.



Papygrams describe in a most unusal way some of the geometrical transformations (Del Grande, Jones, Lowe and Morrow, 1971). For example, given a square ABCD, which transformation does each Papygram describe?





By applying the transformations of translation, reflection, rotation and glide reflection to geometric figures, children learn two important things:

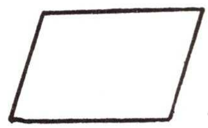
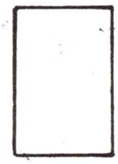
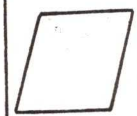
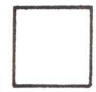
(1) the properties of geometric figures,

(2) the properties of the transformations. (See Coxford and Usiskin, 1971, and Del Grande, Jones, Lowe and Morrow, 1971-72.)

The properties of geometric figures include the

- properties of isosceles triangles
- angle sum of a triangle
- properties of parallel lines
- properties of quadrilaterals.

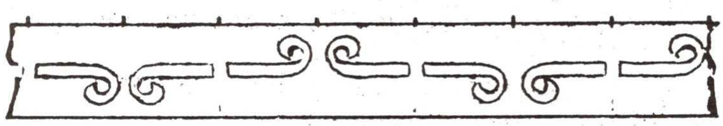
After activities involving quadrilaterals and transformations, Grade VIII children can fill in charts such as the one below.

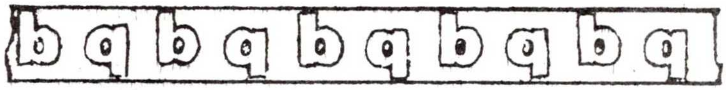
	Parallelogram	Rectangle	Rhombus	Square
				
a. The diagonals bisect each other.	yes			
b. The diagonals cross each other at 90^0.			yes	
c. All sides are congruent.				
d. All angles are congruent.		yes		
e. Opposite angles are congruent.				
f. Opposite sides are congruent.				
g. Opposite sides are parallel.				

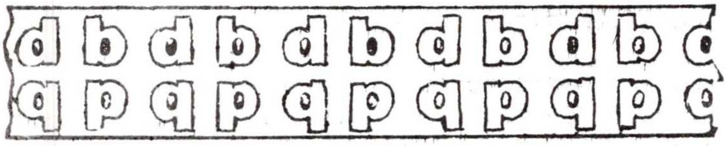
With this background of transformations and figure properties, it is an easy matter to prove the three congruency theorems side-side-side, side-angle-side, and angle-side-angle by showing that one triangle is the image of the other after a combination of translations, reflections or rotations. Thus, one can readily see that deductive geometry, in the traditional sense, can be attacked using the traditional tools along with the powerful yet natural weapons of transformations.

Dieudonné shocked the mathematical world when he said in his address at Cercle Culturel de Rogaumount in 1959, "Euclid must go". This modern approach to Euclid is what he wanted. His pleas have not gone unheard. We've accomplished what he predicted should happen; and it happened through transformations.

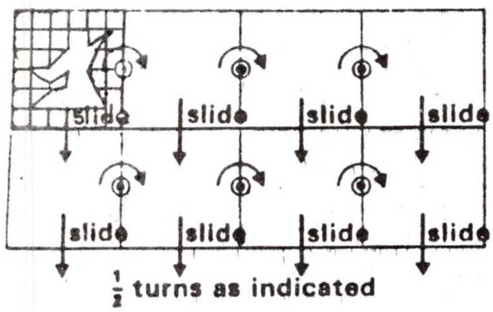
Transformations were never taught to young children in the past, nor were they usually outside the context of geometry. The study of the symmetries of a figure lead to the study of groups and all the fascinating aspects of strip patterns and wallpaper designs (Budden, 1972, Weyl, 1952). The following strip patterns are "generated" from a basic figure using transformations.

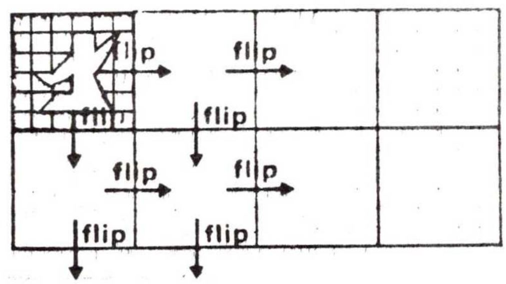






The following diagrams illustrate instructions which will generate two of the many possible wallpaper patterns.

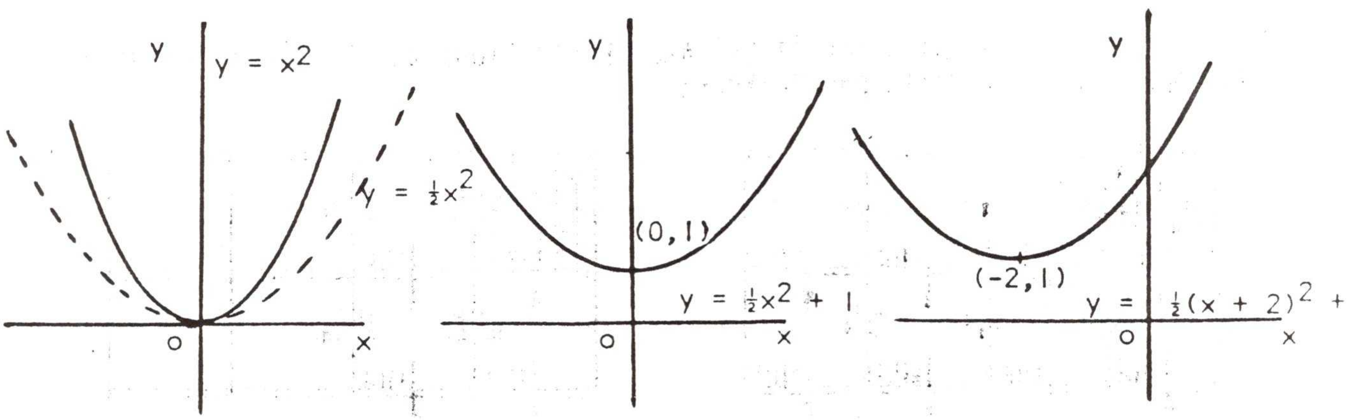




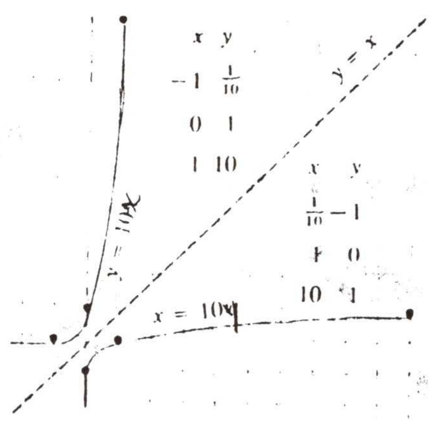
Symmetry leads to crystallography and nuclear structures. In fact, an excellent introduction to transformations is found in the book *Symmetry: a Stereoscopic Guide for Chemists* (Bernal, Hamilton and Rice, 1972). A few pictures from this interesting book follow.

The work of the graphic artist M.C. Escher is primarily based on transformations. His pictures are famous and delight children of junior high school age who try to imitate his rules but with different designs.

The study of the linear and quadratic functions can be made dynamic and meaningful through transformations (Del Grande, Duff and Egsgard, 1970). For example, a parabola with equation $y = ax^2 + bx + c$ can be obtained from the parabola $y = x^2$ by a stretch and a translation. It is no great problem to develop a series of exercises through which a student can discover this fact for himself.



Functions and their inverses are shown to be mirror images of one another in the line $y = x$. The logarithmic function is defined as the inverse of a corresponding exponential function. The graphs of these two functions are shown below and are mirror images of each other in the line $y = x$.



The following gives a set of rules for relating functions to their graphs through transformations.

TRANSFORMATIONS AND CURVE TRACING

f is a function with defining equation $y = f(x)$.

Graph of f is symmetric about the y axis.

$f(a) = f(-a) \quad (x,y) \to (-x,y)$

Graph of f is symmetric about the origin.

$f(a) = -f(-a) \quad (x,y) \to (-x,-y)$

The function af

If $a = -1$, the graph of af is the mirror image of the graph of f in the x axis.

If $a > 1$, the graph of af is a "vertical stretch" of the graph of f.

If $0 < a < 1$, the graph of af is a "vertical compression" of the graph of f.

If $a < 0$, the graph of af is the mirror image of the graph of $|a|f$ in the x axis.

The function f^{-1} $\quad (x,y) \to (y,x)$

The graph of f^{-1} is the mirror image of the graph of f in the line $y = x$.

$h:x \to f(x + a)$ $\quad (x,y) \to (x - a,y)$

The graph of h is congruent to the graph of f.
If $a > 0$, the graph of h is a units to the left of the graph of f.

If $a < 0$, the graph of h is a units to the right of the graph of f.

$h:x \to f(x) + a$ $\quad (x,y) \to (x,y + a)$

The graph of h is congruent to the graph of f and f is translated a units parallel to the y axis.

$h:x \to f(ax)$ $\quad (x,y) \to (\frac{1}{a}x,y)$

If $a > 1$, the graph of h is a "horizontal compression" of the graph of f.
If $0 < a < 1$, the graph of h is a "horizontal stretch" of the graph of f.
If $a < 0$, the graph of h is the mirror image of the graph of $k:x \to f(|a|x)$ in the y axis.

$h:x \to f(ax + b)$, $a > 1$, $b > 0$

$$f(ax + b) = f(a(x + \frac{b}{a}))$$

The graph of h is a "horizontal compression" of the graph of f and $\frac{b}{a}$ units to the left of the graph of $k:x \to f(ax)$.

Discuss the cases $a > 1,\ b < 0$

$0 < a < 1,\ b > 0$

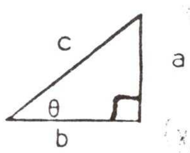
$0 < a < 1,\ b < 0$

$a < 0,\ b > 0$

$a < 0,\ b < 0$

Although trigonometry can be approached using similar triangles and ratios in the early years, the trigonometric functions can be introduced in a meaningful way using a mapping.

The trigonometric ratios are first defined using a righ-angled triangle for $0^\circ < x < 90^\circ$

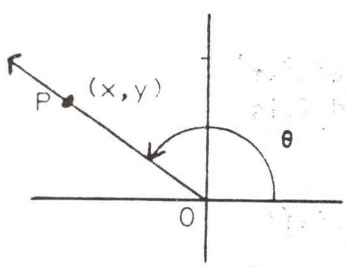


$$\sin\theta = \frac{a}{c}$$

$$\cos\theta = \frac{b}{c}$$

$$\tan\theta = \frac{a}{b}$$

The definition may then be extended to angles of any measure by using the analytic approach



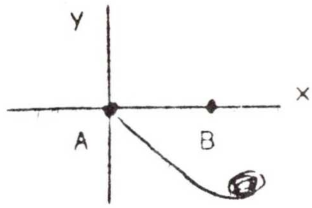
$$OP = r$$

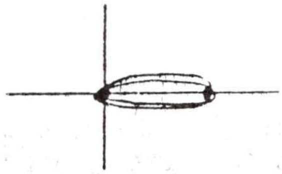
$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$

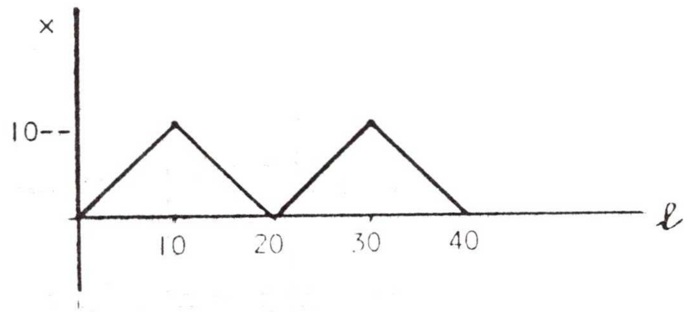
During these early years, we should develop some basic ideas for periodic functions through problems such as the following from Del Grande, Duff and Egsgard (1970) and Del Grande and Egsgard (1972).





- Two pegs are placed at A and B.
- Wrap a string around the pegs as shown.
- Each point on the string maps onto a point (x,y) on the plane.
- Relate the length of string ℓ to x and then ℓ to y.
- Graph the functions ℓ → x and ℓ → y.

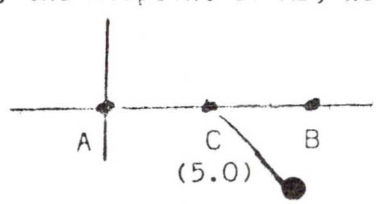
ℓ	0	1	2 ...	10	11	12 ...	20	21 ...
x	0	1	2 ...	10	9	8 ...	0	1 ...

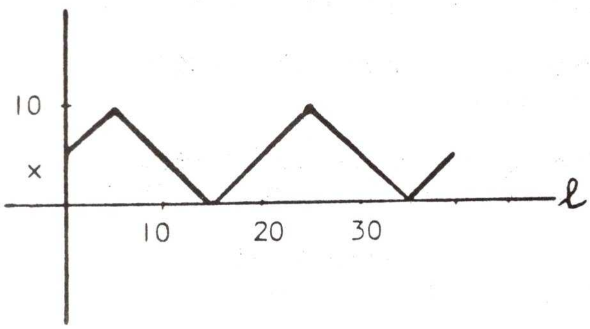


ℓ	0	1	2 ...	10 ...	20
y	0	0	0 ...	0 ...	0

The first graph allows us to study periodicity and amplitudes. Phase shift can easily be shown by using the same pegs but starting the string at some point between A and B. For example, if we start at C, the midpoint of $\overline{AB}$, we ge

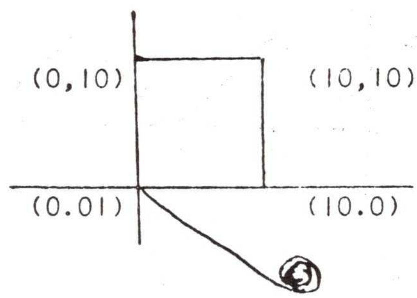
ℓ	0	1	2 ..	5	6	7 ..	10 ...	15 ...	20
x	5	6	7 ..	10	9	8 ..	5 ...	0 ...	5



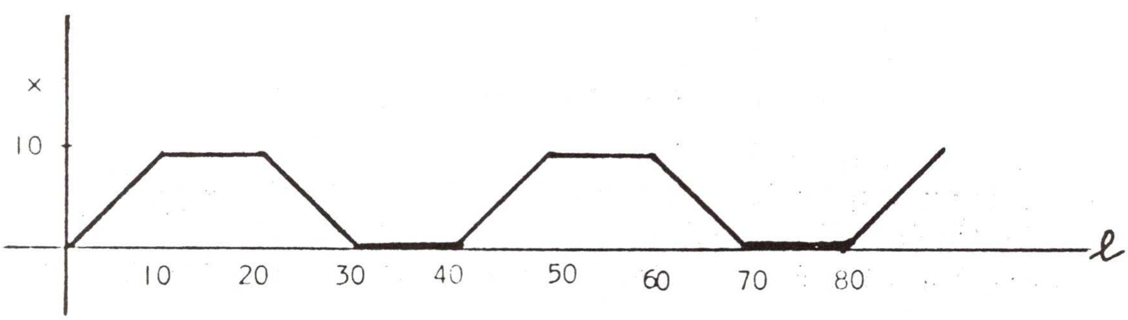


This new graph is a shift copy of the first with the exception of the starting portion. These graphs can be extended to the left by winding the string in the opposite direction and using negative values for ℓ. Periodicity amplitude and phase shift should be studied long before the graphs of the trigonometric functions are introduced.

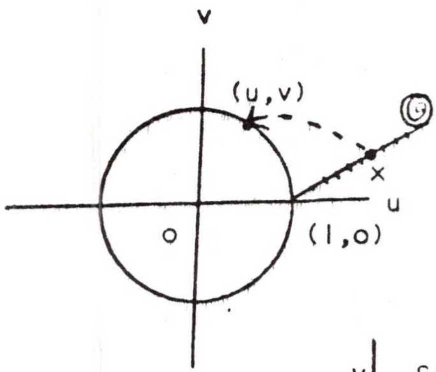
The next exercise might involve the winding of a string about a square.

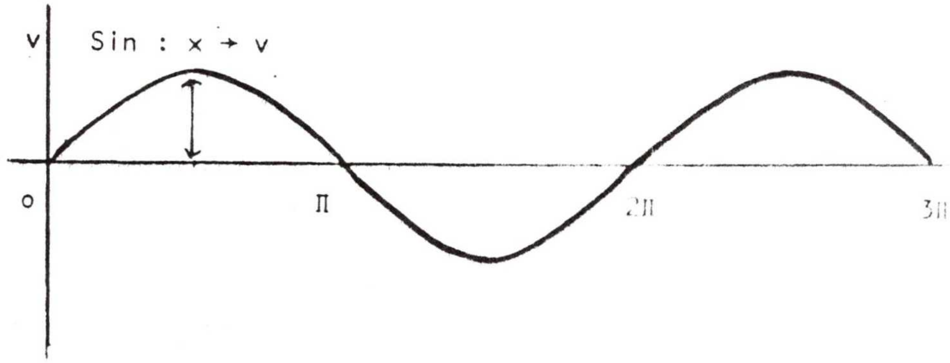


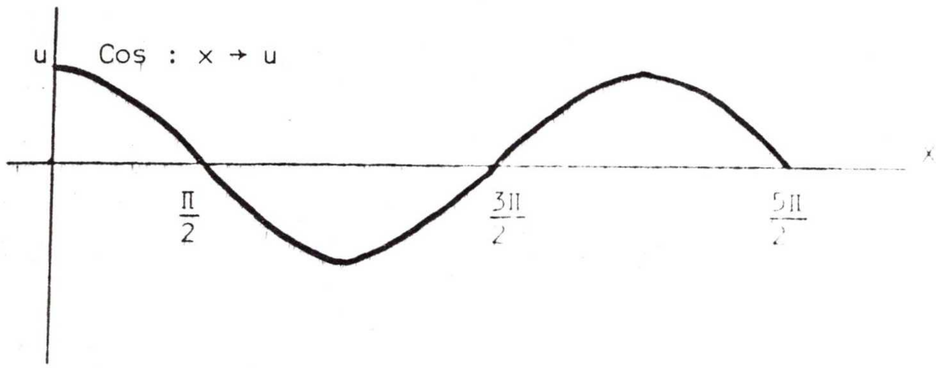
The graph of $\ell \to x$ is as follows.



Finally, a string is wound around a unit circle and the familiar trigonometric functions emerge.



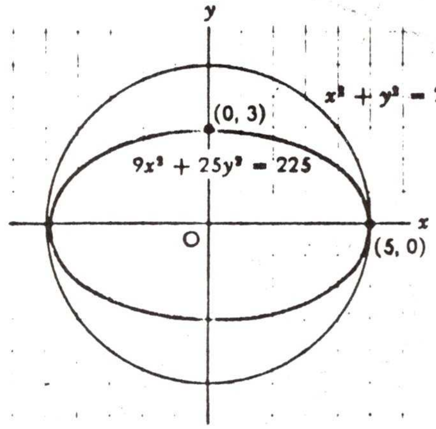




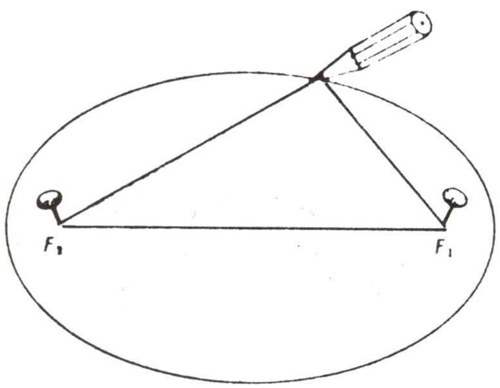
The change in symbols from (x,y) to (u,v) is necessary to arrive at the equations we desire, namely $y = \sin x$ and $y = \cos x$. Notice that x is the length of string and is a *real number*.

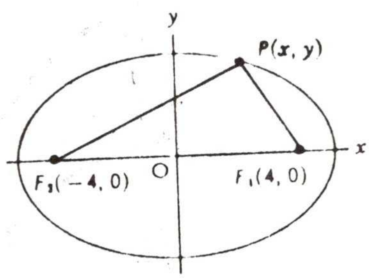
Graphs of functions such as $y = a \sin (bx + c)$ can now be related to transformation. The graph is a sine curve with amplitude $\underline{a}$, frequency $\frac{2\pi}{|b|}$ and phase shift $-\frac{c}{b}$.

Transformations enable us to approach the conic sections in a very interesting way (Del Grande and Egsgard, 1972). Starting with a circle with equation $x^2 + y^2 = 25$, a one-way stretch $(x,y) \to (x, \frac{3}{5}y)$ is applied. The resulting curve is $9x^2 + 25y^2 = 225$.



To show that the image curve is an ellipse, we define an ellipse using the constant sum of the focal radii.





By selecting suitable foci and a suitable sum we obtain the ellipse $9x^2 + 25y^2 = 225$.

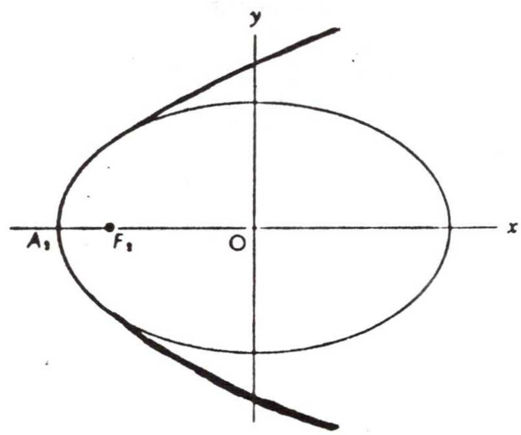
Thus, we show the image curve is in fact an ellipse.

These results can be generalized using the circle $x^2 + y^2 = a^2$ and the one way stretch $(x,y) \rightarrow (x, \frac{b}{a} y)$. To obtain an ellipse with foci on the y axis, we apply the stretch $(x,y) \rightarrow (\frac{b}{a}x, y)$.

Having stated that the graph of a quadratic function is a parabola (Del Grande, Duff and Egsgard, 1970), we show that parabolas have image parabolas under stretches. To show how to obtain a parabola from an ellipse, we start with the ellipse

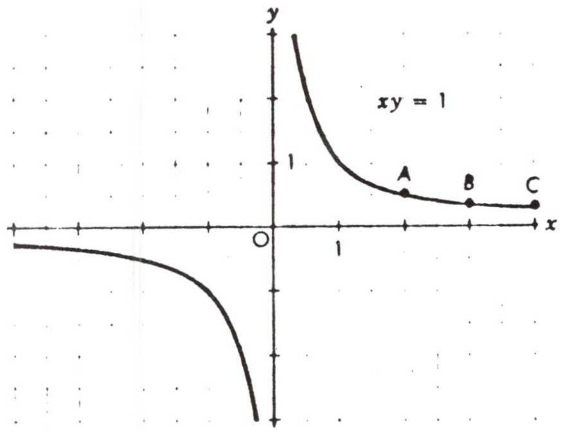
$$b^2x^2 + a^2y^2 = a^2b^2$$

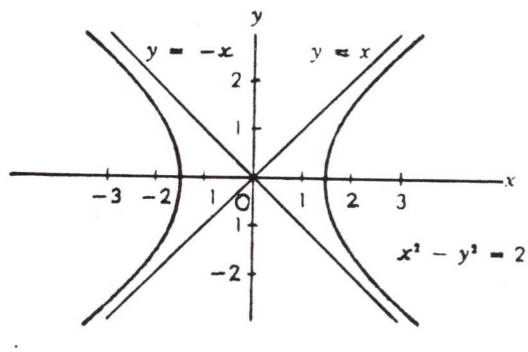
The ellipse is translated so that (-a,0) moves to the origin. We employ a special transformation that holds the vertex at (0,0) and the nearer focus fixed while the other focus moves to infinity along the positive x axis (Del Grande and Egsgard, 1972).



The final result is parabola!

For the hyperbola, we start be stating that xy = 1 is a hyperbola. This is easy to graph and the asymptotic properties of the hyperbola are apparent.



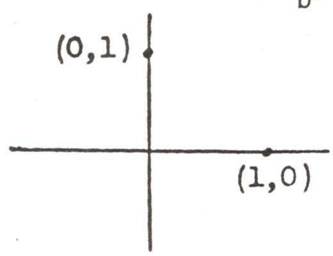


By a 45° rotation clockwise about the origin we obtain the image hyperbola $x^2 - y^2 = 2$. A two-way stretch, $(x,y) \to (\frac{a}{\sqrt{2}}x, \frac{b}{\sqrt{2}}y)$, of the image gives the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where the asymptotes are now $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$, which are the final images of the x and y axes.

It is interesting that starting with an hyperbola and holding one of its foci fixed, a suitable "stretch" will give an image that is a parabola. Thus the intuitive notion is established that when an ellipse is "stretched to infinity" it becomes a parabola, and when "stretched beyond infinity" it becomes an hyperbola.

Transformation gives us an excellent opportunity to introduce and apply matrices (Coxford and Usiskin, 1971). A 2 x 2 matrix can be used as an operator on the vertices of a figure to give the coordinates of the image points. Coordinates of points are written as a matrix

$$(a,b) \rightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$



If $(1,0) \rightarrow (x,y)$

$(0,1) \rightarrow (u,v)$

under a transformation, then the matrix operator is $\begin{pmatrix} x & u \\ y & v \end{pmatrix}$

The matrix operator for reflection in the x axis is obtained as follows:

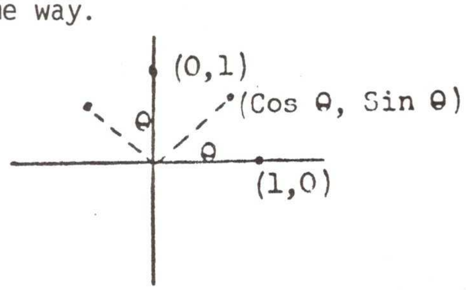
$$(1,0) \rightarrow (1,0)$$
$$(0,1) \rightarrow (0,-1) \qquad M_X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To find the image of $\triangle ABC$ under a reflection in the x axis, where A(1,1), B(2,3) and C(-1,3), we perform a matrix multiplication as follows

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -3 & -3 \end{pmatrix}$$

↑	↑	↑
operator	coordinates of A,B,C	coordinates of image points

The matrix operator for a rotation of an angle θ about (0,0) is obtained in the same way.



$(1,0) \rightarrow (\text{Cos } \theta, \text{ Sin } \theta)$

$(0,) \rightarrow (-\text{Sin } \theta, \text{ Cos } \theta)$

$$R_\theta = \begin{pmatrix} \text{Cos } \theta & -\text{Sin } \theta \\ \text{Sin } \theta & \text{Cos } \theta \end{pmatrix}$$

Composition of transformation is obtained by multiplying matrix operators To illustrate, we use successive rotations of θ and α

$$R_{(\theta + \alpha)} = R_\theta . R_\alpha$$

$$\begin{pmatrix} \text{Cos}(\theta + \alpha) & -\text{Sin}(\theta + \alpha) \\ \text{Sin}(\theta \ \alpha) & \text{Cos}(\theta + \alpha) \end{pmatrix} = \begin{pmatrix} \text{Cos } \theta & -\text{Sin } \theta \\ \text{Sin } \theta & \text{Cos } \theta \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}\alpha & -\text{Sin}\alpha \\ \text{Sin}\alpha & \text{Cos}\alpha \end{pmatrix}$$

$$\begin{pmatrix} \text{Cos } \theta \text{ Cos}\alpha - \text{Sin } \theta \text{ Sin}\alpha \\ \_ \quad \_ \quad \_ \quad \_ \end{pmatrix}$$

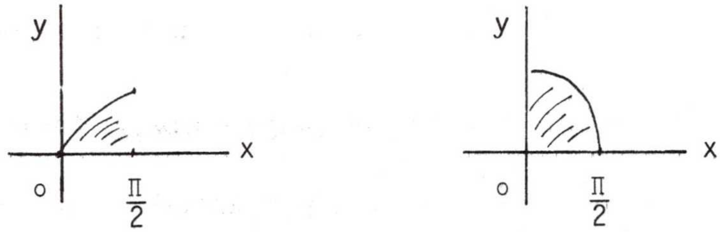
By comparing the two matrices we get the familiar result
$\text{Cos}(\theta + \alpha) = \text{Cos}\,\theta \;\text{Cos}\,\alpha - \text{Sin}\,\theta \;\text{Sin}\,\alpha$

Can you complete the matrix equation to show that
$\text{Sin}(\theta + \alpha) = \text{Sin}\,\theta \;\text{Cos}\,\alpha + \text{Cos}\,\theta \;\text{Sin}\,\alpha$?

Throughout the work on transformations, symmetry appears time and again in most unusual ways. We use an example from calculus to illustrate.

To find $\int_0^{\frac{\pi}{2}} \text{Sin}^2 x \, dx$ we notice that the areas under the curves of

$y = \text{Sin}^2 x$ and $y = \cos^2 x$ in the interval 0 to $\frac{\pi}{2}$ are reflection images of each other.



$$\therefore \int_0^{\frac{\pi}{2}} \sin^2 x \;\, dx = \tfrac{1}{2}\left[\int_0^{\frac{\pi}{2}} \sin^2 x \;\, dx + \int_0^{\frac{\pi}{2}} \cos^2 x \;\, dx\right]$$

$$= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x) \;\, dx$$

$$= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} 1 \,.\, dx$$

$$= \tfrac{1}{2}x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

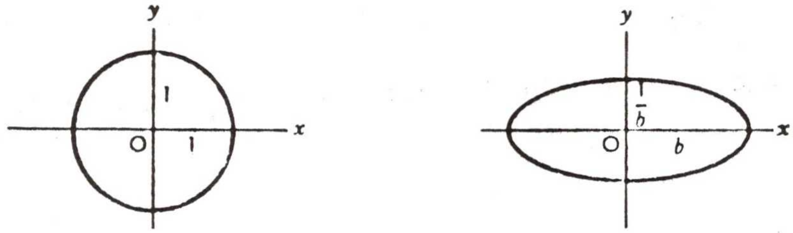
$$\therefore \int_0^{\frac{\pi}{2}} \sin^2 x \;\, dx = \frac{\pi}{4}$$

A second and interesting example from calculus shows the power and simplicity of transformations (Del Grande and Duff, 1972).

It is easily shown that the transformation

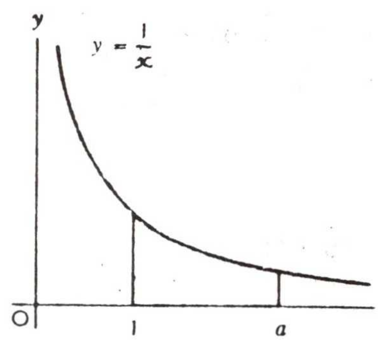
$$(x,y) \rightarrow (bx, \frac{y}{b})$$

preserves area. For example,



the circle has an ellipse as image but the areas are the same!

Under this two-way stretch $(x,y) \rightarrow (bx,\frac{y}{b})$ the curve $y = \frac{1}{x}$ maps onto itself. If the area in the interval $1 \leq x \leq a$ under the curve $y = \frac{1}{x}$ is defined as $\ell n\ a$



we can show that

$$\ell n\ a\ b = \ell n\ a + \ell n\ b$$

which is an important property of a logarithm.

By using areas, the result that

$$\frac{d \ln x}{d x} = \frac{1}{x} \text{ follows.}$$

Results such as these indicate that we can make mathematics more meaningful to our students to whom we have been entrusted to reveal the beauty, sense and fascination of mathematics.

CONCLUSION

The use of transformations produces a great quantity of mathematics suitable for young children and leads to a set of axioms in geometry on which a logical structure can be built. Transformations lead naturally into the ideas of relations, mappings and functions, especially for composition and inverses. Transformations clarify the addition of vectors, help to motivate matrix multiplication, provide better proofs of results in trigonometry, and make the study of curve tracing or graphical representation of functions both dynamic and lucid. Transformations is one of the main themes and is a unifying force throughout the whole of school mathematics.

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Mathematics and the Low Achiever

Many studies have been designed to give us information on low-achieving mathematics students. The results of these studies will be summarized. Also some conjectures will be presented on operational procedures that seem to be effective.

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I have been working on the problem of slow learners for 25 years, and I have a few conjectures which I will share with you. I also feel that I have had some success in working with slow learners.

I will define the low achiever as a person who, for some reason or other, is two or more grade levels below those on which we would expect him to be. These individuals may be categorized further as under-achievers, slow learners or persons with emotional problems. For the purposes of this paper, I will not make these distinctions. We can plan a suitable mathematics program for the low achiever without these more restrictive categories.

A SUMMARY OF THE CHARACTERISTICS OF LOW ACHIEVERS

Generally low achievers do not enjoy competition in the mathematics classroom. They do not seek challenges. They lack inventiveness and a questioning nature. They have a poor self-concept and poor study habits. They are easily confused. They are unable to express their ideas in writing. It takes a longer time for them to learn a concept, and they forget very easily. They have a short attention span and a low frustration level. They have reading difficulties. Often they have social and emotional problems. They have short-range goals. There is a tendency for them to leap to conclusions, and it is difficult for them to generalize. They are unable to follow directions. They are frequently absent, and they don't bring their materials to class.

We can't blame the students for having these characteristics. We must love the students in spite of their deficiencies. If we don't care for the students, they have a very small chance of succeeding.

SUGGESTIONS FOR TEACHING SLOW LEARNERS

Immediately after Sputnik, nearly all mathematics educators were primarily interested in programs for the capable mathematics student. Max Sobel was an exception. Here are some suggestions he had for the classroom teacher.

> Whereas all students crave security and have a need to succeed, the slow learner is especially vulnerable in this respect. Years of consistent failure in the early grades make him prey for any sort of meaningless trial-and-error scheme just to get an answer and satisfy the demands of the teacher.
>
> It is important that we make an effort to motivate the student. At ages thirteen to fifteen there is resistance to learning unless the subject matter is of interest to the student and meaningful to him; again, the slow learner can become especially resistant unless his interest, dulled by years of failure, is aroused.
>
> Junior high school youth in general, and slow learners is particular, are eager to grasp and adopt patterns of work providing them with security and independence.
>
> To summarize briefly: the slow learner in the junior high school has the same characteristics as other pupils of the same age, the same basic needs and interests. However, more than the average child, he needs to be given the chance to experience success and approval; more than the others he needs to feel that he is a member of the group with a contribution to make; he needs status; his confidence must be reestablished, his interest stimulated, his attitude towards mathematics made favorable, his ego flattered [Sobel, 1959, p.348].

Sobel (1959) goes on to outiline 12 specific suggestions for guiding the learning activities of the slow student.

1. Because of a very short attention span, the activities of the slow learner must be varied.
2. Concrete presentations must be emphasized.
3. An emphasis on practical applications is important.
4. These students must be allowed to compete with themselves, and their achievement should be measured in terms of individual growth.
5. Topics must be taken up in spirals - not taught once and then forgotten.
6. Where possible, subject matter should be correlated with work in other classes.
7. Drill is essential, but it must be meaningful and not rote.
8. Verbal materials in the text must be developed orally.
9. Frequent reviews are necessary.
10. In his need of security, the slow learner appreciates and does best in a situation where classroom management is routine.
11. Successful student materials should be exhibited to provide a feeling of success.
12. The final item concerns the procedure used to start the school year, whether it be in grade seven, eight, or nine. There is little doubt but that most slow learners in the junior high school are in dire need of a meaningful re-teaching of arithmetic [p.349].

In reference to the last suggestion, don't make the mistake of spending all of the first week in school reviewing. If a student returns to school with even a little enthusiasm and is asked to do the same "old stuff" in which he failed before, his enthusiasm will quickly die. Give him something a little different so that his interest, as little as it is, will last as long as possible.

Greenholz and Keiffer (1970) made some recommendations and comments about inner-city children. Since inner-city students are often low achievers, I will quote quite liberally from their article. Occasionally I will react to some of the things they say.

> The teacher should give clear explanations and avoid vague generalizations. For example, it is better for the teacher to say, "Only one child may go to the pencil sharpener at a time", than to say, "Let's not have so many at the pencil sharpener at a time." ... A teacher must realize that some words that are shocking to him may be standard terms to inner-city pupils [p.589].

How we handle the "shocking" language can make a big difference. Members of our society are becoming more accustomed to these language variations. If we are going to do anything with the inner-city children, we have to accept them for what they are and for what they say. Perhaps we can do a little to help them develop a more polite language, but we must exercise care.

> Children in the inner-city handle money, do the family shopping, and buy their own clothes earlier than middle-class children do. Practical problems involving the prices of purchases and sizes of various

> containers may be the foundation on which the teacher will build a further development of mathematical skills and understandings [Greenholz & Keiffer, 1970, p.589].

For example, I had a low-achieving boy in one class who had a paper route. One day I experienced some problems relating to a paper route. This boy solved them quickly. Other students were working with paper and pencil and wondered how that boy was able to do so well. Whenever possible, we should select problems consistent with the learner's experience, something he can do, something in his language, and something he is interested in. This means we have to get close to the child in some way and then build the lesson on what we learn about the child.

> Precise diagnosing of each pupil's strengths and weaknesses must precede plans for his instruction [p.589].

Jockey for position. We should try to find out where the students are before we make a frontal attack on their ignorance of mathematics.

> He does not respond well on standardized tests. These tests are formulated for, and standardized on, a middle-class population [Greenholz & Keiffer, 1970 p.589].

> It matters greatly that each student experience sufficient success to strengthen his confidence and pride in himself, to improve his self-esteem, and to encourage him to exert effort [Greenholz & Keiffer, 1970, p.590].

> Teacher expectation is a strong motivational factor [p.590].

If we expect a student to learn, he will probably learn something. If we treat him as if he were a "knucklehead", he will probably fulfill that expectation as well.

> Success or failure in mathematics is closely related to a student's ability to develop the reading skills required by the subject.

> It is not enough to drill the student on a collection of discrete rules, manipulations, and procedures prescribing how to accomplish certain specific tasks. He must learn why the procedures are appropriate and which ones to select in the problems he encounters [Greenholz & Keiffer, 1970, p.590].

As a general rule, if we drill the low achiever on material he doesn't understand, he will forget it in a few weeks and then our efforts will have been wasted.

> Let the pupil measure, experiment, try out his ideas and reach generalizations as much as possible on his own [Greenholz & Keiffer, 1970, p.590].

> Healthy competition is excellent when both success and failure are possible. It cannot exist where the work of the highly capable and

> that of the very limited are compared. School competition thus is a daily punishment for the less favored who can never win [p.591].

For example, I asked some low-achieving Grade II pupils if they cared to play a mathematical game. They expressed a strong interest. One excited little girl lost twice in a row, and her interest immediately waned. You must use games carefully if they are to help in your teaching. Games are effective learning tools if the reason for losing a game can be blamed on poor luck rather than low ability.

> The kind of school organization that permits teachers and pupils to work together with the fewest interruptions is the most effective. Time to give attention to mathematics each day achieves better results than does an irregular schedule with the passing of several days between successive class meetings [Greenholz & Keiffer, 1970, p.591].

In reference to classroom techniques, Greenholz and Keiffer (1970) add this:

> The classroom teacher must operate on the assumption that a fairly positive and forceful attitude toward classroom descipline will allow, in the long run, greater opportunity for meaningful teaching [p.592].

We cannot let every student "do his own thing". We need to be somewhat rigid, but this can be done with kindness.

> Teachers continue to study every proposed instructional aid as they search for better ways to teach urban children. No machine, however, can teach gentleness, respect, and understanding. These come only from human interaction [p.595].

I am in complete agreement with this final quote. We become educated by working with people, not by working with machines. We should remember this as we use computer-assisted instruction and other forms of programmed materials.

RESEARCH ON LOW ACHIEVERS

During the past 10 years, low achievers in mathematics have been the subject of many research studies, but many of the conclusions have been inconclusive. This state of affairs is especially noticeable in studies involving the use of teaching machines, self-study techniques, calculators, flow charts, vocational mathematics, contract and team-teaching, small group instruction and the use of older elementary students as tutors for younger students. However, Suydam and Weaver (1971) report some research which help to answer the following questions. The remarks which follow are for the most part quotes from their report.

Do special mathematics programs for environmentally disadvantaged students make a difference?

It is not at all surprising to find studies which report that special programs designed to provide special treatments and emphasis for disadvantaged pupils result in higher achievement when compared with "regular" programs which include no special provisions for such pupils.

A mathematics program "specially designed for culturally disadvantaged pupils" emhasizing success experiences, careful development of concrete to abstract levels, use of simple language, reduced reading level and load, and such techniques as discovery, inquiry and experiments in the fourth grade in inner-city schools was compared with a "regular" program. Significant differences favored the experimental group on measures of concepts and overall achievement, and gains for the experimental group were greater than for the regular group on computation and application measures (Hankins, 1969).

Are programs for low achievers effective?

The findings of research on grouping on the basis of achievement have been much more variable than those for grouping on the basis of ability. Differentiated instruction appears, however, to be more effective than total class instruction.

Sherer (1968) found that low-achieving pupils in Grades III through VII taught by author-developed materials, using instructional aids such as drawings, counters, and number lines and charts, showed significantly greater gain in arithmetic achievement than those taught by a traditional procedure.

Hillman (1970) found that Grade V pupils given immediate knowledge of results, either with or without candy reinforcement, scored significantly higher than pupils not given knowledge of results until 24 hours later. Low achievers may profit more than high achievers.

Hillman's study brings to mind a rather effective procedure I occasionally used when teaching low-achieving classes in high school. Before class, I would solve every problem worked out in detail and organized rather neatly on a sheet of paper. When I noticed a student struggling unduly with a problem, I would ask him to show me how he was trying to solve it. If he was making no progress, I gave him the paper with the solution and he could immediately compare my work with his. I also used this technique if students were visiting too much with their neighbors. The talkative student usually took the solution key, stopped talking, and started working. It is important for a teacher of low achievers to find such methods for keeping students "at task" rather than the usual more verbal approaches.

What teaching procedures are most effective for slow learners?

Herriot (1968) found that when pupils in Grades VII and IX who were classified as slow learners studied SMSG materials for two years, they achieved a greater gain than a higher ability control group achieved in one year. Time does seem to make a difference, but the optimum time needed by slow learners to reach a satisfactory level of achievement has not been answered. Nevertheless, this research does seem to suggest that to be effective we should slow down the pace.

Are different materials appropriate for the disadvantaged?

It has been suggested that the use of varied aids, media and materials, along with real life experiences and laboratory techniques, is especially effective with disadvantaged groups. Schippert (1965) found that, in an inner-city school, use of a laboratory approach in which Grades VII students manipulated actual models or representations of mathematical principles resulted in significantly higher achievement on measures of skills than students taught by a discovery-oriented approach using verbal or written descriptions of these principles. Howard (1970) used mathematical laboratory experiences, planned "to facilitate a hierarchy of needed concepts", with environmentally and academically disadvantaged rural children. Such experiences resulted in both achievement and attitudinal gains.

What remedial procedures have been effective?

Most research reports do not give specific information about the nature of remedial programs. We do know, however, that diagnosis and individualization are effective remedial procedures.

Olsen (1969) reported that use of volunteer tutors with boys in Grades II, III, and IV who were under-achievers and who were achieving two or more months below grade level resulted in no significant differences on most measures of self-concept, achievement and intelligence. At the Grade III level, however, those tutored in arithmetic achieved significantly more than those not tutored. Possibly in the junior or senior high school a judicious use of tutors, especially university students, may be helpful.

Summary

The disadvantaged, as well as all other pupils, profit from special attention. This may be in the form of attention from the teacher, the content of the program, the instructional materials, the organization for instruction or other ways.

Rate of learning is but one variable to be considered in providing effective instruction for slow learners. Methods and materials of instruction also must be adapted to these pupils.

Social relevance appears to be more crucial to consider in the case of disadvantaged students; however, little research has attended to this topic. Active physical involvement with manipulative materials, which is believed to be important for all children, may be even more so for the disadvantaged.

MASTERY LEARNING

Before terminating this account of research on the low achiever, I should also say something about mastery learning. Bloom (1971) claims that mastery learning, if practised as he and others propose, could eventually eliminate individual differences. At present, as students progress through our mathematics programs, the span of individual differences gets greater. Bloom is suggesting that we can make these differences vanish if we employ certain precedures. First we must determine the objectives we wish to accomplish and arrange them into a sequence or hierarchy. The next step is to divide the sequence into manageable and fairly discrete levels. Each level would have entry behaviors which students must have mastered before being allowed to go to the next level. If this is done and we pay careful attention to certain factors in the affective domain of the student relating to success and failure, give consideration to the development of the self-concept of students, use good instructional strategies such as a variety of explanations and cues, let the students become actively involved, provide reinforcement at the right time, then the individual difference as pictured in Figure 1(a) for the first level will become in the last level the individual differences as pictured in Figure 1(b). Bloom also suggests that the time spent by students at a particular level would vary depending upon when they attain mastery. Of course, if this plan were followed throughout the elementary school, the common practice of homogeneous grouping according to *age* would no longer be possible.

Figure 1(a)

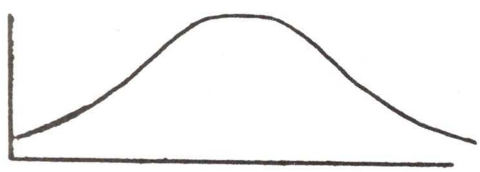
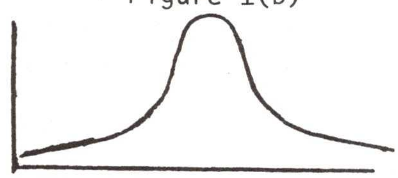


Figure 1(b)



TEACHERS OF LOW ACHIEVERS

Who teaches the low achievers in your school? Is he the best teacher with the most experience or is he the inexperienced first-year teacher who has not yet learned how to defend himself? Usually it is the latter teacher who is given this difficult assignment and often without assistance or moral support from the experienced teachers in the building. I am afraid that we, the experienced teachers, rather than the administrators are to blame for this unfair practice. If the mathematics staff in a building were to suggest to the administrator that low-achieving classes should be shared by the experienced teachers including the department chairman, I am certain the administrator would be pleased to comply. Apparently we feel that getting an easier assignment is, in some way, a promotion. Somehow we must eliminate this "upside down" value system. Really a person who teaches low achievers should earn the right to teach them and should then receive recognition from the profession for doing so.

Here are some characteristics of teachers of low achievers. He has an innate respect and concern for the pupils, he firmly believes that the pupils are capable of learning and that learning results from interaction of pupils with the teacher. He is patient and is determined to provide pupils with some success experiences. A teacher of low achievers must also have a sense of humor and a high frustration level. He must be satisfied with small gains. Even though he may be the best teacher in the school, he recognizes that he needs assistance with multi-media teaching techniques, pacing and sequencing, and in other areas.

Assistance is needed in multi-media techniques because if we are not careful, a visual device may lead students to a generalization somewhat different from the "textbook" generalization which we want them to learn. For example, we can teach subtraction of integers to low achievers using a number line and relate it to football games and other things in which students are interested. If a student comes up with a rule, it will probably not be the "change the sign and add" rule that we may want to teach.

We need assistance in pacing and sequencing not because we do not pay attention to the developmental aspect of the concepts from the point of view of the students. For example, we use the repeated addition concept of multiplication in the lower grades. We indicate 2 x 5 means two fives, but we cannot treat multiplication of fractions that way. It does not make sense to say that 2/3 x 3/4 means two-thirds three-fourths. Somewhere along the line we have to gradually change the concept of multiplication to one which generalizes to include fractions.

The use of diagnostic techniques and designing realistic course goals is another area in which we need help.

Teachers of low achievers should not have to teach them all day. They should have at least one class of high achievers. They need the change of pace!

SUGGESTION TO TEACHERS OF LOW ACHIEVERS

Experience in Mathematics Ideas, published by the National Council of Teachers of Mathematics, contains many ideas for teaching low achievers. Any material based upon activity, individuality, success, meaning and novelty will probably be successful.

A teacher of mathematics for low achievers should know something about teaching reading. He has to pay careful attention to the readability of materials. Students need help with notation, but we must not burden them with notation.

A frequent change in activity is needed. It is wise to plan about three different activities for a 50-minute period. You might have everyone together for the first activity and then give students a choice for the next two.

Plan sequential instruction based upon feedback from students. For example, one student told me that he didn't know anything about fractions but he needed to learn to work with them so he could learn more about engines. I designed some lessons based on his interest and level of understanding, and he learned enough about fractions to do the things he wanted to do.

Local color makes a big difference. Textbooks don't provide it, *you* have to provide it. For example, I prepared a unit on my bank chequing account. I used one month's transactions. My cheques were written to local business concerns. I "overdrew" my account to provide some experience with a negative balance. The students liked the unit, I think, because they were familiar with the business establishments who were getting paid. They probably were also interested in the way their teacher spent his money. I was willing to sacrifice my privacy on financial matters if students would work in class.

Other suggestions worthy of consideration are:

- Have a planned maintenance program.
- Use non-verbal approaches as much as possible.
- Develop a classroom environment which will help students respect themselves and their classmates.
- Use a laboratory approach to instruction.

COMMENTS CONCERNING A SCHOOL'S TOTAL MATHEMATICS PROGRAM

Every school should have mathematics classes in which a student can achieve success and yet be challenged. Of course, this should apply to low achievers as well as other students. There should be a minimum number of courses in the school solely for low achievers, and in most of these the students should be prepared to take courses in the regular sequence. However, every effort should be made to be certain that students have the necessary entry behaviors before enrolling in a regular course. The purpose of at least some of the special courses for low achievers should be to help them attain the entry behaviors required for the regular courses. It is cruel to place students in classes where the chances of failure are almost certain.

Classes for low achievers should not have an enrollment of more than 20 students. Many of these students do need, at least part of the time, individualized instruction administered individually or in small groups. I would keep these groups small even though the enrollment in some of the classes of more capable students might then become as high as 35 or 40 students.

Classes for low achievers should be provided at several different levels. In general, students should be directed into these classes as soon as deficiencies are detected.

In the Eugene Public Schools we have provided students with a variety of regular and mini-courses to suit individual needs of all students. We feel that this move has resulted in no appreciable loss during the past five years in total enrollment in a secondary school's mathematics enrollment. Students are not required to take more than nine years of mathematics. Nevertheless, over 90 percent of the students who graduate do take at least one course beyond the minimum requirement. We feel that the emphasis on courses for low achievers has helped to keep enrollment from falling.

If your school has not made a significant effort to provide courses for low achievers, I would like to challenge you to try. There are more and better materials available today than five or ten years ago. I also think you will find that the effort will be rewarding.

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Secondary School Mathematics from a Piagetian Point of View

A case will be made for rethinking how mathematics is presented in secondary schools in the light of insights which have emerged from Piaget-related research. Practicable alternative approaches to specific topics will be described.

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INTRODUCTION

High school mathematics teachers tend to see students' mathematical backgrounds as very poor and their attitudes towards mathematics as improper, irrational, and negative. On the surface, this seems to be an anomaly in view of the considerable efforts which have been expended in upgrading the elementary and junior high school mathematics programs. But could the "upgrading" have been misguided?

It is the writer's opinion that curricular reforms in school mathematics have encouraged teaching approaches which run counter to natural learning processes.

Piaget's theory of intellectual development clearly describes *natural* or *true* or *living learning* as originating from the *child* and *his own* interests and drives. Too often the child comes to view "school learning" as a process of mastering this or that skill "for the teacher" or "to get by" or "to beat the system", whereas in real life the child learns from active interaction with his environment, doing things that are vitally interesting to *him* and for *his own* welfare. He explores, experiments, and modifies his behavior and conceptions of the world by means of basic self-fulfilling drives.

It is essential that we organize the content and approaches in secondary school mathematics courses so that each student is able to build more effectively on the considerable talents he has already developed pursuing "real life" learning outside the classroom. To do so, we need to have some understanding of the ways in which our students learn and how their learning strategies are affected by their level of intellectual development at any given time.

*CONCRETE VERSUS FORMAL OPERATIONS**

Secondary school students are able, by and large, to operate with mathematical ideas at either a *Concrete operations* level or a *Formal operations* level, or both.

According to Piaget's research, the dominant thought structure for the majority of children in the age range of seven years to eleven or twelve years is that of the *Concrete operations* stage. However, the work of Lovell (and others) with British children suggests that it is only the very brightest children who progress beyond the concrete operations stage by the age of twelve when they are working with mathematical concepts. The majority of the British students do not emerge from the concrete operations stage until age fourteen or fifteen and some never do in the context of mathematical tasks (Lovell, 1971).

However, as will be pointed out later, the pegging of ages to stages is much less important than realizing that children's and adult's thinking patterns do repeatedly progress through stages like those described by Piaget.

What is *concrete-operations-stage* thinking like? As described by Hermine Sinclair (1971), one of Piaget's Genevan colleagues, in the stage of concrete operations,

> ... the child can think in a logically coherent manner about objects that do exist and have real properties and about actions that are possible; he can perform the mental operations involved when asked purely verbal questions and when manipulating objects [pp.5-6].

*For a more complete summary of Piaget's stages of intellectual development and a discussion of implications for elementary and secondary school mathematics education, see Harrison, 1969.

The concrete operation child "... can manipulate and think about real objects but he cannot work with hypothetical entities [Sinclair, 1971, p.6]."

The operations available to the child in this stage of development include *classification* (putting objects together in a class and separating a collection into sub-classes), and *seriation* (ordering things, like numbers, or events in time).

Again as Sinclair (1971) has said:

> These operations are transformations that are reversible, either through annulment (as in the case of adding, annulled by subtracting) or through reciprocity (as in the case of relationships: A is the son of B, B is the father of A) [p.7].

Using their classification skills, concrete operational children can successfully make comparisons between a general set of objects and its subsets; they can determine elements in the interaction of given sets; they can find missing elements in double-entry tables. Their seriation notions enable them to cope successfully with transitivity arguments such as: if $A > B$ and $B > C$, then $A > C$ [p.9].

During the concrete operations stage, the types of reasoning made possible by these operations become more powerful and are applied in more and more difficult contexts, paving the way for much more general *formal operations*.

As Lovell (1971) has indicated, "from around 12 years of age in the brightest pupils and from 14 to 15 years in ordinary pupils, we see the emergence of formal operational thought [p.7]." This stage is characterized by the development of formal, abstract thought operations with which the adolescent can reason in terms of hypotheses and not only in terms of objects. Prior to this level of development, the child thinks concretely rather than reflectively, dealing with each problem in isolation and not integrating his solutions by means of any general theories from which he could abstract a common principle. In contrast, the adolescent is most interested in theoretical problems and constructing theoretical systems (Piaget, 1968). The adolescent can identify all possible factors relevant to a problem under investigation, and he can form all possible combinations of these factors, one at a time, two at a time, three at a time, and so on. He can form hypotheses, construct experiments to test the hypotheses against reality, and draw conclusions from his findings. He need no longer confine his attention to what is real but can consider hypotheses that may not be true and work out what would follow if they were true. That is to say, in addition to considering what *is*, he can consider what might be. The hypothetico-deductive procedures of mathematics and science have become open to him (Piaget, 1964; Inhelder, 1962; Adler, 1966; Berlyne, 1957).

LEARNING CYCLES

An interesting interpretation of Piagetian theory is the view that the *concrete operations* used by an individual are "concrete" in the sense that they are mental operations involving some system of objects and relations that is *perceived* as real by the person. What is "concrete" is relative to the person's past experience and mental maturity. While the kindergarten child considers the union of two beads with three beads as a concrete operation but the addition of 2 and 3 as not, the introductory algebra student considers 2 + 3 as concrete but not x + y. The student of introductory abstract algebra considers the additive group of integers to be concrete but not so the concept of an abstract group. So the progession goes, and it is evident that "concrete" operations are used not only in the concrete operations stage, in which they are the most advanced operations of which the child is capable, but also at all succeeding levels of learning. In the development of new concepts at any level it is essential to proceed from what the learner perceives as concrete to what to him is abstract (Adler, 1966). Indeed, as Ausubel and Ausubel (1966) said,

> Even though an individual characteristically functions at the abstract level of cognitive development, when he is first introduced to a wholly unfamiliar subject field, he tends initially to function at a concrete-intuitive level [p.407].

A similar point of view has been taken by Dienes (1966) in his postulation that the learning of abstract concepts can be thought of as occurring in cycles which can be regarded as microscopic copies of Piaget's developmental cycle - that is to say, the concrete operations to formal operations cycle (at least) repeats at higher and higher levels of abstract learning.

GENERAL WAYS OF KNOWING

A recurring theme through Lovell's (1971) excellent paper "Intellectual Growth and Understanding Mathematics" is the characterization of Piaget's work on intellectual development as describing the gradual development, in a person, of *general ways of knowing*. General ways of knowing have to be actively constructed by the child through active interaction with his environment. Once a child's experiences make him aware, for example, of the relationship between a class and a sub-class, he never loses that idea in mental health. The quality of a child's *general ways of knowing* determines the manner in which, and the extent to which, he will be able to assimilate any particular knowledge he is exposed to in school settings.

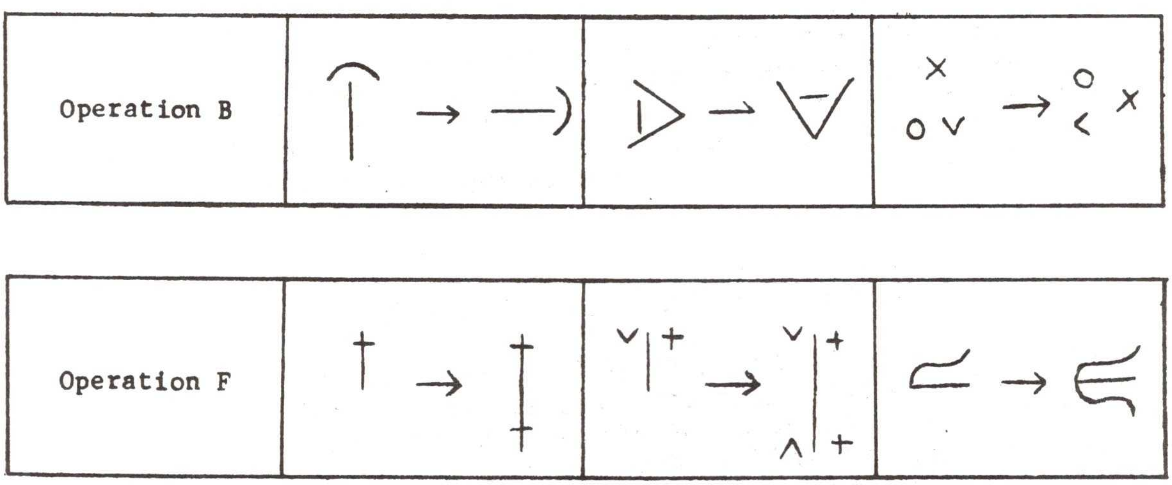
SKEMP'S REFLECTIVE INTELLIGENCE

Richard Skemp (1958), a psychologist at the University of Manchester, has stated that the chief ability required in mathematics at the secondary school level is the ability of the mind to become aware of and to manipulate its own concepts and operations, an ability he calls *reflective intelligence*. Reflective intelligence notions tie in very closely with characterizations of thinking at the formal operations stage. Skemp has devised tests to measure student abilities to reflectively manipulate concepts and operations. Consider, for instance, his operations test. In the first part of the test (SK6, Part I), the subject is required to operate on test figures using operations illustrated on a Demonstration Sheet by means of three examples. In the second part (SK6, Part II), the subject is required to demonstrate "combining", "reversing", and "reversing and combining" operations on test figures. Since to do this test, the subject must have discovered what the ten basic operations in SK6, Part I, are, the operations are explained to the subjects after the administration of Part I but before Part II. There are five "combine ", five "reverse", and five "reverse and combine" items in SK6, Part II. Some sample operations and sample items from Skemp's operations test are reproduced on the following page.

In a study involving 50 fifteen-and sixteen-year-olds, Skemp (1958) found an amazingly high correlation of 0.72 between the students' scores on this test and their scores on a general certificate of education mathematics examination (something like a C.E.E.B. mathematics examination at a younger age).

In 1966, the writer administered Skemp's tests to two classes of students at each of the Grades V through XI levels (a total of 340 students). A plot of their mean scores on Skemp's reflective intelligence test (SK6(2)) by student age levels is made in Figure 1. Overlooking the fourteen-year-olds (most of whom were frantically preparing to write external Grade IX examinations at the time tested and their mean scores were not statistically significantly lower than twelve-and thirteen-year-old means in any case) one could say that, in general, the SK6(2) mean scores increased with increasing age. Such evidence gives further support to the notion of a gradual development with age of more sophisticated levels of thinking or "general ways of knowing".

SK6: DEMONSTRATION SHEET (Sample Operations)*



*Sample Operations and Sample Items from Skemp's SK6 test are reproduced from Harrison, 1967, pp. 312, 313, 319, 320.

SK6: PART II (Sample Items)

In Part II, the problem is to combine the operations on the Demonstration Sheet, or to do them in reverse, or both. When combining operations, they are to be done in the order given (that is, "Combine C and G" means "Do Operation C first and then do Operation G.").

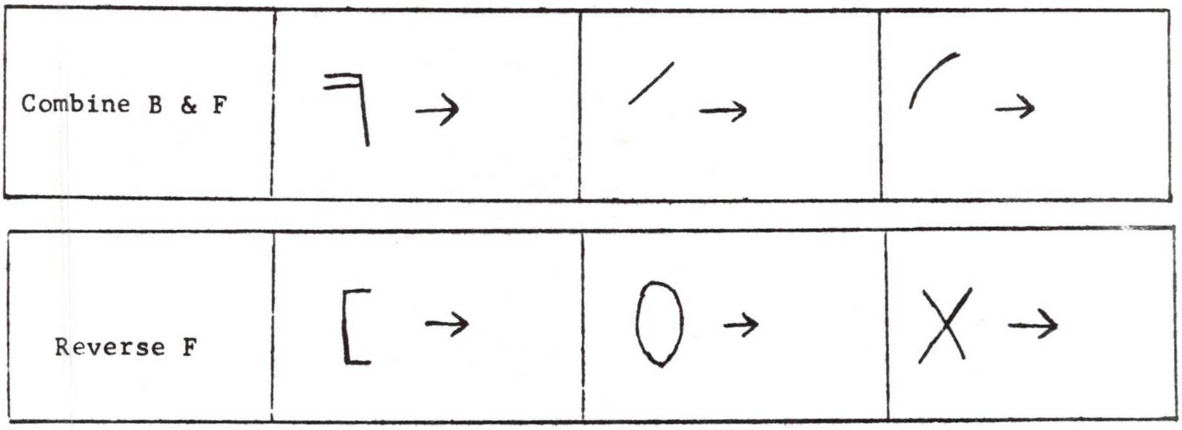
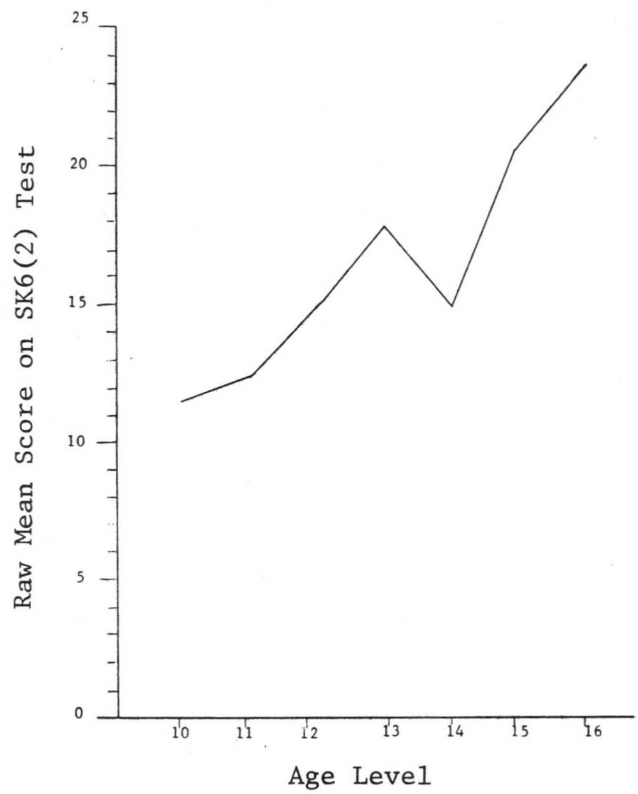


Figure 1
THE RELATIONSHIP BETWEEN MEAN SK6(2) SCORE
AND AGE LEVEL OF STUDENTS TESTED*



*Reproduced from Harrison, 1967, p. 225.

MATH CONCEPTS STUDIES

Lovell (1971) has gathered evidence from numerous research studies to support his contention "... that it is the development of ... general ways of knowing which determines the manner in which taught material is understood [p.10]."

For instance, many studies with British and American pupils have

> ... confirmed that apart from very able 12-year-olds, it is from the beginning of junior high school onwards - the actual age depending on the ability of the pupil - that facility is acquired in handling metric proportion [for example, constructing a rectangle similar to but larger than a model]. Many pupils may not be able to do this until 14 to 15 years of age and some never [Lovell, 1971, p.8].

For example, in one study cited by Lovell it has been found that it was not until the onset of formal operational thought, around 14 years of age, that the majority of British students tested "... were able to dissociate, completely, area and perimeter of square/rectangle, and realize that under certain changes area is conserved and not perimeter, while under other changes the reverse is true [Lunzer cited by Lovell, 1971, p.6].

Still another example has arisen from Reynold's investigation of "The Development of the Concept of Mathematical Proof in Abler Pupils", involving students at the equivalent of our Grades VII, IX, and XII. This study showed that answers "... that were characteristic of the concrete-operational stage of thinking appeared regularly, but the answers also indicated an increasing ability to use formal-operational thought with age [Lovell, 1971b, p.77]."

FUNCTION STUDIES

Following up Piaget's earlier studies of the development of the concept of functions involving laws of variation, H.L. Thomas, at Columbia in 1969, explored the understandings of more general mathematical functions that had been attained by very capable Grade VII and VIII students studying Secondary School Mathematics Curriculum Improvement Study (SSMCIS) materials (average age: 13 years; mean I.Q.: 125; very capable in mathematics). In the Grade VII SSMCIS materials, the concept of a function is introduced as a *mapping* of set A to set B, using the word "image" to refer to the object in B assigned to an element of A. The three essential elements of any mapping are described as "... a first set A whose members are assigned images, a second set B from which the images are selected, and a rule or process which assigns to each element of the first set exactly one element of the second set [Thomas, 1969, pp. 25-26]." In the Grade VII SSMCIS materials, arrow diagrams, rules, ordered pairs, and graphs are used in treating mappings, composition of mappings, inverses, translations and dilations, all in the context of developing the concept of function.

Thomas (1969) administered written function task tests to 201 Grade VII and VIII SSMCIS students and carried out detailed interviews with 20 selected students to assess their grasp of the notions about functions to which they had been exposed. Analyzing the responses, Thomas identified four stages in the development of the concept of a function which will be described later. Of the 201 written test subjects, 55 (27 percent) were rated as having attained an understanding of the function concept at the two highest stages, while 164 (82 percent) could be said to have attained a minimal concept of function. Thomas' (1971) assessment of these results (even though the individual interviews were more encouraging; 13 out of 20 showed mastery of the basic concept of a function as a special relation) was that

> It was ... a shock to this investigator to find that, in a group of students who had supposedly been carefully introduced to the concept of function, many could not distinguish functions from non-functions in simple and concrete situations. At the same time these students could carry out many of the processes associated with the function concept.
>
> One might speculate on this basis as to whether students should be allowed to work with the processes associated with function and only later learn to discriminate sharply those objects that are functions. This has, indeed, been a traditional route. Current thinking, however, runs counter to this approach [p.7].

Orton (1970), at the University of Leeds, carried out a cross-sectional study of the development of the concept of a function by individually interviewing 72 subjects ranging in age from 12 to 17; eight boys and eight girls in the upper half of the ability range in mathematics from each of the forms equivalent to our Grades VIII through XI, and eight very select mathematics students from the equivalent of our Grade XII. By Grade VIII these students had a background of sets, operations on sets, ordered pairs in various contexts, and graphs of ordered pairs. Beginning in Grade VIII, they were introduced to relations by means of arrow diagrams and mappings illustrating such relations as "is a brother of". Domain and range were defined and then a *function* was defined as a relation in which each member of the domain has only one image. Then graphing of functions was covered, followed by inverse functions and linear and non-linear functions (School Mathematics Project, Book 2, 1966, pp. 153-170). In each successive grade, relations and functions were repeatedly worked with.

Through tasks which had to be completed, the students were required to recognize functions, distinguish between functions and relations, and pick out the domain, range, and set of images in a wide variety of situations in which the relations considered were described by means of arrow diagrams, graphs, ordered pairs, tables and equations. A sample Orton function task employing an arrow diagram is reproduced on the following page.

Based on Thomas' stages and the information from his own interviews, Orton described the following stages in the development of the concept of a function.

STAGE I

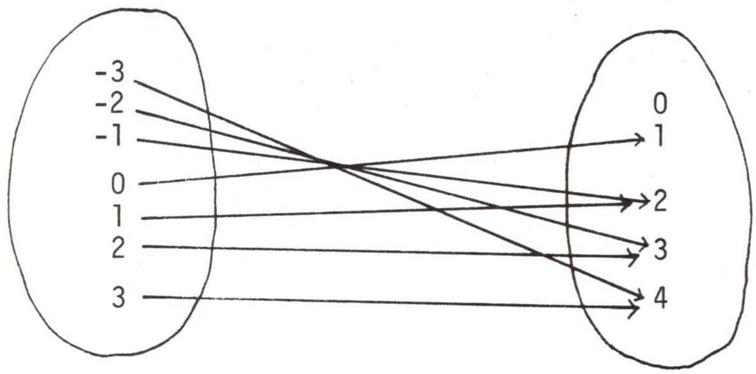
- concrete, intuitive
- can handle processes when arithmetic, or in arrow diagram or table
- concept of function as specific type of relation not mastered
- limited extension of notions in ordered pair graphs

STAGE II

- basic criterion for relation to be a function still not mastered
- good grasp of relational aspects of function concept in that able to find images, pre-images, sets of images, and domain.

SAMPLE ORTON FUNCTION TASK*

1. Study this arrow diagram for a relation which maps
   {-3, -2, -1, 0, 1, 2, 3} into {0, 1, 2, 3, 4}



(i) Write down each image of 2.

(ii) Write down each number that has 2 as its image.

(iii) Write down the domain for this relation.

(iv) Write down the set if images.

(v) Write down the range for this relation.

(vi) Is this relation a function?

STAGE III

- can identify whether a relation is a function or not in several types of representation
- mastery of basic concept of function
- care not always taken to check uniqueness of images or correct domain for inve

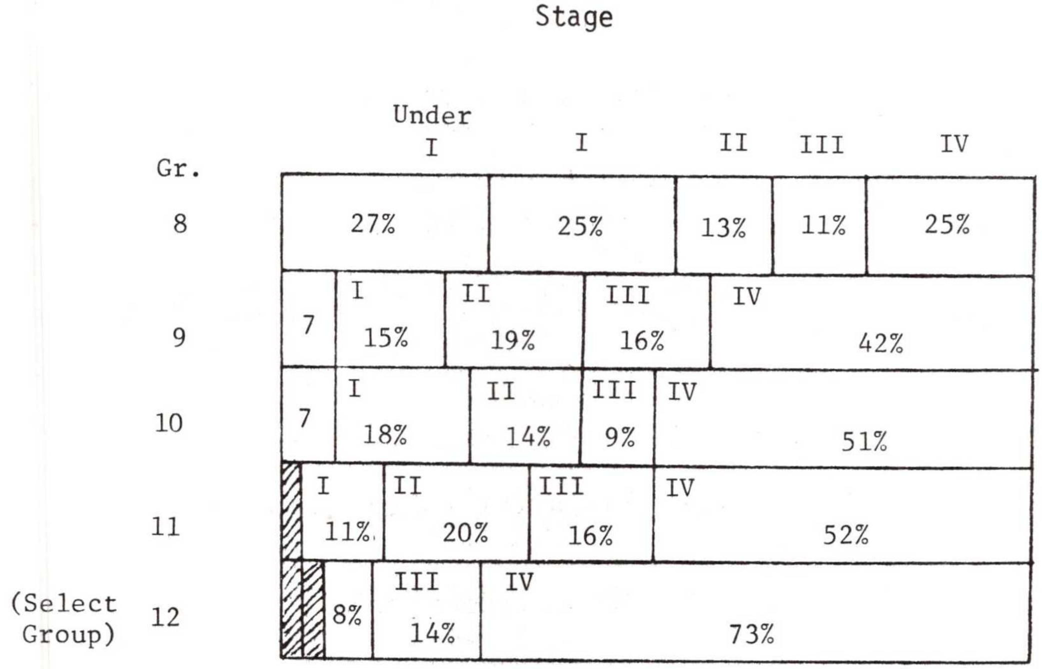
STAGE IV

- mastery of basic concept of function to greater degree of generality than in Stage III
- all representations of relations and their inverses classified as functions or not with precise analysis of the uniqueness criterion [Lovell, 1971a, pp. 17-

*Lovell, 1971a, pp. 25-26.

Figure 2 contains a summary of Orton's findings:

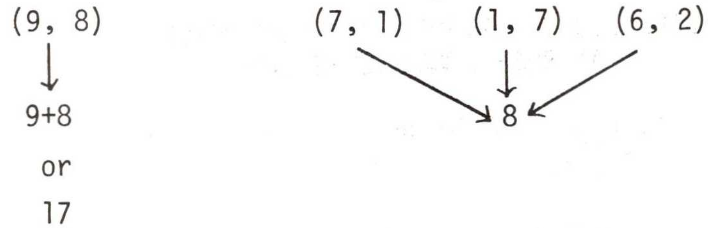
Figure 2
LEEDS STUDY (ORTON): % OF RESPONSES
AT EACH STAGE BY GRADE*



The percentage of responses at the various stages at each grade level certainly supports Orton's statement that "The growth of understanding of the concept of a function takes place slowly and over a long period of time [Orton, 1970, p. 121]". At least this certainly seems to be the case in the age range sampled when the concept of a function is embodied in a fairly abstract, concise definition.

Bernice Andersen, an M.Ed. student at the University of Calgary, replicated Orton's study with 72 Calgary junior high and senior high school students. Six boys and six girls at each of the grade levels VII through XII with ability from average to above average were interviewed using Orton's tasks with minor modifications. Beginning in Grade VII, these students were given an intuitive background for the concept of function by working with sets, ordered pairs, and graphical representations of ordered pairs. They were also given, in Grade VII only, a very brief intuitive introduction to the notion of a function without the term being mentioned in the context of a very limited number of examples of one-to-one and many-to-one mappings such as these:

*Lovell, 1971a, p. 19.



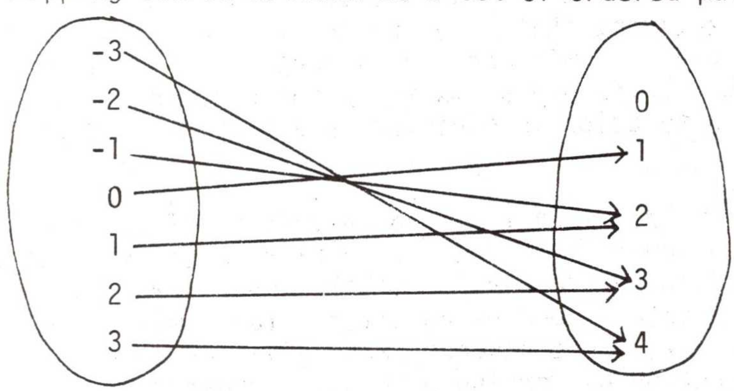
Functions, as such, did not come up again until they were formally defined in Grade XI after a unit on relations. In Grade XI the definition presented to students was:

> A function on a set A is a relation on A such that for every element of the domain there corresponds a unique element of the range [Beesack, 1966, p. 67].

Since the Calgary students through Grade X had not been introduced to the terms "function", "relation", "domain", or "range", Orton's tasks were reworded in terms of "mapping", "set of ordered pairs", "set of first components", and "set of second components" as in the redraft of question 1, which follows.

Figure 1
SAMPLE MODIFIED ORTON FUNCTION TASK*

1. Study the arrow diagram below. The arrows indicate a mapping of {-3, -2, -1, 0, 1, 2, 3} into {0, 1, 2, 3, 4}
   The mapping can be written as a set of ordered pairs.



Refer to Figure 1

(i) Write every ordered pair that has 2 as its first component.

(ii) Write very ordered pair that has 2 as its second component.

(iii) Tabulate the set of all first components.

---

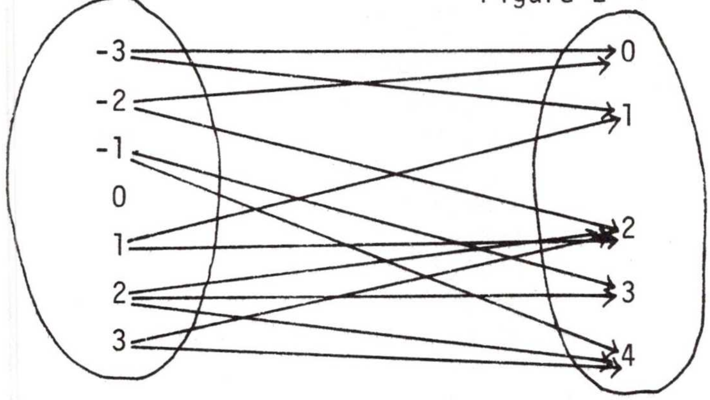
*Andersen, 1971, pp. 83-84.

(iv) Examine the set of second components illustrated by the arrow diagram. Tabulate the set of all second components that *are paired* with members of the other set.

(v) Tabulate the set of all the numbers in Figure 1 that could have been used as second components.

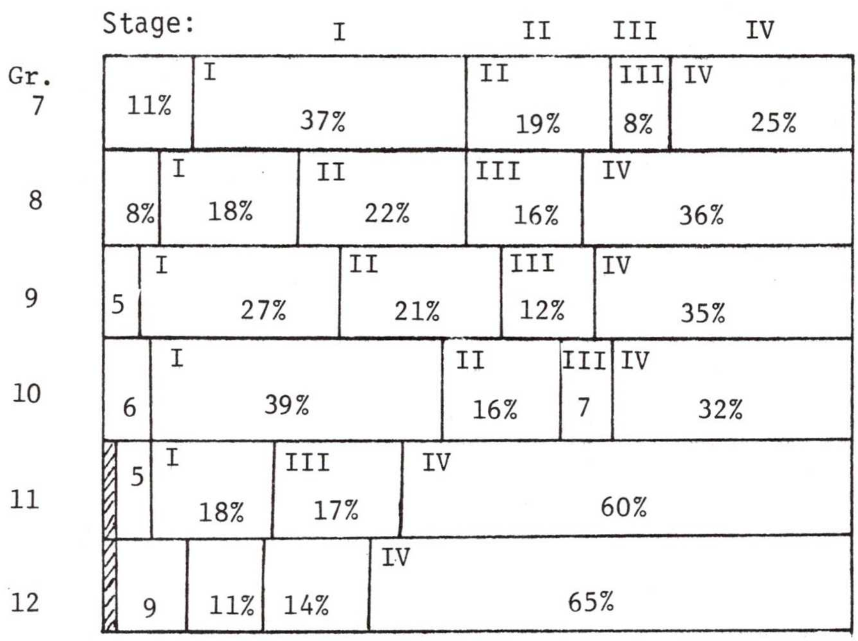
(vi) Is the set of ordered pairs indicated by Figure 2 a mapping? Discuss.

Figure 2



Orton's interview procedures, scoring and assignment of responses to stages were closely followed in the Calgary study. Figure 3 summarizes the results.

Figure 3
CALGARY STUDY (ANDERSEN): % OF RESPONSES
AT EACH STAGE BY GRADE*

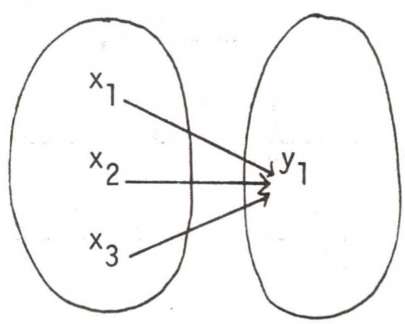


*Andersen, 1971, p. 31.

Comparing the results of the Calgary study with those of the Leeds study, focussing on the percentages of responses at the Stage IV level, it appears that the Calgary students in Grades VII and VIII (who were not introduced to a formal definition of function at all, except in the context of the modified Orton tasks) were able to sort out the function notions in Orton's tasks just as well as the Leeds eighth graders who had been "taught" a formal definition of function. There is no marked increase in the percentage of Stage IV responses in the Calgary results until Grade XI, whereas the Leeds results show a gradual growth under repeated coverage of function properties coupled with the formal definition. It is in Grade XI in Calgary schools that the formal definition of a function has been given in the past, and, as can be seen, the percentages of Stage IV responses at the Grade XI level for Calgary and Leeds students are very similar. Keeping in mind that the twelfth-grade-equivalent Leeds students were a very select group of mathematics majors while the Calgary group was not, the Grade XII Calgary results also appear comparable to those of the Leeds study. In fact, statistical comparisons between the Calgary and Leeds mean function task scores across the grade levels showed no significant differences. Admittedly, comparisons such as the foregoing are fraught with all kinds of pitfalls, but considering that, as far as could be determined, the Calgary sample was certainly not more capable academically than the Leeds group, the patterns in the findings do seem to suggest that the three years of formal work with functions prior to Grade XI might not be time well spent. Perhaps a better use of the time would be to have students explore function ideas in very concrete contexts (such as those suggested later in this paper) building a firm intuitive foundation for the later formal definition.

Many of Orton's Grade VIII students had not appreciated the basic definition of function and wanted to define function as a relation which produced a pattern when plotted as an ordered pair graph. Accordingly, he recommended that, if it is desired to use the modern definition of function, functions involving proportionality, which produce a particular kind of graphical pattern, should not be introduced too soon as pattern confuses the issue if it is mentioned before the basic definition of function is appreciated. (Interestingly enough, the Calgary Grade IX students study a section on graphs of formulas involving direct and inverse variation.)

Orton also found that many students in all of the year groups (whose notions of functions were in terms of one-to-one or many-to-one relations) confused many-to-one and one-to-many.



Some students saw the above many-to-one relation as one-to-many because there was only one arrow leaving each member of the domain, but *many* arriving at a single image. Because of this common confusion, Orton (1970) says that "it is better to attempt to define a function in terms of uniqueness of images of members of the domain, as, for example [in the routine way favored by some subjects], that in an arrow diagram, [of a function] only one arrow leaves each member of the domain [p. 140]."

The Grade VIII and IX students were generally not able to interpret the graphical representations of relations with any confidence. Difficulties were frequently encountered in the finding of images for given pre-images and, vice-versa, finding the domain and range, and converting the graph of a relation into an arrow diagram or set of ordered pairs. Orton (1970) hypothesized that " ... in the early stages of the acquisition of the conept of a function, and the concept of an (x,y) graph, a situation which involves both concepts is too difficult [p. 142]."

In discussing tasks involving the differences between a relation and a function, Orton (1970) made the following comment:

> The unsatisfactory nature of responses, the number of subjects who thought there was no difference between a relation and a function, the number who thought there was no connection at all between a relation and a function [especially at the Grade VIII and IX level], must be considered, from a teaching point of view, to leave a great deal to be desired. If functions are to be appreciated through a study of sets and relations, and then at the end of such an introduction no clear idea of the connection between a relation and a function is held, then some of the point of the approach is lost [p. 142].

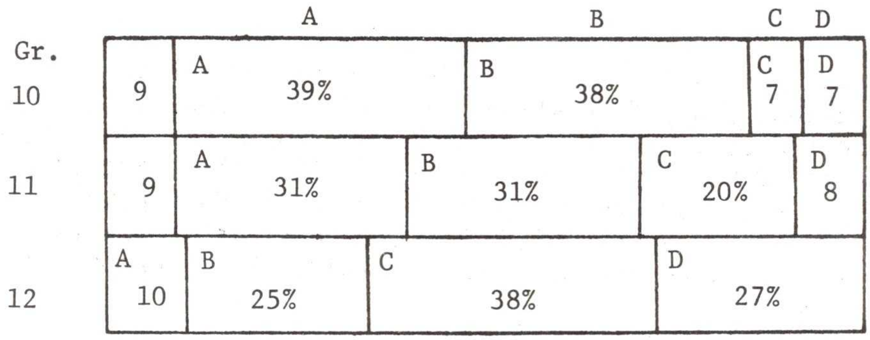
The main reason for this seemed to stem from the fact that the subtle distinction between "relation" and "relationship" was not appreciated by the younger subjects.

Many excellent insightful further observations about the difficulties encountered by students in studying functions are included in the appendix of Orton's (1970) thesis and in Lovell's (1971a) description of Orton's study. Any teacher introducing or working with functions should at least read Lovell's description of Orton's study.

The Grades X, XI, and XII students in Orton's study had been taught *composition of functions*, and they were interviewed further using tasks dealing with composition of functions and inverses. The tasks included f-notation, discrete domains, and equations. The percentage distribution of responses by grade and stage is recorded in Figure 4.

Figure 4

COMPOSITION OF FUNCTIONS
% DISTRIBUTION OF RESPONSES BY GRADE AND STAGE*



Orton (1970) interpreted the above findings in the following way: "It appears ... that the ideas associated with compositions of relations and functions and their inverses ... are not generally understood by other than the most able subjects in the [eleventh and twelfth years] [pp. 132-133]." This even after better than two years of working with functions. So, again, a gradual growth pattern is indicated when a new abstract concept is being learned. No matter how well a difficult concept is taught, we have to learn to allow for this gradual growing awareness in the student's mind about how everything fits together.

SMP, BOOKS A TO H

It is encouraging to note that an alternative version of the School Mathematics Project (SMP) series (namely, SMP Books A to H), aimed at a less select group than SMP Books 1 to 5 (which have an early formal approach to function much like that described in Orton's study), has been produced. In Grade VII in this new series, intuitive notions about relations are introduced, leading gradually into first notions about mathematical relations. This is followed by graphs of relations mapping diagrams, arrow diagrams and inverse mappings in Grade VIII. As the teacher's guide notes, the words domain, codomain and range are purposely not used at this stage since many pupils find them confusing. A *mapping* is described as " ... a special kind of relation in which each member of the starting set is related to exactly one member of the finishing set [SMP, *Book D*, 1970, p. T230]." The authors have decided in the SMP A to H series to keep the mathematical language as simple as possible; hence the use of the word function is avoided. There is apparently a very deliberate de-emphasis on verbal precision in the SMP A to H series. One cannot help feeling that the trend away from rigor toward more emphasis on preparatory, intuitive experiences at the junior high school is very healthy and certainly seems to be supported by the kinds of research summarized on the preceding pages.

*Lovell, 1971a, p. 19.

MATHEMATICS CURRICULA

The results of studies such as those to which we have referred have inescapable implications for designers of mathematics curricula. As Beilin (1971) has pointed out:

> Mathematicians who choose to teach a sequence of mathematical concepts and functions on purely *a priori* bases may encounter great difficulty having these concepts learned. Logical relations are not inevitably paralleled by psychological relations. Unfortunately, little effort has been expended in testing the relations between the conceptual systems of mathematics and the cognitive system of the child except in the most limited of circumstances [p. 118].

Indeed, it has been an all-too-common tendency for textbook authors and teachers to try to "do the whole job" in teaching an abstract concept at first exposure, when the majority of the students are really not intellectually mature enough to effectively assimilate the ideas.

Again, a warning from Beilin (which has also been given by Skemp and Lovell): "When the mathematical idea to be learned depends on a level of logical thought beyond that which the child possesses, the idea is either partially learned or learned with much difficulty and his grip on the idea is tenuous [Lovell (citing Beilin), 1971, p. 3]."

The frustrating thing about students who don't really grasp the basic patterns and ideas in the mathematics they are taught is that they learn by rote enough of what they think we expect, enough to get by on an exam, say, but they do not build the intuitive insights and understandings necessary for progress to more and more abstract ideas. The frustrating thing about teachers, especially those with strong mathematics backgrounds, is that when they find that a student is confused, they explain the idea in increasingly tidier and more abstract terms, which the student is unable to assimilate. The teacher, having the concept firmly in mind, has difficulty imagining what it would be like to assess the situation without the benefit of the concept.

In Alberta schools we have had "first generation" modern mathematics textbooks in the junior high schools with "transitional" and, more recently, "second generation" modern mathematics textbooks in the senior high school programs. A somewhat incongruous result is that the treatment given a particular topic in the senior high school text is often less rigorous and much easier to understand than the treatment of the same topic in junior high.

For example, a thorough coverage of exponents and radicals occurs in Grades VIII and IX and then re-occurs in Grade X with virtually no additional sophistication and a somewhat more straightforward approach. The Grade X teachers say they have to reteach the topic from scratch. Why?

The writer was in a Grade IX class recently in which a girl, who had transferred in from another province, was having trouble deciding what to do to simplify

$$4\sqrt{2} - \sqrt{2}$$

She was asked what kind of mathematics she had been studying in the other province, to which she replied, "modern algebra". So, the writer said, "then you know how to simplify 4x - x." She wrote 3x immediately. Then the writer rewrote $4\sqrt{2} - \sqrt{2}$ under the above expression, but she saw no connection, she just shook her head and looked bewildered.

DAVIS' APPROACH TO VARIABLES, RELATIONS AND FUNCTIONS

Our Grade IX students would be better off if they had early experiences with placeholders and variables in the way Davis (1964) approaches them in the Madison Project materials -- not just

$$2 \times \square = 5$$

but all the interesting, fun things that can be done with placeholders and functions, such as

(a) Nora's Secrets

Can you find the truth set for the open sentence:

$$(\square \times \square) - (5 \times \square) + 6 = 0 \ ?$$

"Nora says she knows two secrets about this kind of equation. Do you know what she means [Davis, 1967, p. 112]?"

(b) Guessing Functions

Davis has successfully led Grade V children to develop "finite differences" strategies for coming up with rules that would generate the following tables of values. The "differences" are shown and the rules so discovered are written below the tables.

$\square$	$\triangle$	differences
0	2	
		) 3
1	5	
		) 3
2	8	
		) 3
3	11	
		) 3
4	14	

$$(3 \times \square) + 2 = \triangle$$

$\square$	$\triangle$	first differences	second differences
0	3		
		) 1	
1	4		) 2
		) 3	
2	7		) 2
		) 5	
3	12		) 2
		) 7	
4	19		

$$(\square \times \square) + 3 = \triangle$$

Not only do children exposed to Madison Project materials come up with rules for tables of values of linear, quadratic and exponential functions, but they also derive rules for the patterns they see in graphical representation of these functions and explore all of the very interesting relationships and patterns that exist in the various modes in which functions can be represented - for example, a rule (or formula), table of values (or tabulation of ordered pairs), or graph. Sigurdson and Johnston (1968, 1970) provide an excellent application of this kind of approach at the Grade XI level.

(c) Madison Project "Independent Exploration Material" (often referred to as Davis' "shoeboxes"). These shoeboxes contain materials and instruction cards designed to produce data for graphs and "function guessing".

(d) Approaching functions with Cuisenaire Rods as in Davis' "Centimeter Blocks" shoebox and in the ways described by Gail Lowe (1972) (e.g., using the white rod as a stamp to cover: individual rods, rods placed end-to-end ("trains"), rods placed side-by-side, side-by-side and staggered, "pyramids", etc.).

The preceding are only a very small sample of the rich sources of ideas and materials for enabling children to explore concepts like functional relationships in a very concrete and interesting way (an excellent annotated compilation of manipulative materials currently available can be found in *Fabric of Mathematics*, Laycock and Watson, 1972).

Although this paper has focused on functions, similar cases could be made for the development of concepts such as mathematical proof (Sample Research: Reynolds, 1967; Lovell, 1971b; Sample Approaches: Davis, 1967), and proportionality and probability (Lovell, 1971c).

SUMMARY

One interpretation of the presently available evidence from classroom research conducted along Piagetian lines is that, at least until the end of junior high school for most students (and even longer for some), the main focus in presenting mathematics in schools should be on providing rich concrete experiences as a foundation for meaningful formalizations in the high school years. Children can certainly begin working with functions, for example, in elementary school but in a very concrete context, and they should have frequent access to concrete embodiments of functional relationships until each child *himself* is ready to progress to a formalized, generalized, abstract conception of what principles are embodied in the many related concrete experiences he has had.

If the reader has any doubt as to what can be accomplished under a student-oriented, active-learning approach, Davis' *Experimental course report: Grade nine* (Davis, 1964a) would make very interesting reading indeed.

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What Really are the Basics?

Rather than to continue the "redipping" process to teach the fundamentals, the presenter will attempt to show and discuss other strategies that have worked with children in levels 3-12.

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If we are to look upon education as building whole people so that they can function in a democratic society, we must re-examine what is fundamental. I suppose if we were to ask almost anyone what the fundamentals are, he would cite the three R's meaning reading, writing and ciphering. This is in keeping with the notion that the school is the place for book learning, that the rest of the organism does not and should not grow in school but should do it somewhere else.

With a scientific approach to education we obviously need a new definition of what is fundamental. It seems to me that *anything is fundamental without which the organism cannot thrive*. I do not say "cannot live", because organisms of all sorts are extremely tenacious of life. Just being alive means very little when compared to becoming a thriving, adequate human being. In fact, there are people who have never developed any psychological selves, others who through brain damage have lost what they had but who live on in a physical sense.

With this definition we can see that the so-called three R's are not fundamental. This has to be granted because we know so many people who have thrived without one or all of them. There are many fine human beings in the world who do not have them. For example, a large percentage of our population, including many college graduates, have almost no skill in mathematics. These people are considered successful because they have lived well. They often have raised fine families, met economic obligations, extended love to many people.

The three R's are skills which facilitate living in our present society. They are, of course, desirable, and we should teach them as much and as well as we can, but they are not necessarily essential to the good life, and we should not teach them so insistently and so aggressively that we diminish the individual's ability to grow toward humanness.

WHAT ARE THE FUNDAMENTALS?

Following is a list of six things which seem to be essential if the human organism is to thrive. Undoubtedly, the list could be extended. Since some of these must and do come at the same time, it is difficult to establish a proper order for them. Some have to be provided in early infancy, but all are continuously essential through life and, therefore, must be provided by the school for that time when the school is a prominent part of life.

1. The first fundamental is *other people.* The infant is born with the equipment for becoming human. Nature provides that no human being shall be deprived of at least one other person. The human infant is born completely helpless and has to be cared for by another human being, usually his mother, if he is to survive. His undeveloped cortex can be built in no other way. This is the meaning of the long infancy in the human species. The lower animals are, at least in most cases, not so dependent on their mothers. They do not need this cortical development because they have little or no cortex and in many instances are able to fend for themselves immediately. The baby chick, for example, can start pecking for its food almost as soon as it is out of the shell. In fact, it comes equipped with the instinct to peck. The lower animals seem to come into life with a good many instincts so that they can survive without assistance. The human infant comes equipped with almost no instincts at all, seemingly to compel others to provide for it. Thus, nature provides another person for the beginning of life at least. This need is continuous throughout life, because the human potentiality for psychological growth is continuous.

2. In order to have other people, we need *good communication* between at least one adult and the very young. This seems self-evident, but the whole business of communication is more complicated than has ordinarily been supposed. The way in which the mother communicates with the babe and in which other adults do so is of greatest importance. Communication is not a one-way affair. Too often it has been believed that to just send messages constituted communication. For example, the radio and television experts call themselves communications people, but all they do is broadcast, and they may have no receivers at all. Teachers often labor under the delusion that to send, to lecture, to tell, is to communicate. There never is any communication until what is sent has been received, and the condition of the receiver is more important than what is sent.

All of us, beginning at birth and throughout life, look out upon our surroundings to see whether those about us seem likely to help or threaten us. When the human environment looks facilitating, we tend to open up and to be receptive, that is, more accessible to communications. If the human environment seems threatening, we tend to withdraw, to build barriers for our protection. We become less receptive, less accessible. Sometimes in early infancy the avenues of communication become entirely closed, and such children are then called autistic. These children, due to their view of their world, have cut off all communication, and they become mere physical organisms. This is what can happen under extremely adverse circumstances. There are all degrees of communication, from the autistic to the open self. The establishment of facilitating communication is certainly one of the fundamentals without which no organism can thrive.

3. In order to establish communication, to have other people, the human being must have other people in a *loving relationship*. If he is to develop into a person who can maintain human relationship, he must be a loving person. By some strange device, nature has arranged for mother love which often "passeth all understanding". Of course, this does not always apply, because the mother sometimes has been damaged psychologically and may be so neurotic as to reject her own offspring. Ordinarily, however, this is not the case, and in normal instances love is automatically provided at the start of life.

If for some reason the infant is denied love in the beginning, he builds not love but hostility. This leads to isolation and deprivation of the stuff out of which adequate humans are built. This stuff, of course, is other people, and those who are driven into isolation are deprived of that which they must have if they are to be really human.

Some may be bothered by the use of the word "love" in this regard. Perhaps this is because the word has been used in a romantic sense for so long. This is only one of many meanings given in dictionaries. It is the only word strong enough to express the acceptance needed between two human beings, or among all, for the proper development of human personality. This is the meaning of the admonition "Thou shalt love thy neighbor as thyself [Matthew 22:39]."

Christ was not the only religious leader who advocated love among humans as essential to the good life. This has been done by religious leaders, poets and seers for centuries. Christ's words came many centuries before it was known that man even had a cortex, much less how it had to be developed. It is curious how scientific study has repeatedly verified (or found base for) these great teachings. It is curious, too, how some people can attend churches and synagogues on the sabbath to hear about the importance of love in human relationships and then contend that love is not needed by youth, that what they need is coercion, that those who do not yield to coercion should then be rejected.

4. A fourth fundamental is that each person must have a *workable concept of self*. The word "workable" is used here after some thought and searching. One needs to think well enough of himself so that he can operate. Perhaps none of us escapes the rigors of life without some damage to his concept of his own self. Abraham Maslow describes those who he thinks have not been damaged as "self-actualizing" but says he can name only a very few. Therefore, conceding that almost no one is going to develop unscathed in this regard, we still must have

people who think well enough of themselves so that they can face the vicissitudes of life. When a person does not think well of himself, he is crippled and cannot do anything. Nobody can do anything unless he thinks he can.

This may seem, at first, to support the show-off, the egotist. But such a person behaves in this way because he feels his inadequacy keenly and is trying to cover up by aggressive action. He comes as far from the mark of good human relations as do those who withdraw.

Workable concepts of self are built by the life good to live, in full love and acceptance of one's fellows. The unloved and unwanted become crippled and cannot thrive.

5. Every human being, in order to develop his full potential, must have *freedom*. This requirement is evidently built into the organism. The effort made by humans to achieve freedom is well known from studies of the history of man from the very beginning. While many people have lived and died in various forms of slavery, the masters have always had to be repressive and have lived in fear that the spirits of those they oppressed would break out in reprisal.

This great need is grossly misunderstood by many. It does not mean that anybody in these times has the right to do just as he pleases. The very fact of our living so closely together naturally limits this right. The right to do just as he pleases could only be achieved by a hermit. But one of the fundamentals is the need for other people. In order to have other people, the individual must behave in such a way that, while he has the choices of a free man, other people will not be repelled. This is freedom within the social scene. It is the product of cooperative living.

In order to live so that one can have the benefit of other people, one has to give up certain minor freedoms. Giving up minor freedoms enables one to achieve freedom on a higher level. For example, I am not permitted to leave the street in front of my house unpaved. At the same time, no one else may do this either. This way we do not have unpaved spaces in our streets. Having given up that freedom, I achieve the freedom of driving along streets without being stopped by mudholes which might exist if everyone were free to decide for himself.

There is enough freedom within the social scene - within cooperative living - to provide for the making of choices. We do not have to accept either autocracy or anarchy. While the need for freedom seems to be present in all humans, the capacity to exercise it within the social scene has to be learned. It can be learned in an atmosphere of love, democracy, cooperation.

6. Every person needs the chance to be *creative*. This does not mean that everyone should paint a picture or write a symphony. Creativity occurs whenever a person contrives a new way out of a unique dilemma. It is simply meeting the problems of living and inventing new ways to solve them. Most of us do this every day to a certain extent. Creativity is the growing edge of learning and living and is essential to any real life fulfillment. It can only take place in an atmosphere of freedom. In fact, freedom begets creativity; that is, when one is free, he will naturally contrive. When he contrives, he is fulfilled. When he is fulfilled, he may be said to thrive.

CONCLUSION

Here, then, are the fundamentals, at least in part. There are probably others, but these are the ones which occur to me. If our young people have these, they will thrive.

To those who still cling to the three R's as fundamentals, I would say that the three R's are tools good to have, but that they alone never saved a boy from becoming delinquent. The most urgent needs of our youths go much deeper than the three R's. Indeed, the three R's cannot even be learned at all unless at least part of the above fundamentals are met. Some of these needs have to be met in infancy, and there is little the school can do about them except to rear a generation of people who will not reject their own young. But most of these needs are continuous throughout life and can be provided by the school.

We will never solve the problem of the three R's, which seems to vex so many people, until we learn to live with our young in such a way that they can be open to receive such matters.

PRINTED AT
BARNETT HOUSE