THE HAND CALCULATOR IN SECONDARY MATHEMATICS

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Calculation is an integral part of elementary school mathematics. In the junior and senior high school it has, until somewhat recently, been downgraded relative to the ideas, theorems, and formulas of secondary school mathematics. True, numbers were involved in factoring quadratics, in computing areas, and so on, but there was a tendency to make numerical situations artificially simple so as not to distract from the central ideas. More recently, there has been a trend to bring computation into the mainstream of mathematics development with a recognition that calculation can promote the growth of ideas as well as play its former, more pedestrian role. Some examples are found in computer-oriented mathematics texts which have appeared on the market over the past decade. Related to all of this has been the use of computer-related techniques of flow-charting and setting down precise algorithms for performing classes of calculations. The computer enforces a precision in the writing of algorithms which goes substantially beyond that required in more informal communication. The computer also focusses attention on the ideas related to computing while it takes over the more mechanical aspects of the computation itself.

Although it is generally recognized that the computer can make the kind of contribution referred to above, many secondary school mathematics students have limited, if any, access to the computer. The hand-held calculator makes possible an intermediate stage between conventional text book presentations and the more numerical approach of a computer-oriented course. With the increased popularity of these calculators, their use in mathematics courses becomes increasingly feasible. Teachers, and others, are attempting to determine how best to employ these fascinating electronic devices. Currently, we appear to be at a stage where a sharing of ideas is needed. Following are some areas which the writers believe warrant further exploration:

1. the development of mathematical concepts contained in the regular curriculum through an approach based on calculation;

2. the exploration of enrichment problems which have a strong numerical component with a view to generating mathematical ideas;

3. the use of the calculator to increase the accuracy and to reduce the tedium of arithmetic calculations, particularly repetitive ones, where calculation is supplementary to the ideas being developed; and

4. the development of computer-related techniques such as flow-charting and the writing of algorithms motivated by the availability and use of the hand calculator.

The selection of problems below illustrates the above applications. An emphasis is on the first application in problems 1, 2 and 3 (provided that they are used in the introductory phase), 4, 6, 18; on the second in problems 5, 7, 8; on the third in problems 9a, 10, 11, 12, 13, 14; and on the fourth in problems 9b, 15, 16, 17. Clearly most of the problems involve more than just one of the four areas listed above.

The lowest grade level at which the average student will likely be equipped with the ideas for solving the problem is indicated beside the problem. Where the problem appears to be accessible to any interested and competent secondary school mathematics student, it has been labelled with a "G," for general interest. In many instances the teacher may be able to make modifications that will enable students to attack the problem at a grade level below that indicated.

The solution to the problem depends not only on the teacher and student but upon the particular calculator available as well. In giving sample solutions, the writers do not claim that these are the best and, further, they have made assumptions about the calculator available to the student. Most of the problems can be done readily with a calculator which provides a memory along with the four arithmetic operations. For a few, additional functions such as the square root are useful.

1. (a) Given that measurements are accurate to the nearest cm, find the greatest and the least possible value of the area of a rectangle whose sides are measured as 56 cm and 83 cm. How many significant digits should be given in the product? How is the product best shown? (Grade VIII)

   (b) During a science experiment, students were required to complete the following calculation based on measurements they had made:

   $$K = \frac{2.4 \times 3.6}{1.8 - 1.6}$$

   All measurements were made to the nearest 0.1. What is the largest and smallest possible value of K? (Grade VIII)

2. Newton discovered a method for calculating the square root of a number to any desired accuracy by using successive approximations. The following example shows how you would use his method to find $\sqrt{151}$

STEPS	CALCULATIONS (On a calculator with 10 display digits)
(i) Guess $\sqrt{151}$	12
(ii) Divide $\sqrt{151}$ by the guess	$12.58\overline{3}$
(iii) Thus $12 < \sqrt{151} < 12.58\overline{3}$	

Use the average of these two numbers as the next guess.	$12.291\overline{6}$
(iv) Repeat steps (ii) and (iii) until the divisor and quotient are equal and retain this value for $\sqrt{151}$.*	(ii) 12.28474576 (iii) 12.28820621 (ii) 12.28820524 (iii) 12.28820572 (ii) 12.28820573 (iii) 12.28820572

*As this example indicates, the divisor and quotient may never be equal. In this case the last digit oscillates between 2 and 3. In cases such as this you could retain either number or round the answer back one digit.

(a) Use this mehod to find $\sqrt{17}$, $\sqrt{7.3}$, and $\sqrt{1295}$. (Grade VIII)

(b) Write a flow chart to demonstrate this method of finding square roots (Grade VIII).

3. (a) Given the function $f(x) = 2x^2 - 5x + 6$ and the point P (2, 4) on its graph, calculate the slope of the secant line $\overleftrightarrow{PQ}$ where Q has the following x-coordinate:

   (i) 3

   (ii) 2.5

   (iii) 2.1

   (iv) 2.01

   (v) 2.001

   (b) Make a conjecture about the slope of the tangent line to the graph at P. (Grade XI)

4. (a) Do each of the following and compare the result with what it should be theoretically.

   (i) $(457 \div 3) \times 3 =$

   (ii) $(458 \div 3) \times 3 =$

   (iii) $(459 \div 3) \times 3 =$

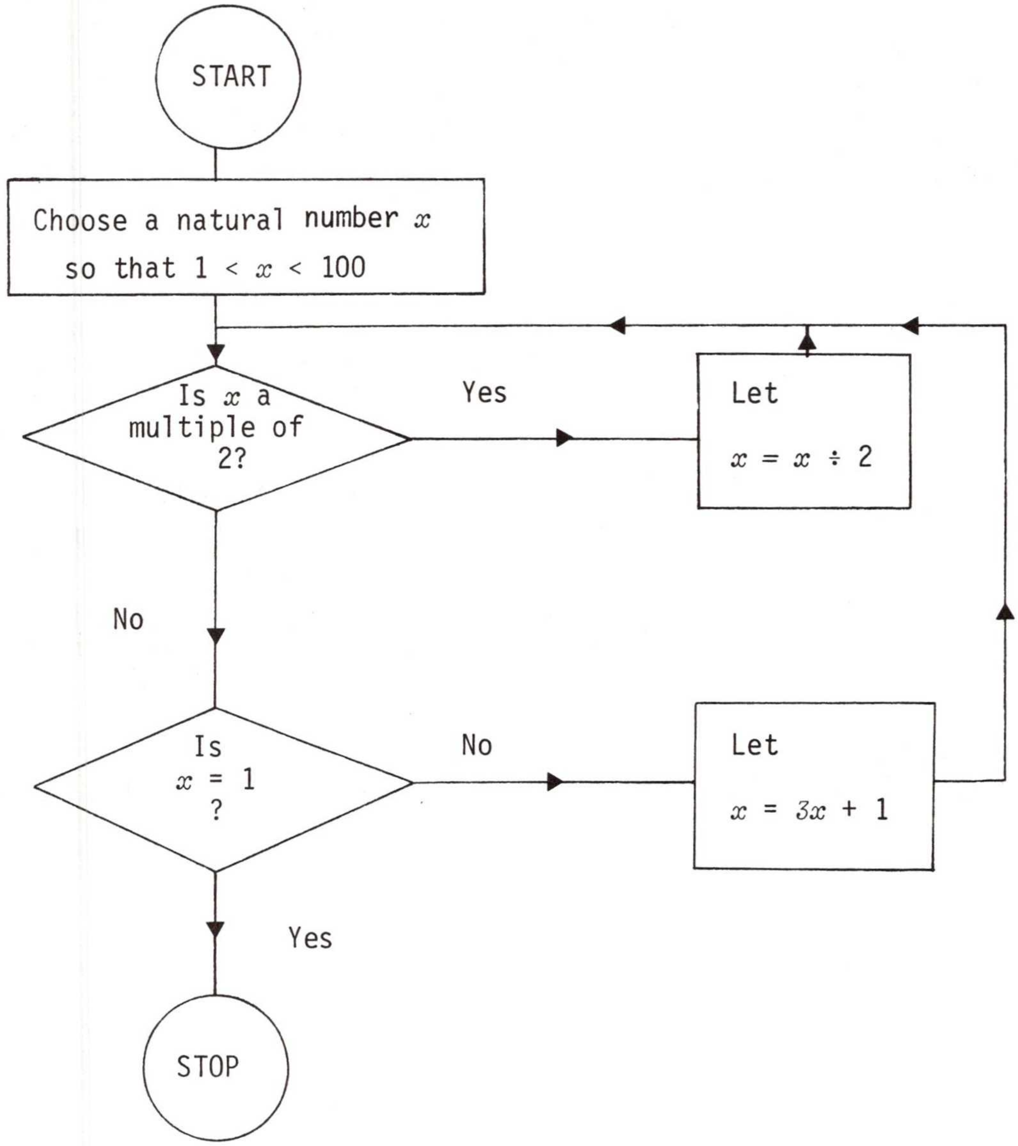
   (iv) $(15 \div 497) \times 497 =$

   (v) $(7 \div 1234) \times 1234 =$

   (b) Decide whether your calculator rounds or simply drops digits when there are more digits in the quotient than can be shown in the display. (G)

5. In some problems about natural numbers you are shown that a certain result occurs for several specific natural numbers. You may then be asked if this result holds for every natural number. A problem of this type is given in

flow chart form below. Work through the calculations for a few natural numbers, starting with small ones.



Pool your results with other class members. Cross-check any doubtful cases. Can you make a conjecture? Would checking 1,000 cases, all of which confirmed your conjecture, convince you that your conjecture is true? Would checking 1,000 cases prove your conjecture? What kind of a result would convince you that your conjecture is wrong? Discuss these questions with other class members. (G)

6. Solve

$x + y = 5$ (1)
$xy = 2$ (2)

by the following method of approximation, giving the solution for x and for y to 6 decimal place accuracy. First get a crude approximation to the answer, say, $x = 4$ and $y = 1$ (which satisfies the first equation, but not the second). Now, using the second equation to find y when $x = 4$ we get $y = \frac{2}{4} = 0.5$. The second equation is now satisfied but the first one no longer is. Now using this y-value in (1) we refine the approximation for x, getting $x = 5 - y = 4.5$. Continue in this manner (alternating between the two equations to refine successively the y-value and then the x-value) until you are satisfied that you have the solution to 6-decimal accuracy. (Grade XI)

7. In the Fibonacci sequence the first two terms are "1" and each successive term is the sum of the two previous ones. Thus the third term is 1 + 1 or 2, the fourth is 2 + 1 = 3, and so on. The first few terms are 1, 1, 2, 3, 5, 8, 13, . . .

   (a) Use the hand calculator to extend the sequence to the 20th term.

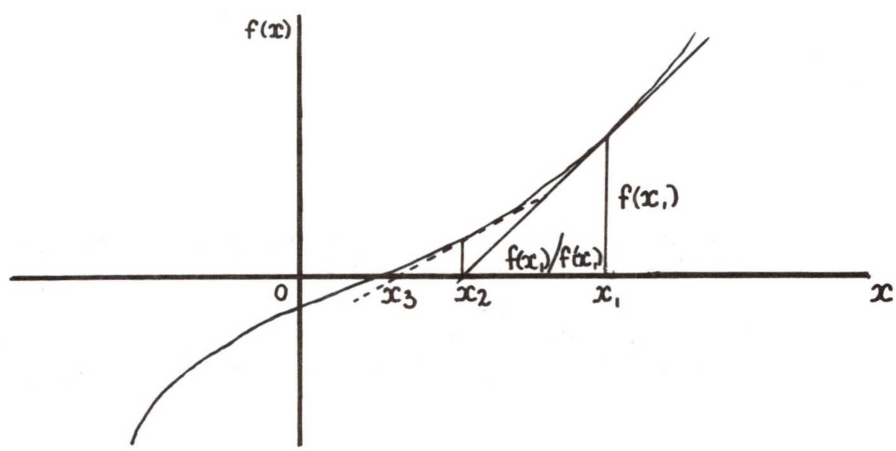
   (b) Find, and write down, the ratio of each term to the following one for the first 19 terms of the sequence. What do you notice?

   (c) The Greek "golden ratio" is expressed by $(\sqrt{5} - 1) \div 2$. Calculate this to 6 decimal place accuracy and compare with results obtained in (b). (Grade VIII)

8. The Newton-Raphson rule for successively refining approximations to a root of an equation $f(x) = 0$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The geometric picture looks like this:



With $x^3 + 3x^2 + 1 = 0$ and $x_1 = -3$, apply the Newton-Raphson rule twice to refine the approximation. (Math 31)

9. (a) Throughout history, many people have developed different approximations for π. Test each of the following to determine the number of significant digits in its decimal form.
(Recall $\pi = 3.141592653589...$) (G)

	Year	Developed by	Value for π
(i)	1800 - 1600 B.C.	Babylonians	$3\frac{1}{8}$
(ii)	1580 B.C.	Ahmes Papyrus (Egypt)	$\left(\frac{16}{9}\right)^2$
(iii)	200 B.C.	Archimedes (Greece)	$3\frac{1}{7}$
(iv)	150 A.D.	Ptolemy (Greece)	$\frac{377}{120}$
(v)	470	Tsu Chung-chih (China)	$\frac{355}{113}$
(vi)	510	Aryabhata (India)	$\frac{62832}{20000}$
(vii)	1220	Fibonacci (Italy)	$\frac{864}{275}$
(viii)	1580	Tycho Brahe (Europe)	$88/\sqrt{785}$
(ix)	1583	Simon Duchesne (Europe)	$3\frac{69}{484}$
(x)	1685	Adamas Kochansky (Poland)	$\sqrt{\frac{40}{3} - \sqrt{12}}$
(xi)	1769	Arima Shūki Sampō (Japan)	$\frac{428\ 224\ 593\ 349\ 304}{136\ 308\ 151\ 570\ 117}$ *

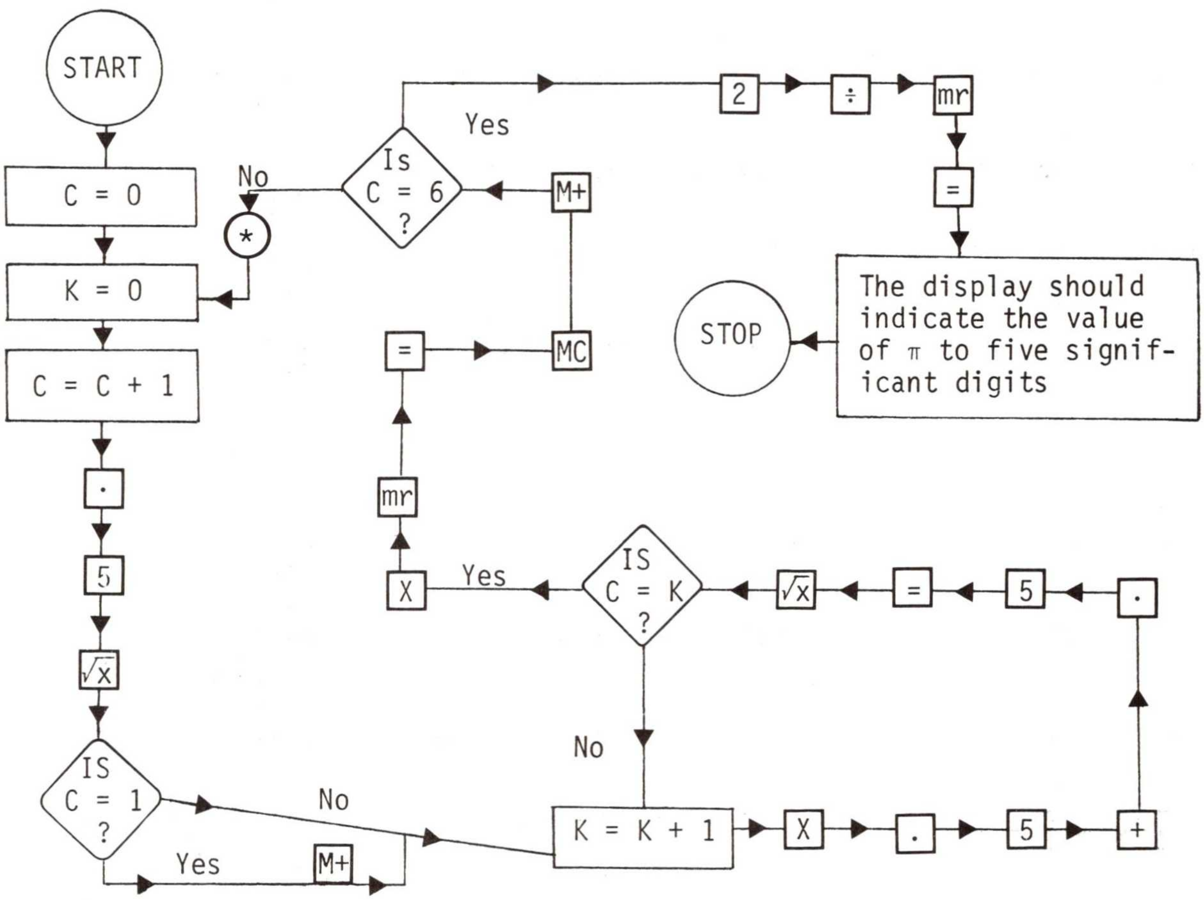
*Since this value likely exceeds the display digits in your calculator, you may have to retain only as many digits as your calculator will allow. The value will still be quite accurate.

(b) Quite a number of methods were developed for calculating π as accurately as one pleased. Unfortunately these methods involve excessive calculations; thus they were of little practical use for determining the value of π until the invention of the modern computer.

Francisco Vieta (1593) used the following continued product:

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}} \cdot \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}} \cdot \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}}} \cdots$$

Use Vieta's method to calculate π to show 5 correct digits. The following flow chart indicates the sequence of buttons you would press. (A square root function and memory are required.) (Grade X)



*Check point: you can check your readout with the answer key.

10. Following is a list of prime numbers between 1 and 100:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,
43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

(a) Using the above information and a hand calculator, show how you could systematically check any whole number less than 10,000 to determine if it is prime and, if not prime, to determine its factors.

(b) Find the prime factors of each of the following numbers using the scheme you have devised in (a) above. If the number is not factorable, state that it is prime:

(i) 456

(ii) 4356

(iii) 9991

(iv) 9027

(v) 1151

(Grade VII)

11. A student takes a student loan of $2,500 at the beginning of each of the years 1973, 1974, 1975, 1976. Interest on the loan is at 9% per annum, calculated monthly. He intends to repay the loan on a monthly basis commencing on January 1, 1977. Using a hand calculator, find the amount of his loan on that date. (Grade XII)

12. Find the product of the following two matrices:

$$\begin{pmatrix} 1.57 & 3.42 \\ -1.96 & 5.08 \end{pmatrix} \begin{pmatrix} 2.66 & -1.73 \\ -4.17 & 3.56 \end{pmatrix}$$

(Math 31)

13. six $x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

Find sin 31° 17' to 4 decimal places and compare your result with what you obtain from a set of tables. (Grade XII)

14. The binomial theorem yields the following series for $\sqrt{1+x}$, which converges for $|x| < 1$,

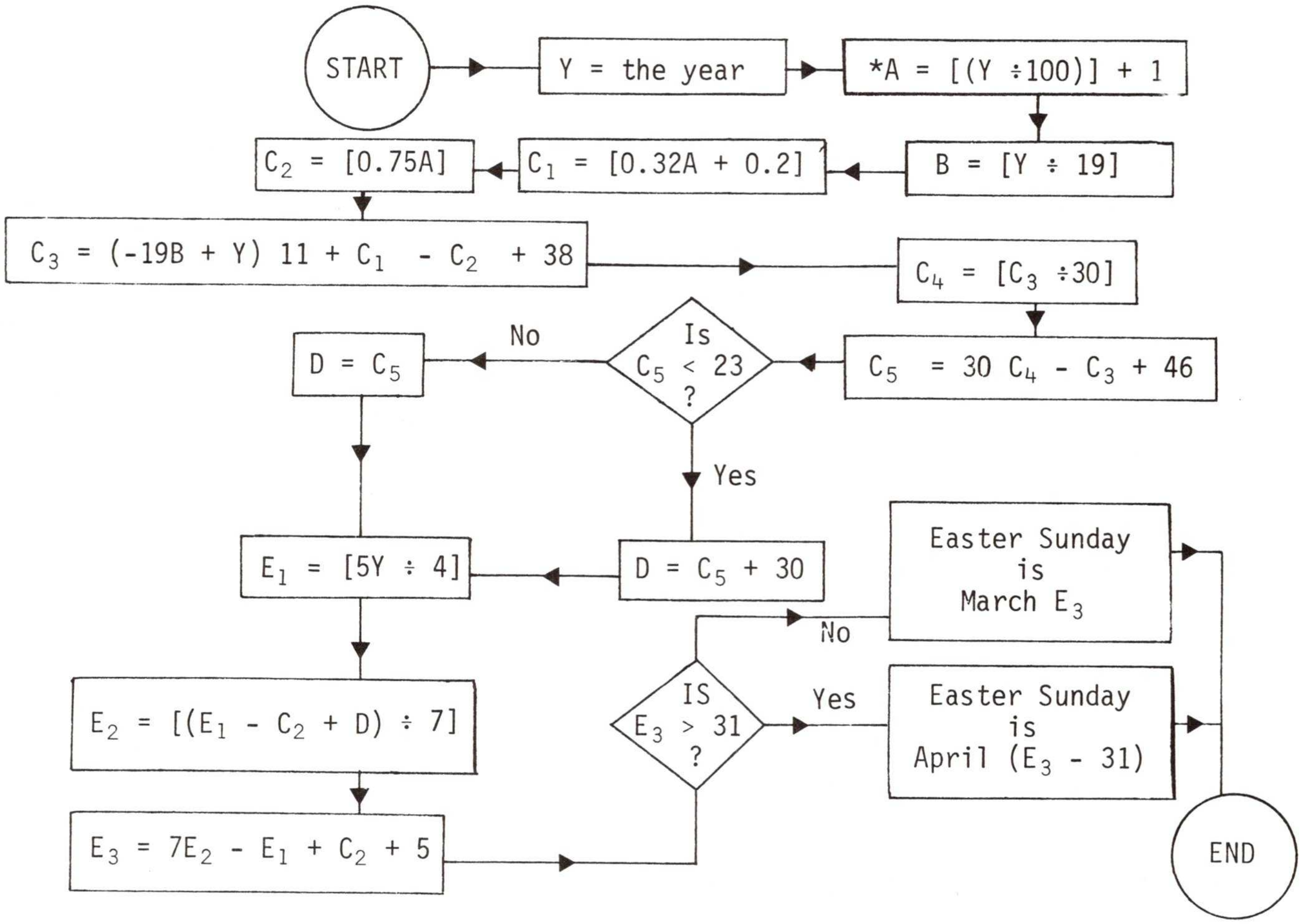
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$$

The first few terms give a good approximation to $\sqrt{1+x}$ if x is small.

(a) Using the first 4 terms, find $\sqrt{1.12}$ to 4 decimal places.

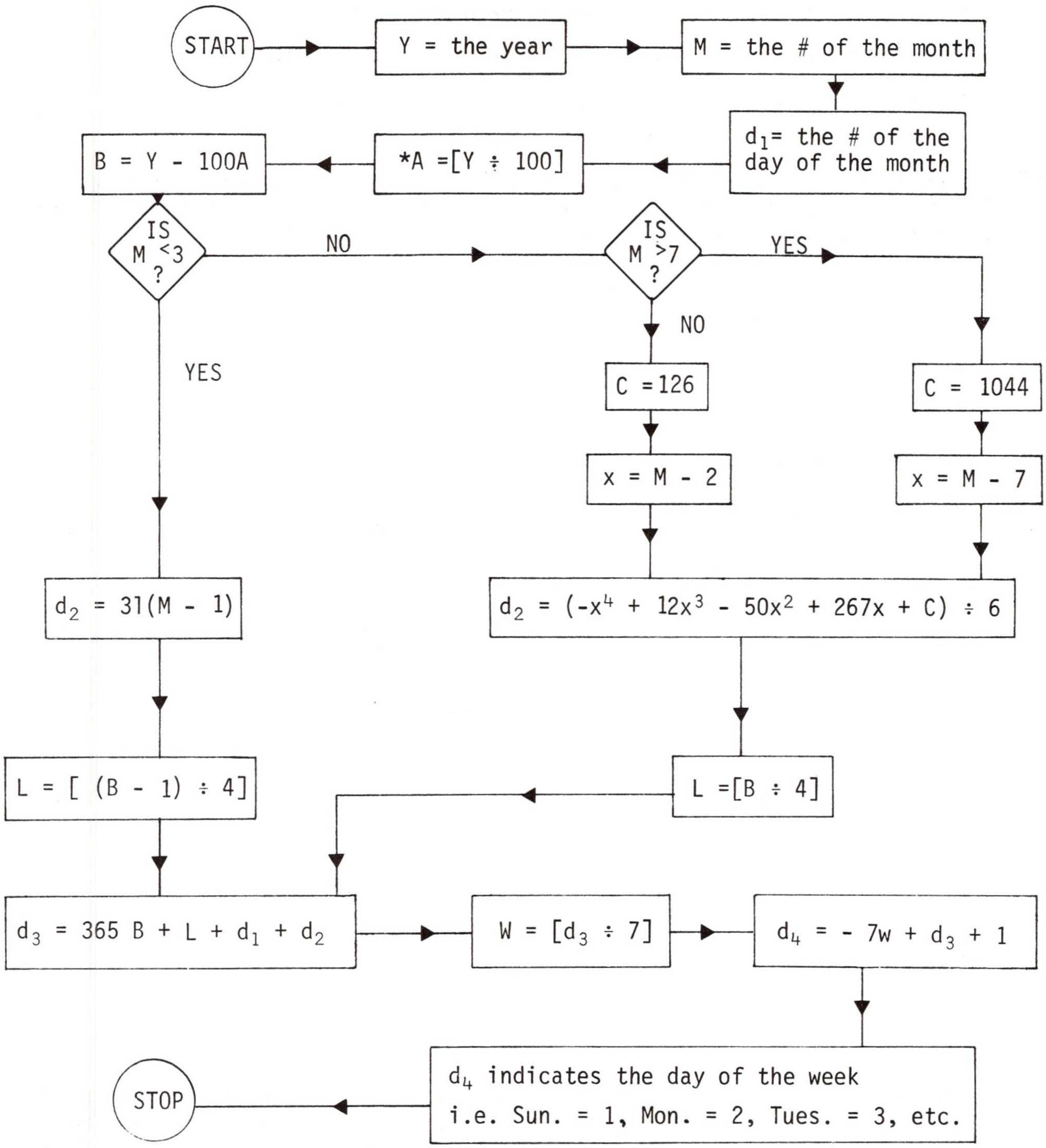
(b) Find $\sqrt{26}$ using the above series. (Hint: note that 26 = 25 x 1.04 so that $\sqrt{26} = 5\sqrt{1.04}$.) Square your result for $\sqrt{26}$ and compare with 26. (Grade XII)

15. The following flow chart can be used to determine the date of Easter Sunday for any year after 1582. Use this procedure to find the date of the next Easter Sunday. [G]



*The symbol [x] refers to the greatest interger not greater than x.

16. The following algorithm will give the day of the week for any date between the years 1900 and 2000. (a) Calculate the day of the week that you were born. (b) How would you alter the algorithm to provide for future centuries? (Grade IX)



This algorithm also gives:

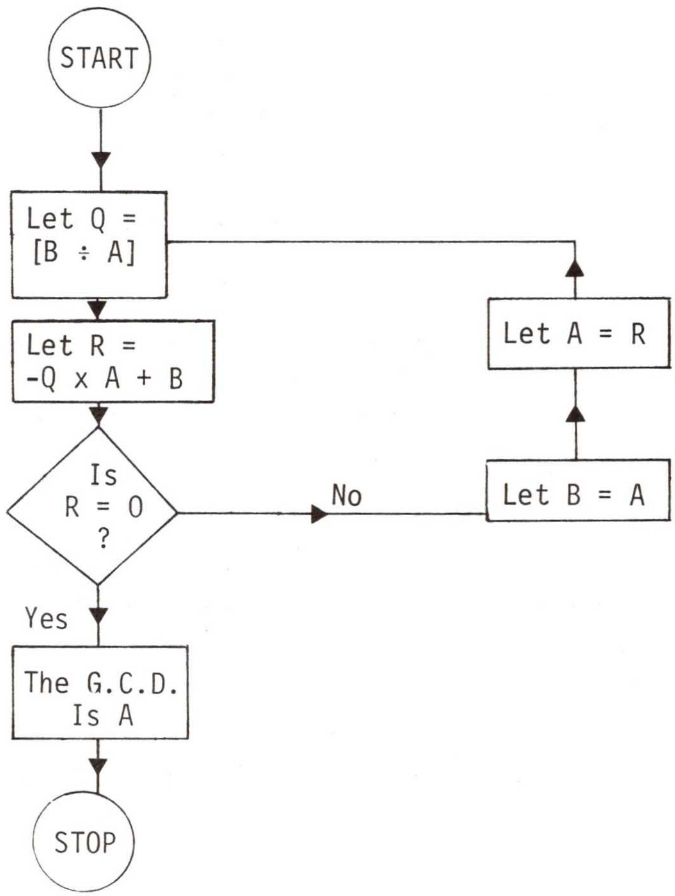
(i) The number of days into the year ($d_1 + d_2$). Add 1 for dates after February 29 in a leap year.

(ii) The number of days into the twentieth century (d_3).

(iii) The number of complete weeks into the twentieth century (w).

*[x] refers to the greatest integer not greater than x.

17. Following is the Euclidean Algorithm, in flow chart form, for finding the G.C.D. of the whole numbers A and B, (A < B). Use the algorithm and a hand calculator to determine the G.C.D. of 4469 and 4687. (Grade VII)



18. A student group did an experiment to get a rough measurement of the length of a molecule of oleic acid. They first obtained a mixture of oleic acid and alcohol in which the oleic acid was 1 part in 1,000 by volume. They dusted the surface of the water in a large shallow pan with lycopodium powder and then put one drop of the oleic acid mixture on the surface of the water near the center of the pan. The drop spread out over the surface in approximately the shape of a disc. (The powder made it easy to see the boundary of the film.) As alcohol dissolves readily in water, the students could assume that the film was entirely oleic acid. The students read in a book that the oleic acid molecules "stand on end" against the surface of the water in a layer one molecule thick.

The student group found that it took 48 drops of oleic acid to increase the volume in a measuring device by 1 cm^3. The diameter of the film was measured at approximately 9 cm.

Using the students' data, calculate the length of the oleic acid molecule, expressing the answer with one significant digit. (Grade XI)

SOLUTIONS AND COMMENTS

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1. (a) If measurements are to the nearest cm, then the least possible value of the area of the rectangle would occur if the true measures were 55.5 cm and 82.5 cm. Thus, the smallest possible area is 4,578.75 cm^2. Similarly, the largest dimensions that would be recorded as 56 cm and 83 cm would be 56.5 cm and 83.5 cm, respectively. Thus, the largest possible area is 4717.75 cm^2. It is clear that only two significant digits should be shown, and that the second digit is in some doubt. Although the product of 56 and 83 yields an area in cm^2 of 4,648, quoting all four digits implies an accuracy that is misleading. The product is best shown in scientific notation as 4.6×10^3 cm^2.

   (b) The largest possible K-value occurs with the maximum value of the numerator and the least value of the denominator.

   The largest possible K-value is $\frac{2.45 \times 3.65}{1.75 - 1.65} = 89.425$

   The least possible K-value is $\frac{2.35 \times 3.55}{1.85 - 1.55} = 27.808$

   Thus, the data do not yield even one significant digit.

2. (a) $\sqrt{17}$

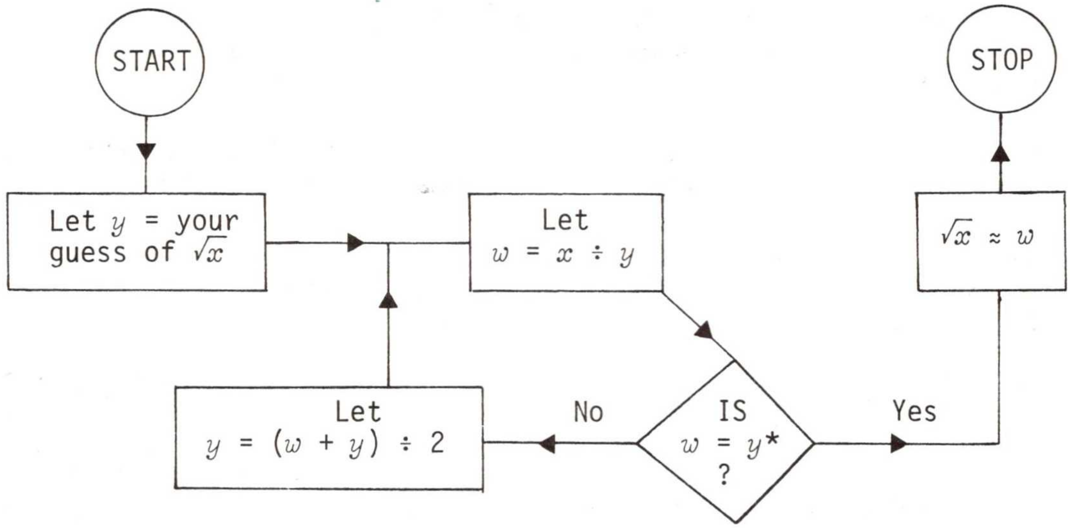
(i)	guess	4 (intermediate results will vary with different guesses)	
(ii)	17 ÷ 4 =	4.25	
(iii)	Average =	4.125	
(iv)	Repetitions:	(ii)	$4.\overline{12}$
		(iii)	$4.123\overline{106}$
		(ii)	4.123105191
		(iii)	4.123105625
		(ii)	4.123105626
		(iii)	4.123105625

$\sqrt{17} \approx 4.123105625$ or 4.123105626

$\sqrt{7.3} \approx 2.701851217$

$\sqrt{1295} \approx 35.98610843$

(b) To find $\sqrt{x}$



*Note the possibility that w may never actually equal y as illustrated in the example.

3. (a) (i) 5
(ii) 4
(iii) 3.2
(iv) 3.02
(v) 3.002

(b) It appears reasonable to conclude that the tangent line would have a slope of 3, since the slope of $\overleftrightarrow{PQ}$ gets closer to 3 as the point Q gets closer to P.

4. (a) Results depend on type of calculator used.

(b) If the results to part (a) are never too large, your calculator drops the extra digits. If they are sometimes too large and sometimes too small, the calculator is rounding off the last digit.

5. A group of students could check all the odd numbers between 1 and 100. In some cases, such as 27, for example, some persistence is needed; the numbers climb to over 3,000, then in a few more cycles tumble down to 1. A group of students could explore ways of sharing the work effectively. For example, if a student is checking 51, he will rapidly turn up 29 in his sequence. If that number has been checked, or if another student has been assigned the task of checking 29, then the first student goes on to the next number of his list. Students may notice that every second odd number, namely, each of 5, 9, 13 . . ., yields a smaller number at the end of one cycle. It is, therefore, unnecessary to check numbers of the form 4n + 1. A reasonable conjecture, based on the results, is that the procedure described will always result in 1 being attained. Checking 1,000 cases gives one some confidence in the conjecture, but this does not prove the conjecture, of course. The conjecture would be disproved if we could exhibit a

single example of a number for which the procedure did not lead to 1. If we could find any number which trapped us in a loop (that is, a cycle of the same numbers over and over again) then the conjecture would be disproved. To the best of the knowledge of the writers this problem is still open, that is, the conjecture is still a conjecture which no one has yet been able to prove or disprove. It will be of interest to some students that such an apparently innocent looking problem turns out to be so difficult.

6.

$x = 5 - y$	$y = 2 \div x$
	1
4	0.5
4.5	0.44444444
4.5555556	0.4390243
4.5609757	0.4385026
4.5614974	0.4384525
4.5615475	0.4384476
4.5615524	0.4384472
4.5615528	0.4384471
4.5615529	0.4384471

7. (a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, . . .

(b) 1, 0.5, 0.6666666, 0.6, 0.625, 0.6153846, 0.6190476, 0.617647, 0.6181818, 0.6179775, 0.6180555, 0.6180257, 0.6180371, 0.6180327, 0.6180344, 0.6180338, 0.618034, 0.6180339, 0.6180339.

The ratios alternately get larger and smaller but seem to be settling down toward a fixed value.

(c) $(\sqrt{5} - 1) \div 2 = 0.6180339$.

8. $f(x) = x^3 + 3x^2 + 1$, $f'(x) = 3x^2 + 6x$

$x_1 = -3$. $f(x_1) = 1$, $f'(x_1) = 9$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -3 - \frac{1}{9} = -3.1111111$$

$f(x_2) = -0.075446$, $f'(x_2) = 10.37037$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -3.1111111 - \frac{-0.075446}{10.37037} = -3.1038$$

9. (a) The number of significant digits is:

(i)	2	(vii)	4
(ii)	2	(viii)	4
(iii)	3	(ix)	3
(iv)	4	(x)	5
(v)	6	(xi)	29
(vi)	5		

If your calculator has 10 display digits, you would get 7 significant digits.

(b) The following records the values obtained on a calculator with 10 display digits.

C	K	* Check Point Read-Out
0, 1	0, 1	0.653281481
2	0, 1, 2	0.64072886
3	0, 1, 2, 3	0.637643574
4	0, 1, 2, 3, 4	0.636875504
5	0, 1, 2, 3, 4, 5	0.636683688
6	0, 1, 2, 3, 4, 5, 6	0.636635746

At this stage $\pi \approx 3.141513828$

If you continue this procedure, you will eventually get as many correct digits as you have on your calculator's display.

10. (a) For any number $N < 10{,}000$ check to see if 2 is a factor (and, if so, how many factors of 2 there are); then move up the list of given primes (3, 5, . . .) checking, as above, as far as is necessary. If, after removing a certain number of factors, the remaining factor is any number on our list of primes, then our factorization is complete. If, in attempting to factor the number (or a factor of that number) by repeated trial - by trying 2, 3, 5, . . . in order - we reach a point where the quotient is less than the divisor, then the number in question is a prime. The display on the calculator readily reveals whether one number is divisible by another.

(b) (i) $2 \times 2 \times 2 \times 3 \times 19$
(ii) $2 \times 2 \times 3 \times 3 \times 11 \times 11$
(iii) 97×103
(iv) $3 \times 3 \times 17 \times 59$
(v) prime

11.

1973	1974	1975	1976	1977
\$2500	\$2500	\$2500	\$2500	

$$A = 2500\,[(1.0075)^{48} + (1.0075)^{36} + (1.0075)^{24} + (1.0075)^{12}]$$

$$A = \frac{2500(1.0075)^{12}\ (1.0075^{48} - 1)}{(1.0075)^{12} - 1}$$

Enter 1.0075

Press [X] [=] [X] [=] [M+] [X] [=] [X] [MR] [=]

Record display minus 1, .0938062, as denominator of A

Press [CM] [M+] [X] [=] [X] [=] [-]

Enter 1

Press [X] [MR] [X]

Enter 2500

Press [÷]

Enter .0938062

Press [=]

Record display 12,575.65 as answer in dollars.

[M+] - add to memory
[MR] - recall memory
[CM] - clear memory

12. $$\begin{pmatrix} -10.0852 & 9.4591 \\ -26.3972 & 21.4756 \end{pmatrix}$$

13. First convert 31° 17' to 31.283 3333 radians by multiplying by π and dividing by 180 and then do the necessary calculation. Check against a table of values.

14. (a) With $x = 0.12$ the terms of the series given yield 1.058308 which checks to 4 decimal places with tables.

    (b) By substituting 0.04 into the terms of the series given we get 1.019804 which when multiplied by 5 gives 5.099020 which checks with tables to 6 decimal places.

15. $Y = 1977$
    $A = 20$
    $B = 104$
    $C_1 = 6$

C_2 = 15
C_3 = 40
C_4 = 1
C_5 = 36
? No
D = 36
E_1 = 2471
E_2 = 356
E_3 = 41
? Yes

Easter Sunday is April 10

16. Example: Suppose you were born April 7, 1960:

Y	= 1960	C	= 126
M	= 4	X	= 2
d_1	= 7	d_2	= 90
A	= 19	L	= 15
B	= 60	d_3	=22012
?	No	W	= 3144
?	No	d_4	= 5 i.e. Thurs.

You were born on a Thursday

17.

A	B	R
4469	4687	218
218	4469	109
109	218	0

The G.C.D. is 109.

So $\frac{4489}{4687} = \frac{41}{43}$.

18. Volume of oleic acid in drop = $\frac{1}{1000} \times \frac{1}{48}$ cm^3

Area of oleic acid film = $\pi \times 4.5^2$ cm^2

Thickness of film = $\dfrac{\frac{1}{1000} \times \frac{1}{48} \text{ cm}^3}{\pi \times 4.5^2 \text{ cm}^2}$ = 3×10^{-7} cm

Length of molecule = 3×10^{-7} cm, approximately.