D. CALCULUS

D.1 Symmetry

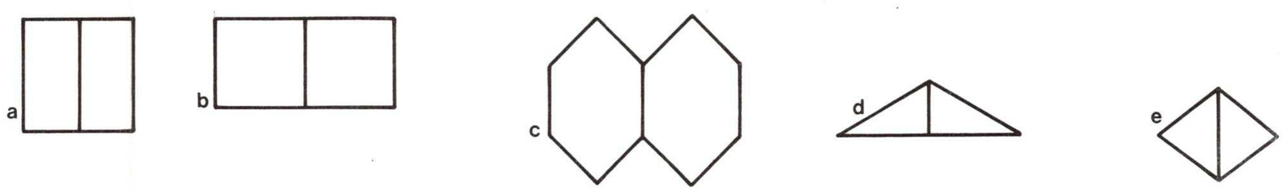
Rediscovery and extension of knowledge of symmetry *(Dyrholm)*

[For individual or small group work. Criterion from Bates, Irwin and Hamilton Teacher's Guide. *"We maintain that open-ended activities are appropriate in the initial stages of developing new concepts. . ." As most high school concepts are not new but extensions of previous concepts, partially-directed activities may be preferred to open-ended activities.]*

MATERIAL Paper and scissors

Cut shapes which you believe to be symmetrical and indicate the line or point with respect to which they are symmetrical. Good for decorating bare bulletin boards. In discussion of figures produced and introduced the terms "bilateral" (line) symmetry and "rotational" (point) symmetry.

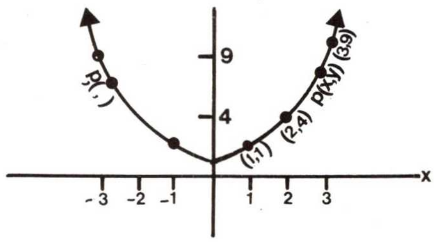
Identify the axis of symmetry for each of the regions below. Compare the areas and perimeters of the regions on each side of the axis. Which figures also possess rotational or point symmetry?



Explore the printed letters of the alphabet to find out which are symmetrical. Can you complete the following chart? (Some letters may fit in more than one category.)

(1) No symmetry	*(2) Symmetrical about vert. axis only*	*(3) Symmetrical about horiz. axis only*	*(4) Symmetrical about both horiz. and vert. axis*	*(5) Symmetrical about a point (rotational sym.)*
Q	T	B	H	H S

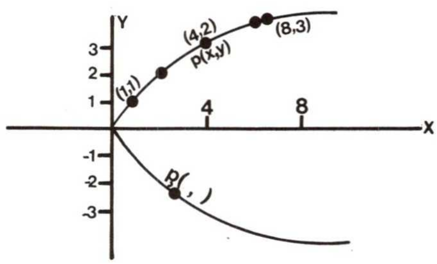
Compare results in columns 4 and 5 and try to develop a rule to explain the results.



The above graph is symmetrical with respect to ______________________________.
Indicate the coordinates of the point symmetrical to each of the given points.
[(x,y) is a general point.]

Given	*Symmetrical Counterpart*
(1,1)	
(2,4)	
(3,9)	

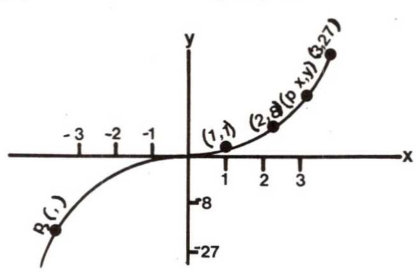
A graph is symmetric with respect to the y-axis if the point P,(,) is on the graph whenever the point p(x,y) is on the graph.



The above graph is symmetrical with respect to ______________________________.
Indicate the coordinates of the point symmetrical to each of the given points.
[(x,y) is a general point.]

Given	*Symmetrical Counterpart*
(1,1)	
(4,2)	
(9,3)	
P,(x,y)	

A graph is symmetrical with respect to the y-axis if the point P (,) is on the graph whenever the point p(x,y) is on the graph.

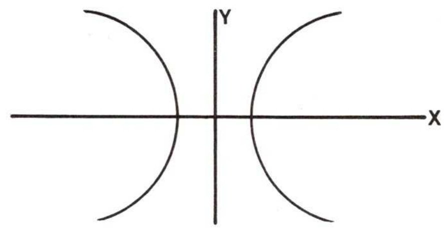


The above graph is symmetrical with respect to ______________________________.
Indicate the coordinates of the point symmetrical to each of the given points.
[(x,y) is a general point.]

Given	*Symmetrical Counterpart*
(0,1)	
(2,8)	
(3,27)	
P(x,y)	

A graph is symmetrical with respect to the origin if the point P (,) is on the graph whenever the point p(x,y) is on the graph.

What type of symmetry does the following graph have?



Does it obey the rule you make when working with the alphabet?

You have worked with the rotational symmetry (respect to a point) and bilateral symmetry (respect to a line). What other types of symmetry do you believe there to be? What types of objects would you use? Can you set up a display?

REFERENCES from *The Mathematics Teacher*

Atneosen, Gail H. "The Schwarz Paradox: An Interesting Problem for the First-Year Calculus Student," March 1972, pp.281-84.
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Shilgalis, Thomas W. "A Theorem on Lines of Symmetry," January 1972, pp.69-72.

D.2 Functions

Derivative of a Function *(Webber)*

Sketch a diagram as you read the following paragraph.

Let us consider the motion of a particle along a straight line. Let O be a point of reference of the line. If we say the particle is moving with a constant speed of 4 ft/sec we know that the object moves 4 feet every second but we don't know the direction of its motion. Therefore, let us consider one direction as the negative direction. If the particle moves with a constant speed of 4 ft/sec in the positive direction, then we say the velocity of the particle is 4 ft/sec or +4 ft/sec. If the speed is 4 ft/sec in the negative direction, we say the velocity is -4 ft/sec. We can also consider the motion of particles whose velocities are not constant.

Note - Velocity is the *rate of change* of distance with respect to time in a specified direction.

Acceleration is the *rate of change* of velocity with respect to time.

Please answer the following questions using graphs, tables, charts, or any other method you wish.

1. Suppose a particle moves along a line such that the distance, s, from 0 is given by s = f(t) = 3t + 2.

   (a) What does the graph of f(t) = 3t + 2 look like?

   (b) What is the slope of the line joining the points on the graph where $t_1 = 1$, $t_2 = 3$?, where $t_1 = 2$, $t_2 = 4$? where $t_1 = -4$, $t_2 = -4.5$?

   (c) What do you notice about the slopes?

   (d) Can you write a general expression for the slope of the line joining the points on the graph where time equals t, and time equals t_2?

   (e) What is the meaning of the slope on the distance-time graph of the above function?

2. If a particle moves along a straight line such that the distance, s, from 0 is given by $s = f(t) = 144 - 16t^2$,

   (a) what does the graph of $f(t) = 144 = 26t^2$ look like?

   (b) What is the slope of the line joining the points of the graph where $t_1 = 0$, $t_2 = -2$?, where $t_1 = 1$, $t_2 = 2$?, where $t_1 = -2$, $t_2 = 2$?, (try any other values you wish.)

   (c) What is the meaning of the slope on the distance-time graph of the above function?

   (d) What is the slope of the line joining the points on the graph where $t_1 = 1$, $t_2 = 4$?, where $t_1 = 1$, $t_2 = 3$?, where $t_1 = 1$, $t_2 = 2$?, where $t_1 = 1$, $t_2 = 3/2$?, where $t_1 = 1$, $t_2 = 5/4$?, where $t_1 = 1$, $t_2 = 9/8$?

   (e) Indicate what happens to the slope as t_2 approaches t_1.

   (f) What is the value of $\lim\limits_{t_2 \to t_1} \dfrac{f(t_2) - f(t_1)}{t_2 - t_1}$ where $t_1 = 1$? $t_1 = 0$?

   (g) What is the meaning of $\lim\limits_{t_2 \to t_1} \dfrac{f(t_2) - f(t_1)}{t_2 - t_1}$?

   (h) Can you write a general expression for $\lim\limits_{t_2 \to t} \dfrac{f(t_2) - f(t)}{t_2 - t}$?

(i) Letting $t_2 = t + h$ how would you write $\lim_{t_2 - t} \dfrac{f(t_2) - f(t)}{t_2 - t}$?

(j) Suppose we write $y - f(x) = 144 - 16x^2$. What is the meaning of $\lim_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$?

3. (a) Using simple polynomial functions such as $y = x^2$, $y = x^3$ can you determine $\lim_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$ for each function?

   (b) Can you now conjecture (guess intelligently) how you might determine $\lim_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$ for polynomial functions?

4. (a) Using the ideas you obtained from the preceding questions and references such as your text, can you explain the meaning of the symbols $f'(t)$, $Df(t)$, $\dfrac{dy}{dt}$ and y'?

   (b) How would you define the <u>tangent to a graph at a point x</u>?

REFERENCES from *The Mathematics Teacher*

Atneosen, Gail H. "The Schwarz Paradox: An Interesting Problem for the First-Year Calculus Student," March 1972, pp.281-284.

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Lipsey, Sally Irene, and Wolfe Snow. "The Appreciation of Radian Measure in Elementary Calculus," January 1973, pp.31-2.

D.3 Conics

Construction of Conics and Their Properties *(Farndale)*

1. If you sliced a solid cone into two parts and looked at the bared surfaces, what shapes night you see?

2. (a) Given two pins, a length of string and a pencil, draw as many different curves as possible. Can you make an ellipse?

   (b) Draw a large circle on paper and a long straight line through its center. If one vertex A of a wooden triangle or set square is made to move along the line, what shape will the third vertex trace out if AB = radius of the circle?

3. Attach a length of string to one end A of a rod AB and pivot the other end at B. Fix the free end of the string at C and discover the curve produced by moving a pencil P along AB in such a way that the string is kept taut and passes between the pencil and the rod.

4. Devise apparatus so that a rod AB can move at right angles to a line L. Fix a length of string to one end A of the rod, and the other end of the string to a fixed point C. What will the curve be that is traced out by a pencil which runs along AB, trapping the string between itself and the rod so that the string is kept taut?

5. (a) Construct any conic. Take any six points A, B, C, D, E, F, in that order, on the conic and join up A to D, B to E, and C to F. What do you notice? Repeat for other conics.

(b) Construct any conic. Take six points A, B, C, D, E, F, in that order on the conic. Let AE and FB meet at P, BD and CE meet at Q, and AD and CF meet at R. What do you notice about P,Q, R?

Conics - Division IV *(McIntyre)*

Although much of the following may be familiar to some students, these definitions should renew old memories and bring the student body to a uniform starting point.

Circular cone - A surface generated by a straight line moving so to intersect a given circle and to pass through a given fixed point not in the plane of the circle.

Element of a cone - The line in each of its positions as it moves around the circle.

Vertex of a cone - The fixed point.

Nappes - Each of two cones formed on either side of the vertex.

The following materials should be placed at student stations in reasonably large quantities: paper cones of various sizes and shapes, plastic models of cones of one or two nappes, a sharp cutting tool (preferably a knife), paper clips, glue, graph paper, pencils, rulers, construction paper of various colors, scissors.

Developmental questions

1. Experiment with the paper cones. Imagine a plane intersecting your cones in different ways. How many different "cuts" can you make? How many cuts go through only one nappe? Both? Are the cuts really different or only different kinds of the same cut? How are they different? How are they the same?

2. How many of the cuts are like a circle? How many are not? If the cut only cuts one nappe and is not like a circle, what is peculiar about how you cut? If the cut cuts both nappes, how do the cuts compare? Does it depend upon the shape of the cones?

3. On the given graph paper, plot a graph of values which satisfy the equation. (Circle). $x^2 + y^2 + 12x = 0$.

How could you cut a cone to form a figure the same shape as your graph?

4. Plot a graph for $3x^2 + 9y^2 = 45$ (ellipse). How would you cut a cone in this shape?

5. Plot $y^2 = 8x$ (parabola). How could you cut this one?

6. Plot $4x^2 - 9y^2 = 36$ (hperbola). How could you cut this?

[*Following a thorough study of the above experiments a classroom study session would come. Students would offer suggestions for finding the equation. Later, the terms will be introduced for the curves, the idea of "e" and so on. I think this type of exercise will be much more meaningful for the students.*]

Quadratic Relations *(Burton)*

I. *Circle* - centered anywhere

II. *Ellipse* - center the origin, axes of symmetry the x-axis and the y-axis

III. *Hyperbola* - center the origin, axes of symmetry the x-axis and the y-axis.

[*The Grade XII students have graphed some circles and ellipses in previous years. A few are familiar with hyperbolae. Most of their previous work started with an equation and ended with a graph. Each individual works on his own. Students are free to consult with friends or fellow geniuses at any time. They may even consult with the teacher. For difficult problems they switch to small groups. To develop general equations, one "neat-writing" student at the board follows class suggestions and directions.*]

I. CIRCLE

Given the line segment AB and the fixed point C -

(1) Find as many points as you can whose distance from C is of length AB.

(2) Is it possible to join these points with a smooth curve? Is it logical to join them? Is it a closed curve?

(3) Do you know the name of this curve? Could you cut a cone to form this curve? What is the name usually used for the fixed length and for the fixed point?

(4) On a sheet of graph paper draw Cartesian axes and label your scales. Do #1 again making AB any integral length and putting C on any lattice point (please don't all of you put C at the origin!)

(5) If $P(x,y)$ is any point on your curve can you find the relationship that must exist between x and y?

(6) Simplify your equation, collecting any like terms.

(7) Show your equation to several others to see if they can decide what your center and your radius were. Can you calculate the radius and center they used without seeing their graphs?

CLASS AS A WHOLE - CIRCLE

Before attempting exercises in the text the entire class working together should be able to develop

standard form - $(x-y)^2 + (y-k)^2 = r^2$

general form - $x^2 = y^2 + 2gx + 2fy + c = 0$.

II. ELLIPSE

Given the line segment AB and the fixed points F_1 and F_2 -

(1) Find as many points as you can so that the distance from F_1 plus the distance from F_2 equals the length AB.

(2) Is it possible to join these points with a smooth curve? Is it logical to join them? Is it a closed curve?

(3) Do you know the name of this curve? Could you cut a cone to form this curve? What is the name of the fixed points?

(4) Do (1) again, putting F_1 and F_2 the same distance apart as before but choosing your own fixed length. Is the question ever impossible? Why? Is the curve ever a circle?

(5) Working in groups of three or four and using the neatest graph in the group [from the first time you did (1)] -

(a) draw a line through F_1 F_2 and use this as the x-axis. Draw the right bisector of F_1 F_2 and use this as the y-axis. The given length AB was 10 cms and F_1 F_2 were 8 cms apart. Label the coordinates of the focal points and of all the intercepts of the curve on the axes. Does the fixed length appear in one or more places on your graph?

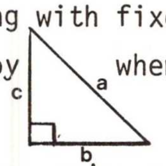
(b) If $P(x,y)$ is any point on your graph, can you find the relationship that must exist between x and y? (How did we solve radical equations last year that had two radical terms?)

(c) If you can't arrive at a simple relationship, use inductive reasoning to arrive at a possible answer. (You do know the x and y intercepts and can see the necessary symmetry from your graph.)

(6) If the fixed length had been 26 units and the focal points 24 units apart, make a sketch placing the axes in position as before in (5). (Join one y-intercept to one focal point.) Calculate a simple relationship between x and y for any point (x,y) on the graph.

CLASS AS A WHOLE - ELLIPSE

The entire class working together should be able to develop $\frac{x^2}{a^2} + \frac{y^2}{b} = +1$

starting with fixed length 2a and focal points 2 c units apart with a, b, c, related by 

when foci are on x-axis and also $\frac{x^2}{b^2} + \frac{y^2}{a^2} = +1$ when foci are on y-axis

The names major axis and minor axis must be given before attempting exercises in text.

III. HYPERBOLA

Given the line segment AB and the fixed points F_1 and F_2 -

(1) Find as many points as you can so that distance from F_1 - distance from F_2 equals the length AB.

(2) Is it possible to join these points with a smooth curve? Is it logical to join them? Is it a closed curve?

(3) Do you know the name of this curve? Could you cut a cone to form this curve? What is the name of the fixed points?

(4) Work in small groups or by yourself. Draw a line through F_1 F_2 and use this as your x-axis. Draw the right bisector of F_1 F_2 and use this as y-axis. The given length AB was 8 cms and F_1 F_2 were 10 cms apart. Label the co-ordinates of the focal points and all the intercepts of the curve of the axes. Does the fixed length appear in one or more places on your graph? If P(x,y) is any point on your graph can you find the relationship that must exist between x and y?

(5) If the fixed length had been 2a and the distance between fixed points had been 2c, make a rough sketch. Can you calculate the relationship that must exist between x and y for any point P(x,y) on the graph.

GLASS AS A WHOLE - HYPERBOLA

Develop $\frac{x^2}{a^2} - \frac{y^2}{b^2} = +1$, foci on x-axis

$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$, foci on y-axis

Discuss asymptotes and their use as a help in sketching a graph if given the equation. The names transverse axis, conjugate axis, and focal radius, must be given before attempting exercises in text.

REFERENCES

Atneosen, Gail H. "The Schwarz Paradox: An Interesting Problem for the First-Year Calculus Student," *The Mathematics Teacher,* March 1972, pp.281-84.

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