Problem Solving with Nim Games

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The development of problem-solving skills should be a primary goal of any mathematics program. Nim games are an excellent vehicle for the development of problem-solving skills at all grade levels. Naturally the sophistication of the games presented to students will be a function of both grade level and previous experience in problem-solving situations.

We will examine selected one-pile and two-pile nim games. One-pile games are usually more elementary than two-pile games. A simple one-pile game has the following rules:

(1) Form a pile with ten markers.
(2) Players alternate turns, each removing one or two markers from the pile.
(3) The player who takes the last marker wins the game.

The teacher should explain the rules of the game and then allow students, working in pairs, ample time to familiarize themselves with the game in numerous contests. We believe adequate time spent allowing students to investigate the game is <u>essential</u> to the development of sound problem-solving skills. While this exploration time may appear to some as wasted and may even try the patience of teachers, it will greatly enhance the chances that:

(1) students will truly understand the rules of the game;
(2) students will appreciate the need for a winning strategy;
(3) students will develop skills for solving problems on their own and thus develop confidence in themselves.

Students who seem lost during this exploration time can be retrieved with challenges from expert players, be they other students or the teacher.

This exploration phase should evolve naturally into what we call the communication phase. Students should be encouraged to share their ideas and conjectures about winning strategies. These communication activities are valuable because they insure that students will:

(1) consider several hypotheses and therefore develop skills in the method of hypothesis construction and evaluation;
(2) practise verbal skills of self expression;
(3) develop social skills in the process of arguing for and against various conjectures;
(4) gain valuable feedback about their own conjectures by having them subjected to scrutiny.

Teachers can structure these important communication activities by asking some leading questions such as:

(1) In analyzing the game should you first consider what happens at the beginning of a game or at the end?
(2) Would it help to keep a record of exactly what happens in a few games? If so, what notation should be used to keep such a record?
(3) If you were to find a winning strategy, what would it look like?
(4) Are there any patterns which seem to develop in the games which can be used to predict a winner?
(5) Can you make a conjecture concerning a winning strategy which applies to at least part of a game?
(6) Having made a conjecture, can you test it by working out several examples and eventually find a logical basis for the conjecture?
(7) How can you expand your conjecture to the whole game? Can you test the resultant conjecture?

By spending adequate exploration time and adequate communication time, including attempts to answer some of the above questions, students will discover that leaving your opponent a pile of three markers will make it possible for you to win. Thus a pile of three markers is considered a "safe" position. Marker piles with one or two markers are considered "unsafe" because the opponent can win if you leave him these positions. A winning strategy consists of the ability to determine whether any position is "safe" or "unsafe." Any move the opponent makes on a "safe" position will leave an "unsafe" position, and from any "unsafe" position there is always a way to leave the opponent a "safe" position. If one plays so that a "safe" position is left after one's turn, a win is guaranteed.

The insights necessary to discover a winning strategy can be found by using the important problem-solving technique of organizing the data in tabular form.

TABLE 1

b	0	1	2	3	4	5	6	7	8	9	10
	s	u	u	s	?	?	?	?	?	?	?

b = the number of markers in the pile
s indicates a "safe" position
u indicates an "unsafe" position

It is apparent that zero markers is "safe"(s) to leave the opponent since this is the goal of the game. On the other hand, one or two are "unsafe" since the opponent will likely remove all the markers and win the game. Three markers in a pile is "safe" since any move the opponent makes on this position leads to an "unsafe" position. Asking the students to complete the table forces them to focus on the next step in finding a winning strategy. Since three markers in a pile is "safe," four or five markers in a pile must be "unsafe." Continuing in this way, students can complete the table and deduce a winning strategy. Students can then test their results against each other or the teacher.

The reader will note immediate possibilities for problems which require students to generalize or change their strategies. Variables in the game are:

(1) number of markers in the pile initially;
(2) maximum number of markers which may be removed in a turn;
(3) which player will take the first turn;
(4) whether the player taking the last marker wins or loses.

As just one example, consider the ten marker one-pile game with the following rule modification. Instead of removing one or two markers in one turn, allow players to remove one, two, or three markers. For this game the player who is forced to take the last marker loses. The table of "safe" and "unsafe" position then becomes:

TABLE 2

b	1	2	3	4	5	6	7	8	9	10
	s	u	u	u	s	u	u	u	s	u

b = the number of markers in the pile
s represents a "safe" position to leave an opponent
u represents an "unsafe" position to leave an opponent

In both of the nim games we have discussed the player who takes the first turn can guarantee a victory. Thus there is no hope for a player who does not get the first turn against a player who possesses and plays the winning strategy.

Two-pile nim provides an excellent opportunity for students to improve problem-solving skills acquired while studying one-pile nim. Again the rules can be modified to provide many problem-solving experiences. A good introductory game has the following rules:

(1) Form two piles of markers with five markers in one pile and ten markers in the other.
(2) Players alternate turns, and during each turn a player may remove one marker from either pile or one marker from both piles.
(3) The player who takes the last marker loses the game.

The learning steps required for the one-pile nim game remain valid for the two-pile game. We believe these steps are vital to the development of sound problem-solving strategies. Therefore we recommend:

(1) ample exploration time to investigate the game;
(2) ample time for students to share their ideas: productivity of this communication phase may be enhanced by forming teams, who work together to decide on the next move;
(3) ample time for organization of data, which, as before, can be prompted by leading questions from the teacher;
(4) ample time for constructing and testing hypotheses.

As we analyze this two-pile game, the concepts of "safe" and "unsafe" positions to leave an opponent are useful.

TABLE 3

b \ a	0	1	2	3	4	5	6	7	8	9	10
0		s	u	s	u	s	u	s	u	s	u
1	s										
2	u										
3	s										
4	u										
5	s										

a = the number of markers in one pile
b = the number of markers in the other pile.

Previous experience with one-pile games should ensure that students can fill out the first row and first column of the table. In order to find more "safe" positions, students must first discover that (1,1), (1,2), and (2,1) are all "unsafe" because they can lead to the "safe" positions of (1,0) or (0,1) in a single move. Thus (2,2) must be "safe" because any move made on (2,2) leaves an "unsafe" postion. The entire table can be completed in similar fashion, since every position which is "safe" leads to three "unsafe" positions. These "unsafe" positions can then be used to find other "safe" positions.

Once the table has been completed, a winning strategy has been found. A problem which requires generalization of this winning strategy is created by changing the number of markers placed in each pile initially. What if one pile has 5,000 markers and the other had 495? Completing a table of this size is not practical. Can students, given ample time and working together, find a simple rule which will indicate whether any given position is "safe" or "unsafe"? Problem-solving skills required for this task include looking for patterns within organized data, making conjectures concerning these patterns, and testing these conjectures. Application of these techniques will lead students to discover the "safe" positions. When only one pile has markers the "safe" positions are those with an odd number of markers in that pile and when there are markers in both piles the "safe" positions are those in which both piles have an even number of markers. After students have had time to investigate the problem, organize data, make conjectures concerning possible solutions, and establish for themselves by examples the validity of those conjectures, rigorous definitions and proofs are appropriate for mathematically mature students.

The goal of the two-pile game may be changed from (1,0), (0,1) to (0,0). That is, the player who takes the last marker wins. For many of the games the winning strategy changes drastically when the goal is changed.

In summary, nim games provide excellent opportunities for teaching problem-solving skills because:

(1) the fact that they involve actually moving physical objects implies that they are easily learned;
(2) the fact that they involve a competitive situation helps to focus the students' attention on the problem;
(3) simple rule changes create a variety of similar problems which allow students to reinforce with practice newly acquired problem-solving skills;
(4) they can be modified in order to find the appropriate game for any age group or maturity level;
(5) they provide problem-solving activities different from those usually found in complex word problems. Problem solving and word problems are too often considered the same thing. While word problems are important, they are by no means the only vehicle for teaching problem-solving skills;
(6) nim games are rewarding for students in the sense that the discovery of winning strategies:
    (a) offers students great satisfaction in the knowledge they have solved a problem through their own efforts,
    (b) allows them to win all games or fully understand why they cannot win, and
    (c) most importantly, develops problem-solving strategies which will be valuable throughout a lifetime!