C. STATISTICS

Permutations and Combinations *(Loose)*

[*This is a topic in the Mathematics 30 text so Grade XII would be the intended grade level. However, as these lessons cover only basic permutations and combinations, they could, in my opinion, be introduced at, say Grade X.*]

Discovery Lesson 1 - Patterns in Arrangements
[*This first lesson could commence after a review of sets and of Cartesian products of sets. It should not be necessary to express the relationship between sets and permutations as many students will likely grasp it after some activity. This work on sets would be the only lead-up necessary, and even this could be optional.*]

OBJECTIVE It is hoped that, by working through the activities, the students will find patterns in arrangements and, using the "blank" method, will be able to solve such problems. It is not necessary for the students to know any general formula, but the better students may discover a law for 'n' objects to be arranged 'r' at a time. However, at the end of the lesson one might mention the n! law to them. In general, then, this lesson will introduce basic permutations.

MATERIALS
- three books (encyclopedias, for example - numbered preferably),
- a number of different colored chips,
- a typed menu with a choice of 2 soups, 3 mix courses, 2 desserts,
- a set of problems on numbered file cards.

PROCEDURE Divide the class into stations by grouping two or three desks together for each. Assign two or three students to each station. They should progress through the cards *in order*, performing the activity suggested, then working through the set of questions which follow it.

FILE CARDS

Card 1 Arrange three books in as many different orders as possible.

Card 2 Suppose there are three slots in which to place books.

___ ___ ___
 1   2   3

a) How many books have you available to put in slot 1?
b) How many choices of books do you have left for slot 2?
c) How many for slot 3?

Do you notice anything? Is there any relationship between the numbers in the blanks and your experimental result?

Card 3 In how many different ways can you and your partner sit in two desks?

Card 4 Represent the desks by two blanks. How many people do you have available to sit in the first desk? How many people are left to fill the second desk? Do you notice anything? How do these blanks compare with your actual result?

Card 5 Take four different colored chips. Arrange them in a pile in as many different color combinations as possible.

Card 6 For the first chip, how many chips do you have to choose from? How many are left to choose from, to put on top of this first one? How many are left now? Have you any choice for the chip on the top? Are there any similarities between these answers and your experimental result?

Card 7 How many different numbers could you form from the digits 4, 8, 9, if no digit is to be repeated (three-digit numbers)?

Card 8 If $\underset{1}{\_\_}\ \underset{2}{\_\_}\ \underset{3}{\_\_}$ represents the different numbers, how many ways can you fill in blank 1? blank 2? blank 3? Do you notice anything? Do any similarities exist?

Card 9 Arrange the material you have collected from your activities in a meaningful way and see if you can discover any pattern.

[*Students will find, hopefully, that one can arrange two objects in 2.1 = 2 ways, three objects in 3.2.1 = 6 ways, four objects in 4.3.2.1 = 24 ways. From this, students should be able to readily say how many ways nine objects can be arranged. If the students have grasped the idea, one could mention n! This simply means, for example, 3! = 3.2.1 = 6 or n! = (n) (n-1) (n-2) down to 1. The students should continue on with the next file cards. They also deal with permutations but the problems are somewhat different and lead to a different discovery.*]

Card 10 Given two digits, 4 and 9, how many two-digit numbers could you make from these if digits can be repeated?

Card 11 Using two blanks to represent the number $\underset{1}{\_\_}\ \underset{2}{\_\_}$, how many digits do you have available to put in blank 1? blank 2? Do you see any relationship between two blanks and the result of your experiment?

Card 12 How many two-digit numbers can you make which are less than 70 but which are also multiples of 5?

Card 13 How many numbers can form the first space in the number? How many in the second space? Do you see any relationship between the numbers and the blanks?

Card 14 You have a menu on your desk. How many different three-course meals could you order from this menu if you were in a restaurant?

Card 15 Let the three courses of the meal be represented by three blanks. $\underset{1}{\_\_}\ \underset{2}{\_\_}\ \underset{3}{\_\_}$. From your previous activities, can you fill in these blanks? Do you see any relationship here?

Card 16 Suppose you are taking a trip to Hawaii. You can make this trip by different routes. You have a choice of two train routes to Vancouver and of three routes to the island: by ship, by plane, by helicopter. How many different types of trips can you make?

Card 17 How many ways can you go to Vancouver? Put this in blank 1. $\underset{1}{\underline{\qquad}}\ \underset{2}{\underline{\qquad}}$ How many to Hawaii? Put this in blank 2. Is there any similarity between your answers and the blanks?

Card 18 You have performed a number of activities now, all of which have been different. Are they similar in any way? Could you solve an arrangement problem now without doing the experiment? Can you describe the "blank method" which you were using in the previous problem?

[*The students could be given a few problems on permutations at this point. By looking at their work, the teacher can determine whether the students have actually discovered the blank method and are using it, or whether they are still solving the problems by experiment. A short review on permutations could follow these problems if desired. Students could proceed immediately to the next lesson or it could be postponed for a couple of days. However, Discovery Lesson 2 should only be used after Discovery Lesson 1 has been covered.*]

Discovery Lesson 2 - Patterns in Selections

[*This lesson is a continuation of lesson 1, therefore, the same materials are needed. However, one more book should be added to the collection. Problems are put on file cards in the same manner as in the previous lesson and students progress through the cards, in order, as before.*]

OBJECTIVE The objective of this lesson is for students to find patterns in selections, and perhaps more important than this, to see the relationship between permutations and combinations.

FILE CARDS

Card 1 How many different selections of two books at a time can you make from a group of four books?

Card 2 In making the first selection, how many choices do you have? In making the second selection how many books are available? Put these in blanks one and two $\underset{1}{\underline{\qquad}}\ \underset{2}{\underline{\qquad}}$. Are any of these combinations of books the same set of books? How many times do the same groups occur together? What effect does this have on the result of your first two steps, that is, the numbers which you put into the blanks? How do both of these compare to your experiment? Can you describe the relationship? Can you explain why the relationship exists?

Card 3 Take three desks and four people. How many different selections of people can sit down? (To make a group different, one or more people must be different.)

Card 4 How many ways are there of filling the first seat? of filling the second seat? of filling the third? $\underset{1}{\_\_\_}\ \underset{2}{\_\_\_}\ \underset{3}{\_\_\_}$. But are all of these different groups of people? How many times does the same group of people occur together? What effect does this number of times have on the result you obtained in the blanks? Describe what you find.

Card 5 You have three books - how many different ways can you make two selections? How many choices of different books have you?

Card 6 Let two blanks represent the two selections. $\underset{1}{\_\_\_}\ \underset{2}{\_\_\_}$. Can you fill in these blanks? What is the relationship between this result and the result of your actual experiment with the books? Can you explain why the relationship exists?

Card 7 A basketball coach has three good guards - John, Bill and Terry. How many ways can he pick two guards for tonight's game?

Card 8 How many people has he to choose from first? Now, how many has he to choose from? $\underset{1}{\_\_}\ \underset{2}{\_\_}$. Has he chosen the same people more than once? Is this reasonable? Why is there a discrepancy between this answer and your actual result?

Card 9 You have five coins (dime, nickel, quarter, penny, half-dollar). In how many ways can you make a selection of two different coins?

Card 10 You have how many coins to choose from first? Put this in the first blank $\underset{1}{\_\_}\ \underset{2}{\_\_}$. How many do you have to choose from now, for your second choice? Put it in the second blank. Are all of these selections different? If not, how many times does the same pair of coins appear? How does this number and the number obtained in the blanks compare to your experimental result?

Card 11 Can you summarize your findings from the preceding activities? Does there seem to be a pattern? Can you relate it to a general problem?

[*It is not crucial that the students discover a general law (this is mainly a question for the better students). However, all students should be able to solve any of this type of problem. A formula is* <u>*not*</u> *necessary. I think, however, that the good students may come up with a general law for selecting 'n' things 'r' at a time. The main thing it is hoped the student will discover is that, for each selection problem he chooses the objects in order, after which he divides by the number of objects which appear together more than one time. It is also hoped that he will discover the difference between permutations and combinations as he works through the different activities. That is, permutations involve order; combinations involve picking things regardless of order. Discovering combinations and permutations by active learning, rather than by abstract formulas, may result in more meaningful concepts for students.*]

Probability *(Webber)*

DICE

Part A - Individual Experiment

Roll a die 25 times and record the outcomes in the appropriate columns

Tabulation Sheet

Roll	*1*	*2*	*3*	*4*	*5*	*6*
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						
14						
15						
16						
17						
18						
19						
20						
21						
22						
23						
24						
25						
Total						

How many times did a 3 turn up? ____ (This is called the frequency of the event and is represented by f_nA for an event A and n trials of the experiment.)

1. Find: (a) $f_{25}3$ (b) $f_{25}5$ (c) f_{25}(an odd number) (d) f_{25}(a number greater than 3).

2. The relative frequency of an event A is f_nA/n. Find the relative frequencies for the events in question 1, and
   (a) an even number or a number greater than 3,
   (b) an even number and a number greater than 3.

Part B -

Combine your results with those of others in the group. Compile the data on the chart provided.

3. Find the following for the statistics obtained by the group:

   (a) f_n3 (b) f_n5 (c) f_n3/n (d) f_n5/n

   (e) $\frac{f_n \text{ (an odd number)}}{n}$ (f) $\frac{f_n \text{ (a number greater than 3)}}{n}$

4. On the group chart, sketch a bar graph to illustrate the frequency of each of the possible outcomes against the number of tosses. Predict the result if the number of trials is 1000.

Part C - Group Experiment

Appoint one group member to act as statistician in recording the results of the chart provided. Take turns rolling the pair of dice until 100 trials have been completed.

5. Calculate (a) the frequency of a sum of seven
   (b) the frequency of an even sum
   (c) the frequency of an odd sum
   (d) the frequency of a sum of two.

6. Calculate (a) the relative frequency of each of the above
   (b) the relative frequency of a sum which is even
   (c) the relative frequency of a sum which is odd.

7. The relative frequency of an event A is an expression of the probability of event A so that $P(A) = \frac{f_nA}{n}$

   From the group data collected, compute
   (a) the probability of a sum of 6
   (b) the probability of an even sum
   (c) the probability of a sum greater than 6
   (d) the probability of a sum which is 2 or odd.

POKER CHIPS

Part A - Individual Experiment

Draw a chip from the bag. Record its color in the appropriate column on the tabulation sheet below. Return it to the bag. Draw again until 25 draws have been recorded.

Draw	Red	White	Blue
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			
Total			

How many times was a red chip drawn? ____ (This is called the frequency of the event A and n trials of the experiment.)

1. Find: (a) f_{25}red (b) f_{25}red (c) f_{25}white

2. The relative frequency of an event A is $f_n A/n$. Find the relative frequencies of the events in question 1, and
   (a) drawing a red or blue,
   (b) drawing a red or white.

Part B -

Combine your results with those of others in the group. Compile your data on the chart provided.

3. Find the following using the statistics obtained by your group:

   (a) $f_n$red (b) $f_n$blue (c) $f_n$white (d) $\frac{f_n\text{red}}{n}$

   (e) $\frac{f_n\text{white}}{n}$ (f) $\frac{f_n\text{blue}}{n}$

4. On the group chart sketch a bar graph to illustrate the frequencies of each color against the colors. From the graph predict the ratio of the three colors in the bag. If the same ratio is maintained and the bag contains 1000 chips, predict the frequencies (relative) of each outcome.

Part C - Group Experiment

Take turns drawing a chip, recording its color and returning it to the bag until 100 draws have been recorded. (Appoint one group member to act as statistician in recording the results on the chart.)

5. Calculate the frequency of drawing
   (a) a red chip
   (b) a blue chip
   (c) a white chip
   (d) a red or blue chip
   (e) a red or white chip.

6. Calculate the relative frequency of each of the events in question 5.

Repeat the above experiment in Part A but do not return the chip drawn to the bag.

7. Calculate the frequency and relative frequency of the probability as listed in question 5 under the conditions outlined in this project.

8. The relative frequency of an event is an expression of the probability of the event so that $P(A) = \frac{f_nA}{n}$

   From the data, compute:
   (a) the probability of drawing a red chip if the chips are returned after each draw; if the chips are not returned after each draw,
   (b) the probability of a red or blue chip if the chips are returned after each draw; not returned after each draw.

CARDS

Part A - Individual Experiment

Draw a card from the deck. Record its suit and kind on the tabulation sheet below. Return the card to the deck before drawing a second card and repeat the experiment until 25 cards have been drawn and the statistics recorded.

	Heart	Spade	Diamond	Club	Total
2					
3					
4					
5					
6					
7					
8					
9					
10					
Jack					
Queen					
King					
Ace					
Total					

How many times was a ten drawn? ____ How many times was a heart drawn? ____ (This is called the frequency of the event and is represented by $f_n A$ for an event A and n trials of the experiment.

1. Find: (a) $f_{25}6$ (b) f_{25}ace (c) f_{25}club (d) f_{25}spade.

2. The relative frequency of an event A is $f_n A/n$. Find: (a) f_{25}ace/25 (b) f_{25}diamond/25 (c) f_{25}face card/25 (d) f_{25}black card/25.

Part B -

Combine your results with those of others in the group. Compile your data on the chart provided.

3. Find the following from the group statistics: (a) $f_n 6$ (b) $f_n$ace (c) $f_n$club (d) $f_n$spade (e) $f_n$queen (f) $f_n$diamond (g) $\frac{f_n \text{face card}}{n}$ (h) $\frac{f_n \text{red card}}{n}$.

4. On the group chart sketch a bar graph to illustrate the frequency of the suit against the number drawn in that suit.

   (a) What is another way of plotting a graph to illustrate the statistics from your experiment? Is this the only alternative?

   (b) As the number of trials increases indefinitely how does the complexion of the graph change?

   (c) What is the relative frequency of getting the ace of clubs in 10,000 trials?

Part C - Group Experiment

Take turns drawing a card but do not return it to the deck until 25 draws have been made. Total the tallies for the 25 draws without returning the cards. (Appoint one student to act as group statistician in recording results.) Repeat the experiment four times until a total of 100 draws is completed.

5. Find: (a) $f_{100}6$ (b) $f_{100}\text{ace}$ (c) $f_{100}\text{club}$ (d) $f_{100}\text{spade}$

   (e) $\dfrac{f_{100}\text{queen}}{100}$ (f) $\dfrac{f_{100}\text{diamond}}{100}$ (g) $\dfrac{f_{100}\text{face card}}{100}$

   (h) $\dfrac{f_{100}\text{red card}}{100}$

   Compare these with the frequencies of the same events obtained in question 3.

6. The relative frequency of an event A is an expression of the probability of event A so that $P(A) = \dfrac{f_n A}{n}$

   From the data, compute:
   (a) the probability of drawing a red card if the card is not replaced after each draw,
   (b) the probability of drawing a 6 without replacement; with replacement,
   (c) the probability of drawing a heart without replacement; with replacement.

COINS

Part A - Individual Experiment

Toss a coin 25 times and record the outcomes in the appropriate columns on the tablulation sheet provided.

Tabulation Sheet

	Toss	*Heads*	*Tails*
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			
Total			

How many times did a head turn up? _____

(Remember, this is called the frequency of the event and is represented by f_nA for an event A and n trials of the experiment.)

1. The relative frequency of an event A is f_nA/n. Find: (a) $f_{25}H$ (b) $f_{25}T$ (c) $f_{25}H/25$ (d) $f_{25}T/25$.

Part B -

Combine your results with those of others in the group. Compile your data on the chart provided

2. Find the following for the statistics obtained by the group:

   (a) f_nH (b) f_nT (c) f_nH/n (d) f_nT/n.

3. On the group chart sketch a bar graph to illustrate the frequency of heads and tails against the number of tosses. (Use the statistics obtained by the whole group.) Predict the result if the number of trials is 1000.

Part C - Group Experiment

Using three coins of the same kind, have each member toss the coins (all at once). Appoint one student to act as statistician to record 100 trials on the chart provided.

4. Calculate:

   (a) the frequency of 3 H
   (b) the frequency of tails
   (c) the frequency of all three alike
   (d) the frequency of 2H, 1T
   (e) the frequency of 2T, 1H
   (f) the frequency of at least one H
   (g) the frequency of 1H or 1T.

5. Calculate the relative frequency of each of the above.

6. The relative frequency of an event A is an expression of the probability of event A, so that $P(A) = \frac{f_n(A)}{n}$. From the data collected by the group compute:

   (a) probability of a head
   (b) probability of a tail
   (c) probability of a head or tail
   (d) probability of at least one tail
   (e) probability of all three alike.

BIBLIOGRAPHY / REFERENCES

Baersig, Teresa M., O.S.F. "An Analysis of Elementary Functions for High School Students," *The Mathematics Teacher,* April 1972, pp.355-70.

Bedford, Clayton W. "Ski Judge Bias," *The Mathematics Teacher,* May 1972, pp.397-400.

Biggs, Edith E., and James R. Maclean. *Freedom to Learn.* Don Mills, Ontario: Addison-Wesley (Canada) Ltd., 1969.

Johnson, Donovan A. *Probability and Chance.* New York: McGraw-Hill, 1966(o.p.).

Naether, Gottfried E. "The Nonparemetic Approach in Elementary Statistics," *The Mathematics Teacher,* February 1974, pp.123-26.

National Council of Teachers of Mathematics. *Arrangements and Selections.* Washington, D.C., 1966.