Romance in Problem Solving

by

H. Laurence Ridge
University of Toronto

So as not to disappoint those who have read the title in a vernacular sense, here is a math dilemma based on matters of the heart.

> On Friday evenings, planes leave Edmonton International at hourly intervals for Vancouver and at hourly intervals for Calgary. Ms. Thema Mattix, a liberated Edmonton math teacher, has an admirer in both of these cities and decides which one she will visit each weekend by taking the first plane out after she reaches the airport. Although school circumstances and variable traffic conditions make her arrival time at the airport completely unpredictable, she finds herself in Vancouver four weekends out of five. How is this possible?

The term 'romance' actually refers to the first stage of Whitehead's three cycles of intellectual activity: "romance, precision, and generalization" (as set out in The Aims of Education). The romantic stage is characterized as follows:

> The first procedure of the mind in a new environment is a somewhat discursive activity amid a welter of ideas and experience. It is a process of becoming used to curious thoughts, of shaping questions, of seeking for answers, of devising new experiences, of noticing what happens as a result of new ventures. This process is both natural and of absorbing interest. (p. 32)

In case you read into these remarks a call for Summerhill-style (total) freedom, Whitehead hastens to add:

> This stage of development requires help and even discipline. The environment within which the mind is working must be very carefully selected... [However,] a block in the assimilation of ideas inevitably arises when a discipline of precision is imposed before a stage of romance has run its course in the growing mind. ... The cause of so much failure in the past has been due to the lack of careful study of the due place of romance. [Precision] has been the

> sole stage of learning in the traditional scheme of education. [To sum up,] without the adventure of romance, at the best you get inert knowledge without initiative, and at worst you get contempt of ideas - without knowledge. (pp.32-33)

In the realm of problem solving as considered in most schools, I see the stage of romance as a period of 'mucking about' - as the British vernacular would have it - brainstorming and experimenting with various strategies virtually uninhibited by rules or algorithms. Call it "development of heuristics" if you will. But - the problems have to be appropriate. The environment must be right - as per Whitehead.

Rather than go right away to the currently fashionable non-algorithmic process problems à la Carole Greenes and others, let's consider a more romantic approach to some traditional text-book-style problems, an approach which may both assist the lesser able student and also offer further insights to the more mathematically able folks.

The treatment presented in this paper is essentially one of numerical analysis which involves the freedom of romance as a reasonable starting point. The methods suggested do lead to precision and even generalization stages. Overall, the message is that much more can be derived from text-book-style problems than arises from the traditional algebraic algorithmic approach. For some readers the approach may smack of too much precision albeit a different form than the traditional algebraic algorithmic approach. From a comparative and realistic point of view, however, the amount of investigative freedom encouraged by the approach is considerable.

Traditionally, when we encourage students to develop 'alternative solutions', our expectations are usually at the precision stage. Here is a case in point.

> Solve in as many ways as you can.
>
> The sum of three consecutive integers is five more than the sum of the least and the greatest of the consecutive numbers.
> What are the numbers?

What we as teachers might expect, or at least, what students would think we expect is a variety of algebraic solutions such as:

I	II	III
Let x represent the smallest of the three consecutive integers. Then the other two integers are x+1 and x+2.	Let x represent the greatest of the three consecutive integers. Then the other two integers are x-1 and x-2.	Let x represent the middle integer of the three consecutive integers. Then the other two integers are x-1 and x+1.

(The third approach is considered to be the most clever since a sum is involved and the constant terms will vanish.)

Now, for students who may be having difficulty with algebraic expressions and equations and in order to introduce what can be an insightful approach, suggest that they try a "guesstimate" of what the three numbers might be. The idea is not to come up with a correct answer via guessing, nor to keep guessing until a correct answer is found, nor to keep making adjustments on the basis of 'too high' or 'too low'. The purpose of this romantic playing with numbers is to investigate the structure underlying the problem. What is really happening to numbers when they are combined as the problem dictates? No matter what trio of consecutive integers is chosen, there has to be a check as to whether this trio solves the problem. For example, should 7, 8, and 9 be selected, the sum of the integers is 24 but when 5 is added to the sum of the greatest and the least, the sum is 21.

It is at this stage that some teacher guidance would be appropriate. Students should be encouraged:

a) to write down what they are doing, and
b) to write it down in unsimplified form
so that patterns can be observed more readily.

The above check of 7, 8, and 9 could easily be done mentally as could checks on several other guesstimates. This style of romantic venture, however, could deteriorate into a guess-til-you-get-there approach which would not serve the solver particularly well in the long run.

The following is a suggested checking format. The assumption here is that for lesser able students it is easier to check a proposed answer than to formalize one in the first place.

GUESSTIMATE: 7, 8, 9

CHECK: $7 + 8 + 9$ $= 24$ $\qquad$ $7 + 9 + \boxed{5}$ $= 21$

The fact that 7, 8, 9 is not the sought after trio pales upon realization from the check that the $\boxed{5}$ must be the middle number of the trio in order to make the sums equal. Should such insight into the 4, 5, 6 solution not arise, the development of a routine for checking any guesstimate leads to a pattern for solution by more precise means.

The romantic aspect lies in the solver's being able to start anywhere and continue in such a fashion, not looking for the answer so much, as for a pattern which may lead to a more precise or insightful approach to the problem. Should a solution arise during such an initial procedure, that is a bonus. It must be emphasized that this romantic playing with numbers is not expected to yield an answer directly.

If a successful trio has not emerged, let's finally take the "numbers that work." For three consecutive integers, we'll need a smallest 'number', a second one or 'number + 1', and a third one or 'number + 2'. (These expressions arise from student experience with specific consecutive triples of integers.) Using the format of the 'check', we have:

$$\text{Number} + (\text{Number} + 1) + (\text{Number} + 2) = \text{Number} + (\text{Number} + 2) + 5$$

[EXPLANATION: The "=" sign
is used because these are
the "numbers that work."
The sums have to be equal!]

or simply,

$$n + (n + 1) + (n + 2) = n + (n + 2) + 5 \text{ , etc.}$$

What we have then, is a romantic approach to a precise method.

'Age problems' can be interesting. Note that this one is set in something of a puzzle motif - again - creating an environment.

> "You've got to be kidding!!"
>
> John is 19 years old and his sister Susan is only 1 year old. In how many years will John be:
>
> a) 7 times as old as Susan?
> b) 4 times as old as Susan?
> c) only twice as old as Susan?
> d) the same age as Susan?

Strategy I: Use guesstimation to help form an equation.

Guesstimate	John's age	Susan's age
8 (years)	$19 + \underline{8} = 27$	$1 + \underline{8} = 9$

$$\text{but } 27 \neq 7 \times 9$$

(After a sufficient number of guestimates to see the pattern of checking ...)

Number that works: n (years) $19 + n = 7(1 + n)$, etc.

Strategy II: Systematic numerical analysis (which adds a dimension of precision to a romantic beginning)

Such an approach may well be suggested during a romantic interlude with numbers. It might be noted by chance that

$$19 + \underline{8} = 3(1 + \underline{8})$$

or that there are other integral age relationships not even mentioned in the problem. This raises the question as to what other integral multiples are possible. (Hence we have a suggestion of generalization even within a romantic context.)

Let's start right from ground zero to see just what is going on.

Number of years from now	John's age (years)	Susan's age (years)	Integral multiple ?
0	19	1	19 = $\underline{19}$ x 1
1	19 + $\underline{1}$	1 + $\underline{1}$	20 = $\underline{10}$ x 2
2	19 + $\underline{2}$	1 + $\underline{2}$	21 = $\underline{7}$ x 3 AHA! - part a solved!)
3	19 + 3	1 + 3	No
4	19 + 4	1 + 4	No
5	19 + 5	1 + 5	24 = $\underline{4}$ x 6 (Part b)
6	19 + 6	1 + 6	No
7	19 + 7	1 + 7	No
8	19 + 8	1 + 8	27 = $\underline{3}$ x 9
9	19 + 9	1 + 9	No
10	19 + 10	1 + 10	No
11	19 + 11	1 + 11	No, but get 30 = $\underline{2.5}$ x 12 (How
12	19 + 12	1 + 12	No much farther to go?
13	19 + 13	1 + 13	No
14	19 + 14	1 + 14	No
15	19 + 15	1 + 15	No
16	19 + 16	1 + 16	No

We might note at this time that only odd numbers added to 19 will give an even result (divisible by 2).

17	19 + 17	1 + 17	Finally! 36 = $\underline{2}$ x 18 (Part c)

It is interesting to note that no matter what the age difference, if any two people live long enough one will be twice as old as the other, and, regardless of their birthdays, will be twice as old for a period of time totaling one year.

Now, how long will it take to get to 1, that is, the same ages?

Although the question in part d is absurd on the basis of common sense (since there is always a difference of 18 years) it leads to a more precise consideration of the relations among numbers based on the above analysis.

Number of years from now	John's age (years)	Susan's age (years)	Integral multiple
n	19 + n	1 + n	k

If n and k are "numbers that work," then

$$19 + n = k(1 + n) \quad (\#)$$

Now, to be 'precise', let's see what limitations there are on k.

$$k = \frac{19 + n}{1 + n}$$

As n gets increasingly large, $\frac{19 + n}{1 + n}$ gets closer to 1. More precisely,

$$\frac{19 + n}{1 + n} = \frac{\frac{19}{n} + 1}{\frac{1}{n} + 1} \longrightarrow 1 \text{ as } n \rightarrow \infty$$

(n increases indefinitely)

$$\left(\text{since } \frac{1}{n} \text{ and } \frac{19}{n} \longrightarrow 0\right)$$

(A practical application of this phenomenon can be seen with older people. The older they get, the less difference there appears to be in their ages even though the numerical difference is constant.)

To see mathematically that k cannot be one even though the ratio $\frac{19 + n}{1 + n}$ can be made as close to one as we please by taking n large enough, write equation (#) as $19 + n = k + kn$ and solve for n to get

$$n = \frac{19 - k}{k - 1}, \text{ so long as } k \neq 1.$$

Hence for integral n, k cannot be 1. This treatment could be generalized and thereby lead to the concept of 'limit of a sequence'.

Another way of looking at the relative behaviours of n and k is to apply a number theoretic approach when

$$k = \frac{19 + n}{1 + n}$$

and write it in the form $k = 1 + \frac{18}{1 + n}$ upon division by $1 + n$. From this form it can be seen that k is integral only when $n + 1$ is a factor of 18, that is, when n = 0, 1, 2, 5, 8, and 17. These values of n correspond to k = 19, 10, 7, 4, 3, and 2, respectively. Similarly, the equation

$$n = \frac{19 - k}{k - 1}$$

can be written in the form

$$n = -1 + \frac{18}{k - 1}$$

From this form it can be seen that n is integral only when $k - 1$ is a factor of 18, the difference in the ages. This condition yields the same pairs of integers as above.

Here we have seen a case of an initial romantic investigation suggesting a more systematic search of a relationship within a seemingly inocuous problem which in turn has led to an intuitive treatment of the limit of a sequence. Once a romance has been started who knows where it may lead? Oh, yes. Each plane for Calgary leaves 12 minutes after a Vancouver plane.