
THE POCKET CALCULATOR AS A TEACHING AID

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Since its first appearance in the market about five years ago, the pocket calculator has quickly become everyone's home tool, as common as the transistor radio or the cassette tape recorder. The increasing variety of calculators, combined with their generally declining prices, has made them available to more and more people. As a result, even people who previously had no interest in mathematics have begun to express an interest in the properties of numbers, the operations with them, and the relations that exist among them.

The stimulation of exploring with these calculators apparently has a psychological effect on people's attitude toward numbers and arithmetic. The boring, time-consuming, and ever-dreaded use of tables - whether they are the ordinary multiplication tables of elementary school days or the logarithmic and trigonometric tables of high school - suddenly disappears. Instead, one plays with a small, pocket-sized, attractive instrument, operated by the touch of a few keys, and generates answers that appear electronically in the display.

Undoubtedly, the introduction of the pocket calculator will bring about a drastic change in the field of elementary computations, that is, computations that are too trivial to be processed by a computer, yet too time-consuming for manual calculation. This trend will have an obvious bearing on the teaching of mathematics in schools. Not only will the use of most numerical tables become unnecessary, but the teaching of many topics in elementary as well as college mathematics will be given new insight, interest, and even fun by the use of these instruments. It is the purpose of this article to offer several examples and suggestions on how this might be accomplished.

ARITHMETIC AND ALGEBRA

The proofs of all the basic formulas and identities of elementary algebra can, and should, be supplemented by testing their validity for various numerical values of the symbols appearing in them. This is where the pocket calculator is

of great help. Not only will it increase students' faith in the formula, but it will also encourage them to test and search for new relations and patterns.

As a first step, however, some simple experiments should be performed in class in order to test the calculator's logic in processing sequences of operations. Calculators may be classified into two categories according to their algebraic hierarchy, that is, according to the method used to process sums of products. In the one category, an expression such as $ab + cd$ is computed exactly as we would understand it, for example, $1 \times 2 + 3 \times 4 = 2 + 12 = 14$. In the other category, the calculator will interpret the same expression as $(ab + c) \times d$, that is, $(1 \times 2 + 3) \times 4 = 20$. Generally, the more sophisticated models, such as Texas Instruments' SR-50, belong to the first category, and the simpler types, such as Texas Instruments' SR-10, belong to the second.

A similar situation exists with respect to sums of fractions, sums of squares, sums of roots, and so on. It is very important to be aware of these differences before a class goes on to more complex calculations. The interpretations of several expressions for two categories of calculators can best be summarized in Table 1. The last column in Table 1 indicates the ways the expressions in column one should be interpreted for calculators in the second category.

TABLE 1
TWO INTERPRETATIONS FOR SOME
SPECIFIC ALGEBRAIC EXPRESSIONS

Expression	Interpretation		Correct Input Notation for Category 2
	Category 1	Category 2	
$ab + cd$	$ab + cd$	$(ab + c) d$	$(\frac{ab}{d} + c) d$
$\frac{a}{b} + \frac{c}{d}$	$\frac{a}{b} + \frac{c}{d}$	$\frac{\frac{a}{b} + c}{d}$	$(\frac{ad}{b} + c) / d$
$a + b^2$	$a + b^2$	$(a + b)^2$	$b^2 + a$
ab^2	ab^2	$(ab)^2$	$b^2 a$

A discussion can now be initiated on these differences, and the students can practice with their own calculators to discover their various features.

The explorations with some of the more sophisticated operations of the calculator might turn into exciting discoveries for the beginner. Let us take the reciprocal key as an example. A simple formula like $\frac{1}{1/x} = x$, with which many students are familiar only as a formal means of handling composite fractions, is suddenly given a very realistic meaning. Choose any number (except zero, of course), press the "1/x" key twice, and the very same number appears in the display! The discovery can be made even more exciting by asking the students to watch the results as they press the "1/x" key first an even number of times and then an odd number of times. Almost without noticing it, we have introduced the students to the transformation equations $T^{2n} = I$ and $T^{2n+1} = T^{-1}$, which characterize all involutory transformations such as $y = 1/x$.

Similar experiments can be performed with the square and square-root keys. On some instruments, such as the SR-50, any successive application of two inverse functions will restore the exact original number; whereas in other types, such as the SR-10, the result will be only approximately equal to the original number. In this latter case, it will be interesting to observe what happens when a sequence of x^2 and \sqrt{x} operations is performed on the same number x . Does the error remain bounded, or does it accumulate?

It is also fun to test the calculator's reaction to "forbidden" operations, such as zero division. Some types will protest with a violent flash of the display; other, shyer types will merely indicate with an error sign that the operation is invalid. An interesting situation is exhibited on the SR-50: if you take the common logarithm of zero, you get, as expected, the largest negative number the display can show, -9.99999999 times 10^{99} , which is the instrument's interpretation of minus infinity. If, however, you take the *natural* logarithm of zero, you get the unexpected flashing result -4.342944819 times 10^{99} ! Each type of mini-calculator has its own idiosyncracies.

Finally, many interesting and quite surprising numerical relations can be discovered with the calculator. Who would guess, for example, that e^π and π^e have almost the same value? ($e^\pi = 23.14069 \dots$; $\pi^e = 22.45915 \dots$). Or that $2 - \sqrt{3}$ is exactly the reciprocal of $2 + \sqrt{3}$? Of course, after discovering these relations on the calculator, the class can go on to analyze them and find out their origin. Thus, $e^\pi \sim \pi^e$ because the value of π is very nearly equal to that of $e \ln \pi$, so that $\pi^e \equiv e^{e \ln \pi} \sim e^\pi$. In the second example we have, of course, $(2 - \sqrt{3})(2 + \sqrt{3}) = 4 - 3 = 1$; incidentally, $2 - \sqrt{3}$ is the tangent of 15° and $2 + \sqrt{3}$ is the tangent of 75° , and our relation becomes the trigonometric equation $\tan 15^\circ \times \tan 75^\circ = 1$. There is almost no limit to the ingenuity of numerical relations and properties that can be discovered with the calculator, and it can turn many a boring mathematics lesson into an exciting and unforgettable experience.

TRIGONOMETRY

There are obvious advantages in using the calculator over the conventional trigonometric tables: greater speed and accuracy and the lack of the need to calculate proportional parts. Most important, it is not necessary to reduce any angle to the range $0^\circ \leq \alpha \leq 90^\circ$, for the calculator will do this automatically. Anyone who has ever worked with angles whose measures are greater than 90° knows how time-consuming these reductions can be. These advantages, however, are somewhat counterbalanced by the fact that all angles must be entered either in radians or in decimal form, not in degrees/minutes/seconds. Only in some very advanced models, such as Texas Instruments' SR-51, is there direct conversion from decimal parts of a degree to minutes and seconds.

The pocket calculator can be of great help in demonstrating the validity of trigonometric identities. It is one thing to prove an identity but quite another to grasp the nature of the formula under consideration. Has anyone once bothered to test in the classroom the validity of, say, the formula $\sin^2 \alpha + \cos^2 \alpha = 1$ for various values of α , not just for 30° or 45° ? Most probably, the answer is no. With the calculator, such a test becomes real fun (Table 2). Now it depends

on the specific type of calculator being used whether the result, after "=" key has been pressed, is exactly 1 or 0.999999 ... The latter result is due to the calculator's truncating property, a point that should be explained to the class.

TABLE 2
VERIFYING $\sin^2\alpha + \cos^2\alpha = 1$

Choose any number for α	
Keyboard	Display
α	α
sin	$\sin\alpha$
x^2	$\sin^2\alpha$
+	$\sin^2\alpha$
α	α
cos	$\cos\alpha$
x^2	$\cos^2\alpha$
=	

Another class of identities that can be demonstrated on the calculator are the trigonometric-inverse trigonometric relations. Identities such as $\sin(\arcsin x) = x$, $\arcsin(\sin x) = x$, or $\arcsin x = \arccos \sqrt{1 - x^2}$ appear quite abstract and meaningless to many students. A simple numerical demonstration will at once make these relations meaningful (Table 3).

TABLE 3
VERIFYING $\arcsin(\sin\alpha) = \alpha$

Choose any number for α	
Keyboard	Display
α	α
sin	$\sin\alpha$
arcsin	α

NOTE: The procedure of the operation "arcsin" differs from one instrument to another. Often, it is required to press two keys, "arc" and "sin" or "inv" and "sin."

Finally, a very common error in the first lessons of a trigonometry course is to confuse, for example, $\sin 2x$ with $2 \sin x$. One or two demonstrations on the calculator will remove any notion that the two expressions are the same. Indeed, the correct relation, $\sin 2x = 2 \sin x \cos x$, can easily be demonstrated in the same way.

FUNCTIONS AND LIMITS

In the calculation of numerical values of functions, the pocket calculator is almost indispensable. Even simple functions, such as $y = ax^2 + bx + c$ or $y = \sin(ax + b)$, require quite involved computations, unless the constants a , b , c have integral values and only as long as we confine ourselves to integral values of x as well. But sometimes assigning only integral values to x will make the values of y too sparse for a graph to be meaningful. This is particularly true near a singular point of the function. For example, the function

$$y = \frac{2x + 1}{2x - 1}$$

yields ever-increasing values as x approaches the value 0.5. It is imperative to calculate the values of y for x s very near to this point. This is a task that fits exactly the capabilities of even the simplest calculator (Table 4).

TABLE 4
EVALUATING $y = \frac{2x + 1}{2x - 1}$ FOR $x = 0.7$

	Keyboard	Display	Keyboard	Display	
	0.7	0.7	0.7	0.7	
d	×	0.7	×	0.7	n
e					u
n	2	2	2	2	m
o	—	1.4	+	1.4	r
m					a
i	1	1	1	1	t
n	=	0.4	=	2.4	o
a					r
t	STO	0.4	÷	2.4	
o			RCL	0.4	
r			=	6.	

NOTE: If the calculator has no memory (STO key), one would have to write down the intermediate result, 0.4, manually and then use it again in the final division. In more advanced models, where there is an interchange function ("x = y" key), one may calculate first the numerator, store it, then calculate the denominator, divide it into the result of the numerator, and then press the interchange key (before pressing the "=" key); or, alternatively, press the "=" key and then the reciprocal "1/x" key.

The same calculations should then be repeated for $x = 0.6, 0.55, 0.525$, and so on; then for $x = 0.3, 0.4, 0.45, 0.475$, and so on.

One may use similar procedures to show how the quotient $\frac{\Delta y}{\Delta x}$ of a continuous function $y = f(x)$ approaches the value of the derivative $y' = \frac{dy}{dx}$ as Δx becomes smaller and smaller near the point x_0 where the value of the derivative is sought (if the derivative exists at that point).

Perhaps the most elegant use of the calculator as an educational tool is in the demonstration of limiting processes, such as the convergence of infinite sequences, series, and products. The calculator will not only calculate the values of the terms themselves but will automatically record the value of the partial sum (or the product, as the case may be) at each step. Moreover, one can test the *rate* of convergence of the sequence or series, a crucial question that, unfortunately, is often neglected in teaching the subject in school. Let us mention just a few simple examples.

1. The sequence $\sqrt[n]{a}$, a any positive number. It is well known (Courant 1956, p.31) that

$$\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1.$$

Choose any positive number, then press the square-root key as often as you wish. It is really exciting to observe how the numbers in the display

successively approach one. (We must be aware of the fact, though, that in this way we obtain the sequence

$$\sqrt[n]{a} = \sqrt[3]{a}, \sqrt[4]{a}, \sqrt[5]{a}, \dots,$$

which is a partial sequence of the original one.) Moreover, when the difference between two successive terms becomes smaller than the last digit the display can hold, the final limiting value will automatically appear (in our example, the number 1). This is a vivid demonstration of the fact that, in practice, we can expect a numerical result to be of any significance only as long as the result is larger than the worst inaccuracy of the measuring device used: the final result, in our example, is displayed as 1 and not as 1.000000.

2. The geometric series $1 + 1/2 + 1/4 + 1/8 + \dots = 2$.

Notice (Table 5) that each pressing of the "+" key automatically gives the value of the partial sum that has already been accumulated (the underlined results). Thus we can literally observe the convergence of the partial sums to their limiting value 2. The sequence of operations can best be represented by a flow chart (Fig. 1).

TABLE 5
EVALUATING $\sum_{n=0}^k \frac{1}{2^n}$

Keyboard	Display
1	1
+	<u>1.</u>
2	2
1/x	0.5
+	<u>1.5</u>
4	4
1/x	0.25
+	<u>1.75</u>
8	8
1/x	0.125
+	<u>1.875</u>
⋮	⋮

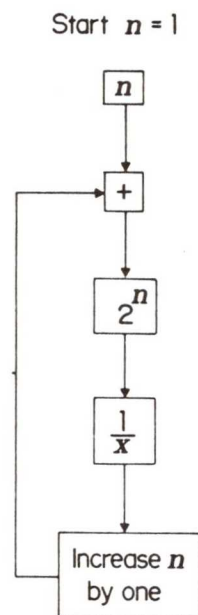


Fig. 1

It should again be noticed that the sequence of operations in Figure 1 will be valid only on instruments that perform directly the addition of products; otherwise, the operations $1 + 2 1/x$ will be interpreted as $(1 + 2)1/x$; that is, we shall get for the first two terms $1/3 = 0.3333$ instead of $1 + 1/2 = 1.5$. In such a case, we must calculate each term separately and add it to the previous terms.

We can now contrast this example with an example of a *divergent* series,

such as the harmonic series $1 + 1/2 + 1/3 + 1/4 + \dots$, and observe how the partial sums gradually increase. This would be a vivid demonstration of the fact that the condition for a series to converge, namely, that $|a_{n+1}| < |a_n|$, is necessary but not sufficient.

Other series representing important constants can likewise be tested and compared. Thus, one could compare the well-known limit for e ,

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n,$$

with the Taylor series for e , $e = 1 + 1/1! + 1/2! + 1/3! + \dots$, and observe how much more quickly the second expression converges than the first. Likewise, it would be interesting to compare some of the many series for π or π^2 that result from the Fourier expansion of various square and triangular functions (Courant 1956, pp. 440, 443, 446). An infinite *product* - which is much more rarely encountered than an infinite series - can also be tested; a well-known example is Wallis's product, (Courant, pp.224,445).

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \dots$$

As a final assignment, the class can be given the task of calculating the numerical values of some of the transcendental functions, such as $\sin x$ or $\ln x$, from their power series - in the case of $\ln x$, one must use the series for $\ln(1 - x)$ or $\ln(1 + x)$. The teacher can, at this point, tell the class how laborious and time-consuming the very same calculations must have been for the mathematicians of the previous centuries, when logarithmic and trigonometric tables were first introduced. Much historical information on this subject exists, such as the works of Klein (1953) and Smith (1953). We are actually introducing the class, through the pocket calculator, not only to numerical analysis but also to some of the more exciting chapters in the history of mathematics!

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REFERENCES

- Courant, Richard. *Differential and Integral Calculus*, Vol. 1. London and Glasgow: Blackie & Son, 1956.
- Klein, Felix. *Elementary Mathematics from an Advanced Standpoint (Arithmetic - Algebra - Analysis)*. Vol. 1. New York: Dover Publications, 1953.
- Smith, David Eugene. *History of Mathematics*, Vol. 2. New York: Dover Publications, 1953.