USING ELECTRONIC CALCULATORS

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In the summer of 1975 Laurie Buxton, Staff Inspector for Mathematics for the Inner London Education Authority, obtained some development money to purchase 200 simple electronic calculators to use in a year's experiment in schools. The machine chosen was the *Citizen 120R*, then the current recommendation to London schools as a four-function machine.

One hundred machines were given to a secondary school to be shared by the mathematics, science, commerce, geography and technology departments. Twenty were given to each of four primary schools.

The aim of the experiment was to see what schools could do with a number of machines that they would not normally buy for themselves, and we were fairly open-minded about the results, being guided mainly by the idea that since calculators were around, they could not be ignored, and we wanted to see what benefit teachers and children could obtain from them, with a reasonable amount of assistance from the inspectorate, the local mathematics center, and the local college of education.

The informal report which follows was published locally in December 1975, as a summary of the first term's developments, and, it was hoped, as a stimulus to the schools involved.

GETTING STARTED

For various reasons, things took some time to get off the ground (or out of the boxes). The secondary school was, rightly, security-conscious about their large number of machines; each had to have a reference number painted on, and wooden boxes were made, each holding just 10 or 15 machines, so that they could be transported easily and quickly checked back. Initial charging up was even more complicated than subsequent recharging. The physics department wired five chargers into each plug. One school's 20 were plugged in at the Mathematics Centre over a weekend. At another I helped the head unpack machines and plug in two in each classroom.

Distribution was easier to plan. Most of the primary schools decided to use machines with their older children, but one thought it would be interesting to have one or two in all classes, including top infants.

After that, it was a case of getting the machines into the hands of children. As with much other apparatus, teachers sometimes felt that they had to be experts themselves before they could teach the children. In fact, when they were persuaded just to give them out without instructions, they found that in about half an hour most 10-year-olds had discovered how to perform all four rules and perhaps other operations as well. (Perhaps it takes longer with older children; a sixth form commerce group who had received careful instructions from their teacher were very unsure about what to do!) The *Citizen* machines, wisely chosen by the Electronic Calculator Sub-Committee of the I.L.E.A. Maths Advisory Panel, lent themselves ideally to discovery since the order in which one pressed the buttons corresponds exactly to what one would write.

INITIAL ACTIVITIES

Ten-year-olds, once they had found out how to operate, soon found things to use the machine for. A pair of boys tested each other on questions like 7 x 5 and 8 + 23. Two girls were checking some subtractions that they had done in their books; strangely, one was doing it by calculating one column at a time, so that for 65 - 23 she would use the machine for 5 - 3 and 6 - 2 separately, and if decomposition was necessary, she did that on paper first! "What year were you born?" asked Victor, and before I had remembered to add on 10 years, he had subtracted it from 1975 and told me my age; then he told me his teacher's age!

However, there were some interesting problems about comprehension, which perhaps had not come to light before in the context of paper and pencil calculations:

- A girl wanted 9 - 7, but pressed buttons for 7 - 9, and complained that the answer was wrong. This prompted a discussion with the teacher.

- Nicola and Louis had calculated 700 ÷ 400 and were worried about the answer. I asked them to do it on paper. It went in once, they decided after some time, and after more discussion they thought that there would be 300 left over, but they were not sure how to write it down. They wrote "700 ÷ 400 1300." We returned to the answer on the machine, 1.75, which they read as "one hundred and seventy-five." I indicated the decimal point. "Oh," they said, "one *point* seventy-five." They didn't know what that meant. I asked them to calculate 800 ÷ 400. "Two point," read Nicola.

- Julia and Andrea were trying to find the highest number. They had entered "111111" and kept pressing the "=" button, which previously had kept adding 111111 but now didn't work. I asked what the highest number was. "I used to know but I've forgotten," said Julia.

- "What's the smallest number?" I asked. "A half," said Andrea. "That's hard," said Julia; "... a quarter ... half of a quarter ... a quarter of a quarter."

- "What's half of 16006?" asked Paul. "8003," I replied. "Thank you," he said! I asked him how he found a half of something. "I don't know," he said; "find half of it and take it away." I asked him how he had halved 16006. "Half of 16 is 8 and half of 6 is 3."

- "What are seven eights?" I asked. The boy used the machine to produce 56. "What are eight sevens?" "56," he said straight away. "Why?" "Because it's the same." "What's 56 ÷ 7?" "Can I do it on the machine?" "If you wish." He used the machine for 56 ÷ 7 = 8, and then for 56 ÷ 8 = 7.

It seemed that all these examples revealed some misunderstanding about arithmetic, partly because the children chose the questions, but mainly because the important part of the work was not in *getting the answer*, as is usual with paper and pencil calculations, but in *framing the question* and *interpreting* the answer. Obviously an earlier understanding of decimals is going to be desirable, but notice Nicola and Louise's incomplete concept of division, which they always referred to as 'sharing', and how this relates to Paul's efforts at halving.

Other activities raised some different problems.

- Simon had entered 78 - 456 and the machine registered '-378'. He thought the answer should be "nought." I asked him to try 1 - 2 and 1 - 3, which produced respectively '-1' and '-2'. Victor thought it might mean "difference." I asked what he would expect from 1 - 4, and he said "a difference of 3."

- "Give me a sum," said Stephen. "Three take away seven," I said. The machine answered '-4'. "Four," said Stephen. "What's this in front of it?" I asked. Stephen didn't know. "What's seven take away three?" "Four." "What's three take away seven?" "You can't do it." "But you *did* it," I said! Stephen was puzzled.

Other children also came across negative numbers in this way, usually on their own because no teacher normally asks 'impossible' questions! Maybe we can teach negative numbers earlier, and perhaps they are easier to teach than fractions. The connection between fractions and division and decimal notation, however, rears its ugly head all too easily.

"4 ÷ 4 = 1," said Victor. "What's 4 ÷ 8?" I asked. Victor was mystified by the answer '0.5'.

Compare the activity of three bright 10-year-olds. They found that pressing 'x' and then '=' squared the number entered, so '3 x=' produced 9, '5 x=' produced 25 and '8.5 x=' produced 72.25. "What must you square to get 10?" I asked. They knew it would be between 3 and 4, and because it would be nearer 3, they tried 3.3, which gave 10.89. 3.2 gave 10.24; 3.1 gave 9.61. So it must be between 3.1 and 3.2 They tried 3.15! This gave 9.9225. 3.16 gave 9.9856 and 3.17 gave 10.0489, so the next thing to try was 3.165. They continued in this way, watching their answers get closer to 10, and one was able to say, for instance, that 9.998244 was closer than 10.004569. Note the exceptional understanding of decimal notation as well as the perfect systematic approach.

By contrast some 13-year-olds were having far more trouble. Some of the class were finding the square root of 63, and no one seemed to have discovered the squaring facility on the machine so they were having to enter a number twice, for example '7.9 x 7.9' rather than '7.9 x='. One knew it should be "seven point nine something," but multiplied 7.9 successively by 6.3, 9.4, 7.9, 9.8 and 8.7. Those who were squaring numbers seemed to do so at random. (One sequence was 7.98, 7.96, 7.95, 7.98, 7.943, 7.912, 7.901, 7.911, 7.900.) "Have you got to start with seven point?" asked a girl; "You ain't gotta," was the reply, "but that's the way to do it."

The less able in the class had been asked to calculate things like 6 + 6 + 6 + 6 + 6 + 6 + 6 + 6 and 6 x 8; they were surprised that these gave the same answer! I entered '8' on one machine and was told it was 8 units; '85' was 8 tens and 5 units; '855' was 8 hundreds 5 tens and 5 units; '855.9' was 8 thousand ... Other simple activities, primary and secondary, have included building up tables by repeated addition, and multiplying numbers by 10 or 100.

Teachers have commented on the stimulation given by the machines, the increase in confidence achieved by the less able, some reluctance to use them by the more able, and the implicit faith in them some children seem to have, even when the answers are wrong!

OUTSIDE MATHEMATICS

The sixth form girls taking commerce seemed to be getting least benefit. As was mentioned above, they were virtually taught by rote, and, perhaps in consequence, were far less confident about their use than primary children. Although the machines stimulated some interest, the girls preferred to work on paper "where you can see what's going on," and some of them expressed some worry about their availability when they started work!

Other subjects seemed very satisfied that the difficulties and wasted time caused by calculation had been removed, so that ideas and principles could more easily be discussed. In engineering technology, third year boys could discuss gear ratios and levers without being held up doing the arithmetic, and since this subject was working toward a mode iii C.S.E., the use of machines could be allowed in the examination. In geography, second year children calculated average rainfall and temperature; the geography teacher used to bring in his own calculator because, except with the most able children, calculations on paper took much too long and killed off all interest in the geography. Fourth and fifth years doing urban geography were able to calculate percentage increases in house prices in Reading, Chippenham, and Bristol to investigate the effect of opening the M4 Motorway, a task virtually impossible without machines.

In science there was enormous benefit, particularly in physics. Previously a whole lesson on density could be taken up calculating the density of one material; now the densities of many different materials could be calculated on the machines. There was increased confidence, and much more time to spend on the science. However, there were still worries about other mathematical ideas, the concepts of multiplication and division, area and volume, and particularly ratio. In other words, it is still important to understand, and the relief from the tedium of calculation seems to uncover the misunderstandings and throw them into sharper relief. One wonders in particular whether ideas like area and volume are more appropriately dealt with in mathematics or in science, and I discussed with the head of the faculty the difference between the ways ratio is dealt with in the two subjects. It is worth noting, however, that the removal of arithmetical difficulties has enabled these discussions to take place.

WHAT NEXT?

In mathematics, once the initial enthusiasm has been used up, the children know how to use their machines, everyone has tackled the square root problem, and both pupils and teachers have run out of ideas, there seems to be a sudden vacuum. Teachers are then reluctant to interfere with their planned schedules by taking in the machines, and, in any case, they have asked, should there be some worksheets to go with them? (One secondary teacher was using a hastily prepared worksheet to test a third year class's use of the machines, based on old-fashioned primary arithmetic, which raised some interesting problems about how many weeks there were in a month!)

There are several types of activity which could be undertaken.

1. If possible, machines should always be available so that whenever calculation is necessary in some other work, they can be used to do or to check it. This raises problems about exercises in mechanical arithmetic, and it may be interesting to see whether children are willing to try these on paper first and check them afterwards. We have not yet explored enough how much arithmetic children actually learn from using the machines, and teachers are naturally worried about any possible deterioration in arithmetical abilities. We also have to investigate what help machines can be toward understanding whatever is left when the actual burden of calculation is removed, as opposed to the help they can be, as illustrated earlier, in revealing what understanding is missing!

2. The practical ideas promoted by Edith Biggs 10 or more years ago do not seem so prevalent these days. One possible reason is that many of the activities in 'weighing and measuring' or statistics often produced numbers that were too unwieldy to deal with, and this either required drastic approximation, or the teacher eventually resorted to artificial examples with easy numbers that 'worked out'.

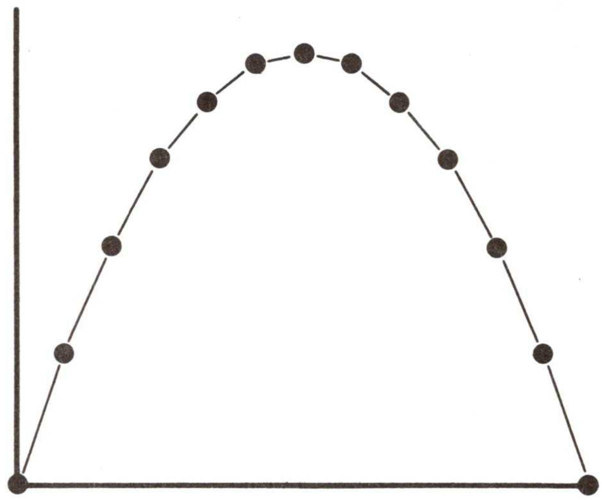
   Now, however, a combination of metrication and electronic calculators makes it possible to go back to practical activities and deal with *any* numbers that crop up. One can concentrate then on (i) the strategies of measurement, (ii) processing the information, (iii) identifying the problems that occur, (iv) deciding on which arithmetical processes are appropriate (with machines teachers can say again, as they used to years ago, that children can add, subtract, multiply and divide but they don't know *which* to do *when*) and (v) interpreting the results. Come back, trundle wheel, all is forgiven!

3. When teaching any mathematical principles, it is customary to keep the numbers simple so that the principles are not obscured by the difficulties of

calculation. Machines make it possible to include the awkward cases too. The following are some examples.

(a) When calculating AREAS OF RECTANGLES, we usually confine children to whole numbers by drawing rectangles on squared paper or making them on nail-boards. Now we can measure any rectangle to the nearest millimetre, and perhaps think of something more sensible to do with the results. A recent piece of work for C.S.E. involved a graph of areas of rectangles with a perimeter of 24 units (graphed against base) where only integral points were plotted, and invariably these were joined up with straight lines.

Machines would enable some intermediate points to be plotted very quickly. A graph of squares raises similar problems.



(b) Normally, calculating PERIMETERS is a fairly futile occupation, but at least we can now deal with anything we can measure. There is an inverse problem to the rectangle one above; fix the area and draw a graph of perimeters; this is more interesting numerically. Perimeters of polyominoes are all whole numbers, but what perimeters can you get from shapes made from tangrams?

(c) When investigating areas or circumferences of CIRCLES, we usually start with whole number diameters. Now we can measure all the circles we have around as accurately as possible.

(d) Operations with MATRICES OR VECTORS invariably involve simple whole numbers. Some applications can usefully involve large numbers or decimals.

(e) CHANGING FRACTIONS TO DECIMALS is usually a limited operation. We will avoid a discussion here of what is involved in dividing numerator by denominator, but having established the principle we feel obliged to limit the denominator to a maximum of 10 or perhaps 12, so that the only really interesting recurring decimals are the sevenths. Now we can investigate thirteenths, fourteenths, and so on. (And see 4d below.) Incidentally, this technique rather spoils the exercises on comparing two fractions by rewriting them, with a common denominator; it is more sensible to express them as decimals!

(f) Any activities involving RATIO are usually limited to easy whole numbers, and perhaps this is why pupils have such incomplete ideas about it. The science department may be happier if we spend some time on the following ideas; volume/surface area ratios for different size cubes, spheres, people, and so forth; mass/volume ratios for different materials; weight/height ratios for people; waist/height ditto; ratio of weight/height to waist/height (are you heavy or fat?); pressure on feet, that is, weight/area of feet; and of course price/weight or price/volume ratios for the same commodities in different shops or in different weeks.

These activities involve no great change, if any at all, in present syllabuses. They just require a little extra thought when dealing with any topic to make sure that some extra calculation, suitable for machine work, is encouraged. Occasionally this will make possible some new ideas.

4. Some mathematical activities can arise as a result of the particular characteristics of the machine.

(a) The finding of SQUARE ROOTS is a typical example. This is an exercise that becomes accessible to younger children because it relies on an understanding of decimal notation rather than on an ability to multiply decimals. A typical introduction to square roots with 12-year-olds used to be exactly the same method, for example, guess the square root of 2, say 1.5; square 1.5 ; 2.25 is too big; try 1.4; 1.96 is too small; try something in between 1.4 and 1.5, say 1.45. A class discussion would clarify ideas about the limiting process, the best strategy, and so on. The trouble was that using paper and pencil it took one lesson to find the square root of 2 to about 4 decimal places. Now, however, the calculation is so much quicker, and many roots can be found, making the principle much clearer, and opening up possibilities for plotting a graph, calculating intermediate points, and discussing alternative methods. Usually one progressed to the Newton-Raphson method, where the first guess is divided into 2 to produce a larger estimate if the first estimate is too small (or vice versa), and so a better estimate would be halfway between the two; this is then divided into 2, and the process is repeated. Strangely, this is a more tedious method on the machine, and perhaps the 10-year-olds' method is more efficient now.

(b) How do you find CUBE ROOTS?

(c) NUMBER PATTERNS abound in the literature, but the numbers are usually simple. Ray Hemmings, in *Mathematics Teaching* No. 48 (p.56), suggested an activity for the old-fashioned mechanical calculator which is easily adapted for the electronic one. Enter, say, 123456, and keep adding it. (The first time you press '+=' and after that only '='.) Record successive entries. Ignore digits after the first six (but you can put these on the display if you wish). Investigate the numbers produced in any way you like. How long before the original number returns, and why? Start with other numbers. The situation is quite different from that of the mechanical calculator, but just as interesting. Try starting with 142857!

The simple number sequences can still be built up. Add successively 1, 3, 5, 7, 9, ... or 1, 2, 3, 4, 5, ..., recording answers. Younger children can add successive 1s, or 2s or ..., and we are back to building up multiplication tables. Try successive multiplication by 2, and see how quickly the numbers grow. Explore successive multiplication by 10. Explore successive division by 2 or 10.

(d) When exploring RECURRING DECIMALS, the sevenths become immediately interesting, and then possibly the thirteenths. One then looks for a recurring cycle of more than 6 digits. The seventeenths are the next longest, with a cycle of 16 digits, but only 11 of these can appear on the display. Problem: How do you obtain the others? (Hint: When you have calculated $1 \div 17$, what happens if you multiply the answer by 17?) When you have solved this, you are ready for longer cycles. For older pupils some are particularly interesting, for example, 1/49 begins 0.0204081632 ...; why? After this 1/24 = 0.0416666 ... is exceptionally curious, if you consider first that it may be equal to 0.04 + 0.0016 + 0.000064 + ..., and secondly that it is the sum of two fractions, respectively equal to 0.041 and to 0.0006666 ...

Incidentally, is there a way of using the calculator to convert a decimal, recurring or not, back to a fraction? Which reminds me that the third year pupils were asked to "reduce 1980/4620 to its lowest terms." What is the best way of tackling that on the machine?

(e) There are some problems which involve the tabulation of figures, especially those that require a MAXIMUM OR MINIMUM. The maximum area for a rectangle of fixed perimeter becomes fairly obvious from the symmetry of the graph (see 3a above), but the inverse problem gives a graph which is not symmetrical. David Hale quotes a similar problem in *Mathematics Teaching* No. 67 (p.20), the one about making an open box from a square by cutting off square corners.

The graph of volume against size of corner is also asymmetrical, and some calculations are necessary around the peak of the graph if the corner size is to be found to any accuracy. Obviously we are getting close to ideas of calculus here, and it is interesting that W.W. Sawyer suggested a numerical approach to differential calculus in *Mathematician's Delight* (Penguin), which is most suitable for calculators.

(f) Many of these examples, like the square root problem, involve the idea of a LIMIT. Limits are not often dealt with numerically, but it is possible to calculate certain numbers from their sequences, series or even continued fractions. For example, a sequence that approaches the square root of 2 is

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \cdots$$

in which m/n is followed by $m + 2n/m + n$. (This can be generalized for other square roots.) There are well-known series which can be summed by successively adding more terms, and the same procedure can be used for continued fractions.

(g) Finding the square root of 2 can be viewed as solving the EQUATION $x^2 = 2$. One can use the same process for solving, say,

$$x^2 + x = 17,$$

that is, guess x and calculate $x^2 + x$ to see how near 17 you are.

Younger children have apparently used machines to solve simpler equations like $x + 3 = 10$ or $2x = 16$ in the same sort of way (perhaps using a 'box' notation for the 'missing number'). They are exploring number facts in a different way from usual and they are also learning the 'guess and improve' technique.

Obviously we have other techniques for solving simple or quadratic equations, which should also be used, but the calculator method has much to commend it intrinsically, and also it will in fact solve any polynominal equation, say

$$x^2 - x^2 + x = 20,$$

though there may be some snags about roots which are near each other.

Another interesting point concerns decreasing functions. If in solving $5x - x^2 = 1$ we guess $x = 4$, then $5x - x^2$ gives 4 which is too much. We may then try $x = 3$, giving 6, which is even larger! A graph will help sort out what is happening if the numbers themselves do not.

(h) Some PROGRAMMING will be necessary. $x^2 + x$ is easy to calculate, because one can square x and then add on x. $x^2 + 2x$ is not so easy, because it looks as if one must square x, record it and clear the machine, double x, then add x^2 from the record (or vice versa). But in fact

$$x^2 + 2x = (x + 2)x,$$

and this can be calculated successively on the machine. Note that this can but $x(x + 2)$ cannot. Similarly the area of a trapezium is $\frac{1}{2}(a + b)h$; when using the machine, $a + b$ must be calculated first. This all gives some point to the distributive, associative, and commutative laws. Other examples are more interesting. How do you calculate

$$\frac{1}{u} + \frac{1}{v} \text{ ?}$$

Although letters have been used here, most pupils when calculating will be dealing with numerical examples. A complexity of operations does not often occur naturally, but we should make the most of it when it does, exploring different ways of performing the calculation, and deciding which is best, either for pencil and paper, or for the machine. Graph plotting is a useful exercise, since whenever we have to evaluate a function for different numerical values we want the most efficient way of doing it.

(i) A useful C.S.E. project could be to produce a small BOOK OF TABLES. This could involve interpolation, some graphical work, and perhaps methods of differences in a simple way, but it would be based mainly on the use of calculators.

All these activities are only suggestions. While some are based on well-tried classroom situations where often a calculator would have been useful, many remain to be explored. I should apologize that they are not in any order of difficulty; primary teachers will have to sort out what is appropriate for them.

We should be pleased to hear from any other schools who are making use of machines.