Recent Advances in Mathematics Education: Ideas and Implications

by

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There have been major changes in mathematics education research over the past decade. Research in education is now highly interdisciplinary, with contributions from cognitive psychologists, workers in artificial intelligence, etc. There are new people, new perspectives, new methodologies -- and most important, new results. Taken as a whole, these results promise to re-shape our understanding of the learning and teaching processes. In this paper I will discuss one aspect of recent work, and its implications.

The three examples I'm going to discuss in this paper seem on the surface to have little to do with each other. John Seely Brown and Richard R. Burton have done a detailed analysis of the way elementary school children perform certain simple arithmetic operations. John Clement, Jack Lochhead, and Elliot Soloway have studied the way that people translate sentences like "There are six times as many students as professors at this college" into mathematical symbolism. My work consists of an attempt to model "expert" mathematical problem solving, and to teach college freshman to "solve problems like experts." Yet all three of these studies share a common premise, and their results tend to substantiate it. That premise is the following:

> There is a remarkable degree of consistency in both correct and incorrect mathematical behavior on the part of both experts and novices. This consistency is so strong that it may often be possible to model or simulate that behavior, at a very substantive level of detail.

The implications of this assumption for both the teaching and learning processes are enormous. First, consider the notion that much of our students' incorrect behavior can be simulated -- and hence predicted. This means that many of their mistakes are not random, as we often assume, but the result of a consistently applied and incorrectly understood procedure. In consequence, the student does not need to be "told the right procedure"; he needs to be "debugged." This idea lies at the heart of the Brown and Burton work. It is also central to Lochhead and Clement's work, where we will see that the simple process of translating a sentence into algebraic symbols is far more complex than it at first appears. The other side of the coin has to do with the consistency of expert behavior. That, of course, is the assumption made in artificial intelligence -- where the attempt is made to model expert behavior in enough detail so that it can be simulated on a computer. If that seems plausible, then another step should seem equally plausible: model expert behavior so that humans, rather than machines, can simulate it. That is, teach students to "solve problems like experts" by training them to follow a detailed model of expert problem solving. That is the idea behind my own work.

1. A Close Look at Arithmetic.

In this section I offer a distillation of Brown and Burton's paper "Diagnostic Models for Procedural Bugs in Basic Mathematical Skills." There is much more in that paper than I can summarize here, and it is well worth reading in its entirety.

The key word in the title of their paper is "bug." It is, of course, borrowed from programming terminology -- and is fully intended to have all of the connotations that it usually does. While a seriously flawed program may fail to run, a program with only one or two minor bugs may run all the time. It may even produce correct answers most of the time. Only under certain circumstances will it produce the wrong answer -- and then it will produce that wrong answer consistently.

Often one discovers a bug in a computer program when it produces the wrong answer on a test computation. One might hope to find the bug by reading over the listing of the program and catching a typographical error or something similar. It is usually easier, however, to trace through the program and see when it makes a computational error. At that point, one knows where the source of difficulty is and can hope to remedy it. If the basic algorithm were simple enough, it might be possible to guess the source of error by noticing a pattern in the series of mistakes it produced. Thus one might be able to find the bugs in a program -- without even having a listing of it. For example, see if you can discover the bug in the following addition program from the five sample problems.

$$\begin{array}{r} 41 \\ \underline{+9} \\ 50 \end{array} \quad \begin{array}{r} 328 \\ \underline{+917} \\ 1345 \end{array} \quad \begin{array}{r} 989 \\ \underline{+52} \\ 1141 \end{array} \quad \begin{array}{r} 66 \\ \underline{+887} \\ 1053 \end{array} \quad \begin{array}{r} 216 \\ \underline{+12} \\ 229 \end{array}$$

Of course, if you don't have a listing of the program, you can never be certain that you have the right bug. However, you can substantiate your guess by predicting in advance the mistakes that the program would make on other problems. For example, if you have identified the bug which resulted in the answers in the previous five problems, you might want to predict the answers to the following two:

$$\begin{array}{r} 446 \\ \underline{+815} \end{array} \quad \begin{array}{r} 201 \\ \underline{+399} \end{array}$$

This particular bug is rather straightforward. We can get the same answers as the program for each of the five sample problems by "forgetting" to reset the "carry register" to zero: after doing an addition which creates a carry in a column, simply add the carry to each column to the left of it. For example, in the second problem, 8 + 7 = 15, so we carry 1 into the second column. That gives us a sum of 4. If the 1 is still carried to the third column, that gives us 1 + 3 + 9 = 13. The same difficulties arise all the way across the board. Using this bug, one would predict answers of 1361 and 700 to the two extra problems.

A student might have this "bug" in his own arithmetic procedure, just as the computer program might. In fact, a child might well use his fingers to remember the carry, and simply forget to bend the fingers back after each carry is added. This would produce exactly the bug above.

The finding of bugs is far more than an exercise in cleverness: it has tremendous implications for the way we teach. The naive view of teaching is that the teacher's obligation is to present the correct procedure coherently and well, and that if anything goes wrong, it is simply because the students have not yet succeeded in learning that procedure. The above example (and many more in the text) suggest that something very different is happening. Suppose a student is making consistent mistakes. The teacher who can diagnose such a bug in that student stands a decent chance of being able to remedy it. The teacher who looks at the student's mistakes and concludes from them simply that the student has not yet learned the correct procedure, is condemned simply to repeat the correct procedure -- with much less likelihood that the student will perceive his own mistakes and begin to appropriately use the correct procedure.

If one makes the assumption that a student's behavior is consistent when it is wrong, then the issue appears to be theoretically simple. You begin with the correct procedure, and then at each step generate what might be considered plausible bugs. Next, you create a series of test problems so that the student's answers to those problems indicate his bugs. Finally, after identifying the bugs, you intervene directly to remedy them.

While this theory may sound remarkably simple, the implementation is actually quite complex. First, it is a surprisingly complicated task to write down all the operations that one has to do to add or subtract two - three digit numbers. Primitive operations involved in subtraction, for example, include knowing the difference between any two single digits, being able to compare two digits, knowing when it is appropriate to borrow, being able to borrow, knowing to perform operations on the columns in sequence from right to left, and many, many more primitive operations. Any flaw in one of these procedures causes a bug which needs to be diagnosed; flaws in more than one procedure cause compound bugs which may be even more difficult to diagnose. Brown and Burton hypothesized the following list of nine common procedural mistakes in the simple subtraction algorithm. When one considers possible combinations of these, things start to get out of hand very rapidly.

$\begin{array}{r} 143 \\ \underline{-28} \\ 125 \end{array}$	The student subtracts the smaller digit in each column from the larger digit regardless of which is on top.

143 -28 125	When the student needs to borrow, he adds 10 to the top digit of the current column without subtracting 1 from the next column to the left.
1300 -522 878	When borrowing from a column whose top digit is 0, the student writes 9 but does not continue borrowing from the column to the left of the 0.
140 -21 121	Whenever the top digit in a column is 0, the student writes the bottom digit in the answer; i.e., 0-N = N.
140 -21 120	Whenever the top digit in a column is 0, the student writes 0 in the answer; i.e., 0-N = 0.
1300 -522 788	When borrowing from a column where the top digit is 0, the student borrows from the next column to the left correctly but writes 10 instead of 9 in this column.
321 -89 221	When borrowing into a column whose top digit is 1, the student gets 10 instead of 11.
662 -357 205	Once the student needs to borrow from a column, he continues to borrow from every column whether he needs to or not.
662 -357 215	The student subtracts all borrows from the left-most digit in the top number.

Based on the premise that students do indeed follow certain consistent procedures, Brown and Burton tested this list empirically with the scores of 1325 students on a 15-item subtraction test. Their data indicates that more than 40 percent of the errors made on the test could be attributed to "buggy" behavior. In particular, more than 20 percent of the solution sheets were entirely consistent with one of their hypothesized bugs. (That is, all of the answers were exactly what that particular faulty algorithm would produce.) Another 20 percent of the solution sheets indicated behavior which was strongly consistent but not identical with such a bug.

Further, the analysis of the students' performance on this test, led to the identification of new "bugs." Of the 1325 students tested, 107 students had a bug in their "borrow from zero" procedure. In consequence, they had missed all 6 of the 15 problems on the test which called for borrowing from zero. In the original interpretation of the data, those 107 students were simply identified as students who scored 60 percent. Later they were identified as students who have not yet mastered the technique of borrowing from zero.

2. A Look at "Simple" Word Problems.

For a number of years, a group at the University of Massachusetts at Amherst has been studying a variety of students' misconceptions in college-level physics and mathematics. This discussion is based primarily on two of their working papers, "Translating Between Symbol Systems: Isolating a Common Difficulty in Solving Algebra Word Problems" by John Clement, Jack Lochhead and Elliot Soloway, and "Solving Algebra Word Problems: Analysis of a Clinical Interview" by John Clement. These papers deal with college-level students, and (at least at first) with subject matter "appropriate" for students at this level. Yet, there are two very strong similarities between this work and the work described in section 1. First, a process which is "simple" to do correctly may be a rich source of potential errors. Second, there is an almost remarkably perverse consistency in the way that students make mistakes -- to the point where remediation is rather difficult, even if one understand what the student is doing. Finally, there is an interesting contrast between the "static" nature of mathematical language and the "dynamic" nature of a programming language.

Since Clement, Lochhead, and Soloway were dealing with college-level students, the authors began with problems of some complexity. One problem, for example, asked the student to determine what price, P, to charge adults who ride a ferry boat, in order to have an income on a trip of D dollars. The students were given the following information: There were a total of L people (adults and children) on the ferry, with 1 child for each 2 adults; children's tickets were half price. The students were asked to write their equation for P in terms of the variables D and L. When fewer than 5 percent of the students given the problem solved it correctly, the authors began to use simpler and simpler problems. After a sequence of increasingly easier problems, they wound up using problems like the ones given in Table 1.

Table 1

1. Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this University." Use S for the number of students and P for the number of professors.

2. Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there are five people who ordered strudel." Let C represent the number of cheesecakes and S represent the number of strudels ordered.

3. Write a sentence in English that gives the same information as the following equation: A = 7S. A is the number of assemblers in a factory. S is the number of solderers in a factory.

4. Spies fly over the Norun Airplane Manufacturers and return with an aerial photograph of the new planes in the yard.

R R R R B B B

R R R R B B

They are fairly certain that they have photographed a fair sample of one week's production. Write an equation using the letters R and B that describes the relationship between the number of red airplanes and the number of blue planes produced. The equation should allow you to calculate the number of blue planes produced in a month if you know the number of red planes produced in a month.

The correct answers for these four problems are (1) $S = 6P$, (2) $5C = 4S$, (3) "There are 7 assemblers for every solderer," and (4) $5R = 8B$. The success rates for these four problems were 63, 27, 29, and 32 percent, respectively.

It might seem at first that the researchers had simply found a bunch of students who were extremely defective in their algebraic skills. However, the students had been given the following six questions:

1. Solve for x: $5x = 50$

2. Solve for x: $\frac{6}{4} = \frac{30}{x}$

3. Solve for x in terms of a: $9a = 10x$

4. There are 8 times as many men as women at a particular school. 50 women go to the school. How many men go to the school?

5. Jones sometimes goes to visit his friend Lubhoft driving 6 miles and using 3 gallons of gas. When he visits his friend Schwartz, he drives 90 miles and used _?_ gallons of gas. (Assume the same driving conditions in both cases.)

6. At a Red Sox game there are 3 hotdog sellers for every 2 Coke sellers. There are 40 Coke sellers in all. How many hotdog sellers are there at this game?

On average, more than 95 percent of these problems were solved correctly. Therefore, the difficulties of these college students were not in simple algebraic manipulations. The difficulties were in translating a statement from a sentence into a suitable algebraic form. Actually, the students were very competent in courses beyond algebra. Clement's paper provides a detailed analysis of the transcript of a problem-solving session with one student who was doing B+ work in a standard calculus course at the time of the interview, and had been able to differentiate the function $f(x) = \sqrt{x^2 + 1}$ rapidly, using the chain rule, without difficulty. Yet, the student was unable to solve any of the problems in Table 1.

As in the Brown and Burton work, the students' errors were remarkably consistent for all of the problems in Table 1. More than four-fifths of the incorrect solutions to the problems were of the form $6S = P$, $4C = 5C$, "Seven solderers for every assembler", and 8R = 5B, respectively. In other words, there was a consistent reversal of the symbols and their role in the equations.

Through an analysis of clinical interviews, the authors identified two major causes for the reversal. The first explanation for the reversal was that the students made a "syntactic" translation of a sentence into algebraic form; i.e., the student reads along the sentence, replacing words where appropriate by algebraic symbols. Thus, "six times as many students" becomes 6S; "as" becomes equals, and "professors" becomes P. The resulting equation is 6S = P.

The second explanation for the reversal was that although the students recognized that an equation does stand for a relationship between two quantities, the way that the students represented that relationship to themselves resulted in a reversal. Many of the students, for example, drew pictures such as:

On one side of the desk is the professor; on the other side are the 6 students. Thus the equality is 6S = P.

To the mathematician, an equation for the "students and professor's" problem is a device which allows him to calculate the number of students given the number of professors, or vice-versa. Since there are 6 times as many students as professors, one must multiply the number of professors by 6 to get the number of students (for example, 10 professors yield 60 students). Thus, S = 6P. Obviously, students do not have this perspective.

In another experiment, the authors provide some dramatic evidence of the difference between the static and dynamic interpretations of an equation. Their "subjects" were 17 professional engineers who had between 10 and 30 years of experience each. The engineers had come to take a course in the BASIC programming language. On the first day of the course, the engineers were asked to write an equation for the following statement:

> At the last football game, for every four people who bought sandwiches, there were five who bought hamburgers.

Only 9 out of 17 of the engineers solved the problem correctly. The following day, without any discussion of the previous problem and the solution to it, the engineers were asked to write a computer program for the following:

> At the last company cocktail party, for every 6 people who drank hard liquor, there were 11 people who drank beer. Write a program in BASIC which will output the number of beer drinkers when supplied with the number of hard liquor drinkers.

All 17 of the engineers solved the problem correctly. The authors further substantiated these results with a study of some college students in a programming course. The notion of programming suggests a possible means of remediation: If we train students to think of an equation as a "program" with inputs and outputs, we may increase the likelihood of their getting the correct answers.

3. A Look at Problem Solving.

Apparently random problem-solving behavior can actually be quite consistent. In the work with BUGGY and with elementary word problems, the focus was on consistent patterns of mistakes, for purposes of diagnosis and remediation. In this section we look at the flip side of the coin. Just as a look beneath the surface discloses consistency in novices' incorrect behavior, a look beneath the surface will also disclose great consistency in the problem-solving behavior of experts. To make the point that experts and novices approach problems in dramatically different ways, consider the following three problems -- all of which are ostensibly accessible to high school students.

Problem 1: Let a, b, c, and d be given numbers between 0 and 1. Prove that $(1-a)(1-b)(1-c)(1-d) > 1-a-b-c-d$.

Problem 2: Determine the sum $\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!}$.

Problem 3: Prove that if $2^n - 1$ is a prime, then n is a prime.

On problem 1 most students will laboriously multiply the four factors on the left, subtract the terms on the right, and then try to prove that $(ab+ac+ad+bc+bd+cd-abc-acd-bcd+abcd) > 0$ -- usually without success. Virtually all the mathematicians I've watched solving it, begin by proving the inequality $(1-a)(1-b) > 1-a-b$. Then they multiply this inequality in turn by $(1-c)$ and $(1-d)$ to prove the three-and four-variable versions of it.

Likewise in problem 2, most students begin by doing the addition and placing all the terms over a common denominator. A typical expert, on the other hand, begins with the observation, "That looks messy. Let me calculate a few cases." The inductive pattern is clear and easy to prove.

The expert who read problem 3 and said "That's got to be done by contradiction" was typical (given the structure of the problem, one really has no alternative). Yet this almost automatic observation by experts was alien to students. A large number of the students to whom I have given the problems either responded with comments like "I have no idea where to begin" or tried a few calculations to see whether the result is plausible and then reached a dead end.

Of course these are special problems, for which expert and novice performance are each in their own way remarkably consistent. While the experts did not consciously follow any strategies, their behavior was at least consistent with these "heuristic" suggestions:

a. For complex problems with many variables, consider solving an analogous problem with fewer variables. Then try to exploit either the method or the result of that solution.

b. Given a problem with an integer parameter n, calculate special cases for small n and look for a pattern.

c. Consider argument by contradiction, especially when extra "artillery" for solving the problem is gained by negating the desired conclusion.

Many of the novices were unaware of these strategies, and many others "knew of them" (that is, upon seeing the solution they acknowledged having seen similar solutions), but hadn't thought to use them. Expert and novice problem solving are clearly different. The critical question is: Can we train novices to solve the problems as experts do?

There are a number of obstacles. First, we have to factor out simple subject matter knowledge: There is no way that one can hope to give the students experience before they have it, or to compensate for it. Rather, we would like to provide the students with strategies for approaching problems with flexibility, resourcefulness, and efficiency.

Second, we must realize that the heuristic strategies described by Polya are far more complex than their descriptions would at first have us believe. Consider the following strategy and a few problems.

"To solve a complicated problem, it often helps to examine and solve a simpler analogous problem. Then exploit your solution."

Problem 4: Two points on the surface of the unit sphere (in 3-space) are connected by an arc A which passes through the interior of the sphere. Prove that if the length of A is less than 2, then there is a hemisphere H which does not intersect A.

Problem 5: Let a, b, and c be positive real numbers. Show that not all three of the terms a(1-b), b(1-c), and c(1-a) can exceed 1/4.

Problem 6: Find the volume of the unit sphere in 4-space.

Problem 7: Prove that if a + b + c + d = ab+bc+cd+da, then a=b=c=d.

These four problems, like problem 1, can be solved by the "analogous problem" strategy. Yet, it is unlikely that a student untrained in using the strategy would be able to apply it successfully to many of these. Part of the reason is that the strategy needs to be used <u>differently</u> in the solution of each problem.

In solving problem 1, we built up an inductive solution from the two-variable case, using the result of the analogous problem as a stepping stone in the solution of the original.

In contrast, analogy is used in problem 4 to furnish the idea for an argument. The problem is hard to visualize in 3-space but easy to see in the plane: We want to construct a diameter of a unit circle which does not intersect an arc of length 2 whose endpoints are on the circle. Observing that the diameter parallel to the straight line between the endpoints has this property enables us to return to 3-space and to construct the analogous plane.

Problem 5 is curious. It looks as though the two-variable analogy should be useful, but I haven't found an easy way to solve it. At first the one-variable version looks irrelevant, but it's not. If you solve it, and think to take the product of the three given terms, you can solve the given problem. So again we exploit a result, but this time a different result in a different way.

Problem 6 exploits both the methods and results of the lower-dimensional problems. We integrate cross-sections, using the same method; the measures of the cross-sections are the results we exploit.

In problem 7 it would seem apparent that the two-variable problem is the appropriate one to consider. However, "which two-variable problem" is not at all clear to students. A large number of those I have watched tried to solve

Problem 7': Prove that $a^2 + b^2 = ab$ implies that $a = b$, instead of
Problem 7": Prove that $a^2 + b^2 = ab + ba$ implies $a = b$.

The description "exploiting simpler analogous problems" is really a convenient label for a collection of similar, but not identical, strategies. To solve a problem using this strategy, one must (a) think to use the strategy (this is non-trivial!), (b) be able to generate analogous problems which are appropriate to look at, (c) select from among the analogies, the appropriate one, (d) solve the analogous problem, and (e) be able to exploit either the method or result of the analogous problem appropriately.

If we assume now that we can actually describe the strategies in enough detail so that people can use them, we run right into another problem. That is: a list of all the strategies in detail would be so long that the students could never use it! Knowing how to use the strategy isn't enough: The student must think to use it when it is appropriate.

Consider techniques of integration in elementary calculus. There are fewer than a dozen important techniques, all of them algorithmic and relatively easy to learn. Most students can learn integration by parts, or substitution, or partial fractions, as individual techniques and use them reasonably well, as long as they know which techniques they are supposed to use. (Imagine a test on which the appropriate technique is suggested for each problem. The students would probably do very well.) When they have to select their own techniques, however, things often go awry. For example, $\int \frac{x\,dx}{x^2-9}$, a "gift" first problem on a test, caused numerous students trouble when they tried to solve it by partial fractions or, even worse, by a trigonometric substitution!

In "Presenting a Strategy for Indefinite Integration" (The American Mathematical Monthly, 1978) I discuss an experiment in which half the students in a calculus class (not mine) were given a strategy for selecting techniques of integration, based on a model of "expert" performance. The other students were told to study as usual -- using the miscellaneous exercises in the text to develop their own approaches to problem solving. Average study time for members of the "strategy" group was 7.1 hours, while for the others it was 8.8 hours; yet the "strategy" group significantly outperformed the others on a test of integration skills -- in spite of the fact that they were not given training in integration, just in selecting the techniques of integration.

The "moral" to the experiment is that students who cannot choose the "right" approach to a problem -- even in an area where there are only a few useful straightforward techniques -- do not perform nearly as well as they "should." If we leap from techniques of integration to general mathematical problem solving, the number of potentially useful techniques increases substantially, as does the difficulty and subtlety in applying the techniques. An efficient means for selecting approaches to problems, for avoiding "blind alleys," and for allocating problem-solving resources in general thus becomes much more critical. Without it, the benefits of training in individual heuristics may be lost.

In consequence of the above, an attempt to teach general mathematical problem solving would need these two components: first, a detailed description of individual strategies, and second, a global framework for selecting these strategies and using them efficiently. One way of presenting such a framework is with a "model" of expert problem solving. That model takes a semester to unfold, so there is no sense in my attempting to summarize it here. What I have done is simply to give the outline of the model (see Figure 1), and a description of the most important heuristic strategies which fall within each of the major blocks of that strategy (see Figure 2).

Of course, documenting improved problem-solving ability is rather difficult. I am slowly amassing evidence, in a variety of different ways, that instruction in problem solving actually can have an impact on students' problem-solving performance. The material on integration provided some evidence of this. A "laboratory study" demonstrated that "problem-solving experience" in and of itself is not enough: In the experiment, two groups of students worked on the same problems for the same amount of time and saw the same solutions, but one saw in addition heuristic explanations of the solutions. The differences in their performances were dramatic. (See "Explicit Heuristic Training as a Variable in Problem-Solving Performance.") Third, there is a large amount of "before and after" data on the students in the problem-solving course. These data indicate both an improved problem-solving performance on the part of the students and an improved ability to generate plausible approaches to problems, as opposed to a control group. There is much data to be analyzed by a variety of different means -- means which were unavailable just a few years ago, and which come from a variety of disparate sources. As one such example, let me discuss briefly the notion of "hierarchical cluster analysis." Consider the following three problems.

Figure 1. SCHEMATIC OUTLINE OF THE PROBLEM-SOLVING STRATEGY

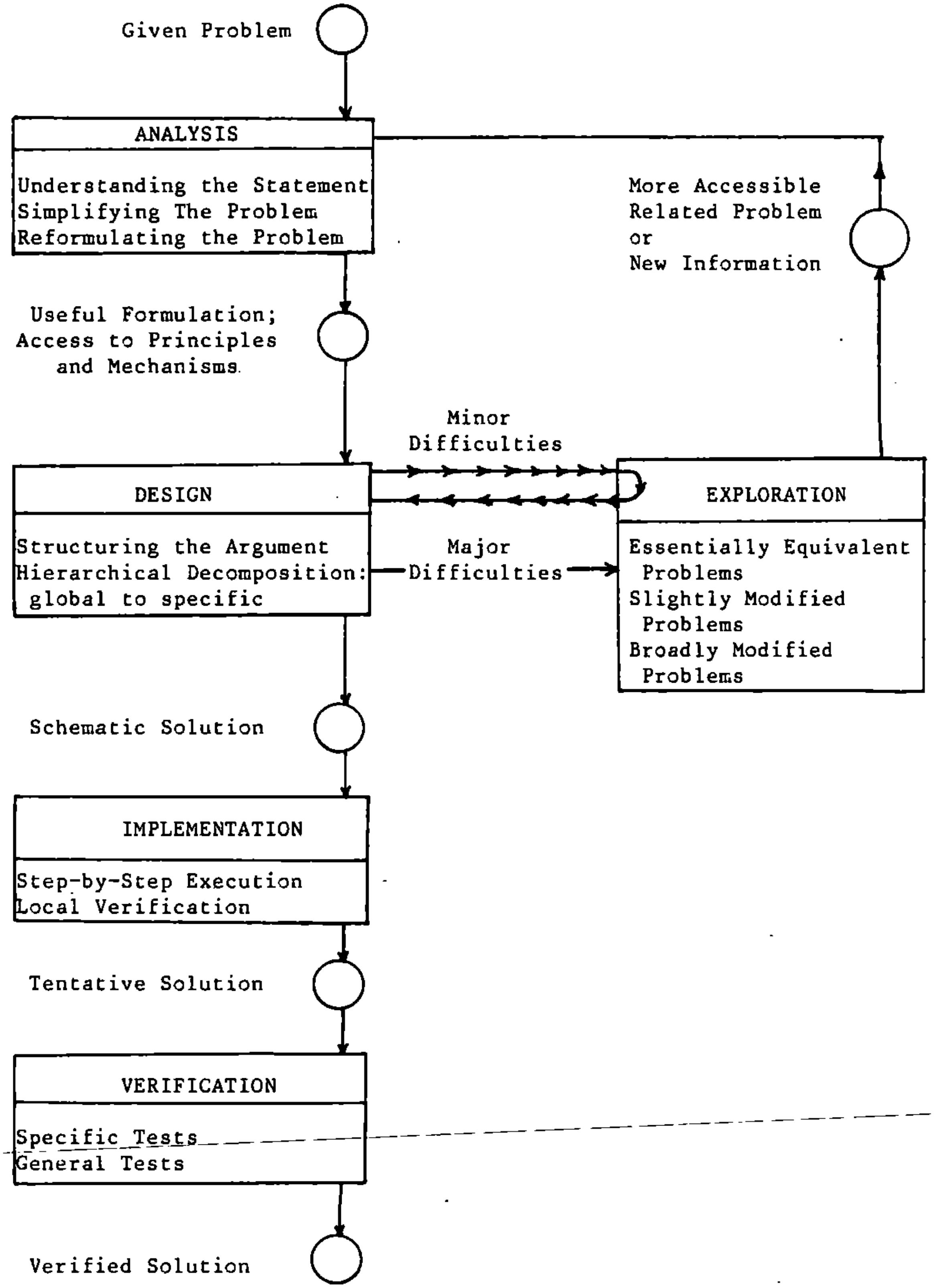

Figure 2. SOME IMPORTANT HEURISTICS IN PROBLEM SOLVING

For Analyzing and Understanding a Problem:

1. Draw a Diagram if at all possible
2. Examine Special Cases
   (a) to exemplify the problem,
   (b) to explore the range of possibilities through limiting cases,
   (c) to find inductive patterns by setting integer parameters equal to 1,2,3,... in sequence.
3. Try to simplify it, by using symmetry or "without loss of generality."

For the Design and Planning of a Solution:

1. Plan solutions hierarchically.
2. Be able to explain, at any point in a solution, what you are doing and why; what you will do with the result of this operation.

For Exploring Solutions to Difficult Problems:

1. Consider a variety of equivalent problems
   (a) replacing conditions by equivalent ones,
   (b) recombining elements of the problem in different ways,
   (c) introducing auxiliary elements,
   (d) reformulating the problem by (i) a change of perspective or notation, (ii) arguing by contradiction or contrapositive, or (iii) assuming a solution and determining properties it must have.
2. Consider slight modifications of the original problem:
   (a) choose subgoals and try to attain them.
   (b) relax a condition and try to re-impose it.
   (c) decompose the problem and work on it case by case.
3. Consider broad modifications of the original problem:
   (a) examine analogous problems with less complexity (fewer variables).
   (b) explore the role of just one variable or condition, the rest fixed.
   (c) exploit any problem with a similar form, "givens," or conclusions; try to exploit both the result and the method.

For Verifying a Solution:

1. Use these specific tests: Does it use all the data, conform to reasonable estimates, stand up to tests of symmetry, dimension analysis, scaling?
2. Use these general tests: Can it be obtained differently, substantiated by special cases, reduced to known results, generate something you know?

Problem 8: Given that lines intersect if and only if they are not parallel, and that any two points in the plane determine a unique line between them, prove that any two distinct nonparallel lines must intersect in a unique point.

Problem 9: Given 22 points on the plane, no three of which lie on the same straight line, how many straight lines can be drawn, each of which passes through two of those points?

Problem 10: If a function has an inverse, prove that it has only one inverse.

Let us take an extreme case. The student who understands virtually nothing of these problems may think that problems 8 and 9 are related because they both deal with lines in the plane. On the other hand, the mathematician sees that both problems 8 and 10 deal with the uniqueness, and are likely to be proved by contradiction. Therefore he may perceive of those problems as being similar.

Suppose 100 students were given these 3 problems, and asked to group together those problems which they thought were related. (They might decide that none of the problems was related or that two of them were, or that three of them were.) One could then create a 3 by 3 matrix, where the i,j-th entry represented the number of students who considered the i-th and j-th problems to be related. A comparison of these matrices before and after instruction, for both experimental and controlled groups, could indicate changes in the students' perceptions of the way these problems were structured mathematically.

In fact, my cluster analysis used 32 problems, with a 32 x 32 matrix for analysis. There were clear differences between experimental pre- and post-test scores, and controlled pre- and post-test scores. Further comparison with "expert" sorting of the problems is also planned. The full tally is yet to come, but the preliminary results are encouraging.

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