Ye Shall Be Known by Your Generations

by

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I. Introduction

A quarter of a century ago with the advent of the "new math," we were persuaded that understanding the structure of mathematics through such pedagogical strategies as discovery learning would attack frontally the most pervasive issues regarding the meaning of mathematics and the roles of the teacher and student as well. That myth has passed for the most part, but we are now bombarded by a new set of slogans as we are cajoled to teach problem solving as our new salvation.

Why have we not been led through the pearly gates in the past, and why is the prognosis not much better now? There are many reasons, of course, not the least of which is that curriculum specialists frequently do not appreciate a valuable intuition that is built into the bones of the best of practitioners: that schools involve a complicated interaction among people whose interests are frequently and fundamentally in conflict, and "diddling" a little bit with a curriculum or with a teaching strategy may bypass some of the most basic components that must be confronted if change is to occur.

Sarason (1971), a clinical psychologist, carefully observed efforts to implement a new math program in a school system several years ago. He articulated a number of characteristics of the school setting that may have accounted for a great deal of the failure of the new curriculum. Among these were:

1. The relation between teacher and pupil is characteristically one in which the pupil asks very few questions.
2. The relation between teacher and pupil is characteristically one in which teachers ask questions and the pupil gives an answer.
3. It is extremely difficult for a child in school to state that he does not know something without such a statement being viewed by him and others as stupidity.
4. It is extremely difficult for a teacher to state to the principal, other teachers, or supervisors that she does not

understand something or that in certain respects her teaching is not getting over to the pupils.

5. The contact between teacher and supervisor (e.g., supervisor of math, or of social studies) is infrequent, rarely involves any sustained and direct observation of the teacher, and is usually unsatisfactory.
6. One of the most frequent complaints of teachers is that the school culture forces them to adhere to a curriculum from which they do not feel free to deviate, and, as a result, they do not feel they can, as one teacher said, "use (their) own heads."
7. One of the most frequent complaints of supervisors or principals is that too many teachers are not creative or innovative but adhere slavishly to the curriculum despite pleas emphasizing freedom. (p. 35)

His main point is that no amount of development and delivery of a new curriculum <u>per se</u> could succeed if efforts were not made to take into account some of the above characteristics. If these characteristics threaten the success of any new curriculum project, how much more must they tend to abort our efforts to implement a problem-solving curriculum -- a curriculum that supposedly not only honors the intelligence of the student, but that suggests a reconception of the authority of the teacher!

But our disinclination to appreciate the complexity of the social context of school is only part of what has doomed earlier curriculum movements to something less than smashing success. Even if it were legitimate to isolate the issues of curriculum from those of the social setting, we have tended to foist a unidimensional view of curriculum issues on teachers who once more frequently intuit correctly that things are more complicated than theorists would have them believe.

Our intention in this paper is to attempt to point out how it is that the slogans of the 1950's and 1960's cannot exist in isolation from those of the 1970's and 1980's, and that any serious efforts at curriculum and instruction reform must search for important linkages. Thus, while the focus of this paper is more modest than the concerns of Sarason, it would be a mistake to implement a program that neglects to incorporate the two areas of concern.

Before beginning our analysis, it is worth admitting a bias that will soon become very obvious. That is, I believe it is a serious error to conceptualize mathematics as anything other than a human enterprise which, among other things, helps to clarify who we are and what we value. That bias will "ooze out" rather than be dealt with frontally at least in the first few sections. It will assume a central position, however, by the end of the paper.

We turn first to a consideration of a concept that was at the forefront of the modern math movement in the 50's and 60's -- that of understanding.

II. On Understanding

How poorly understood both in terms of pedagogical practice and psychological research is the notion of understanding! Let us begin with some comments made by Henri Poincare (1961) in an essay of his in which his focus was on mathematical creativity as a way of seeing some of the difficulty with regard to mathematics:

> How does it happen there are people who do not understand mathematics? If mathematics involves only the rules of logic such as are accepted by all normal minds; if its evidence is based on principles common to all men, and that none could deny without being mad, how does it come about that so many persons are here refractory?
>
> That not everyone can invent is no wise mystery. That not everyone can retain a demonstration once learned may also pass. But that not everyone can understand mathematical reasoning when explained appears very surprising when we think of it. And yet those who can follow this reasoning only with difficulty are in the majority: that is undeniable, and will surely not be gainsaid by the experience of secondary school teachers. (p. 33).

Now it is one thing to attempt to answer reasonable sounding questions but it is frequently far harder to find unwarranted assumptions that relegate such questions to the class of pseudo-questions. In some cases the "excess baggage" is obvious (e.g., "When did you stop beating your husband?"). In other cases it takes the wisdom of centuries to expose pseudo-questions. Mathematics itself is a beautiful example of a discipline in which such "unpacking" required enormous labor pains over hundreds of years. Almost since the creation of Euclidean geometry, questions were asked about derivability of the parallel postulate from other postulates. It seemed that the fifth postulate (through a given outside point one line can be drawn parallel to a given line) was much less fundamental than the others (like "two points determine a line"). For a very long period of time, people tried in vain to prove the fifth postulate from the others. It was, however, not until people began to have the courage to rephrase their questions -- exposing hidden assumptions -- that progress was made. Notice the subtle difference between the following two questions?

> "How can you prove the parallel postulate from the rest of Euclidean geometry?" versus
> "What happens if we assume that the parallel postulate cannot be proven from the rest of Euclidean geometry?"

It was the courage to ask the question in the second way that gave birth not only to non-Euclidean geometry but to a totally new conception of the nature of mathematics.

Enough of a digression! What has Poincare done in inquiring why people have difficulty understanding mathematics? I believe that he has brought along excess baggage that fits somewhere between the obviousness of the husband-beating question and the extreme subtlety of the original parallel postulate question.

In contrasting mere understanding and creating, he assumes that they are different states of mind or different kinds of activities. Understanding mathematics is one thing -- creating is another! What is it that leads us to believe that "mere understanding" is so simple a construct and so divorced from an act of creation?

We have been misdirected partly by a technological input/outgo view of the world to conclude that "coming to understand" is a relatively straightforward matter. The viewpoint is connected to a commonly held myth regarding good teaching. Good teachers are supposed primarily to be able to explain things well and to be able to "get us" to understand things that we could not do well on our own! I would like to explore a more dynamic model of understanding mathematics. I will do so by reflecting on personal experiences in teaching or learning and by examining curriculum as well.

Part/Whole Thinking and Mathematics

We begin with one of the most serious problems in understanding -- that of the attempt to relate the part to a whole or to a context in coming to understand a concept.

Consider the following two problems:

(1) In the set of natural numbers N = 1, 2, 3, 4, 5, ... , we define a prime number as a number with exactly two different divisors. So, 5 is prime because 1 and 5 are its only divisors. 4 is not prime because it has 3 divisors: 1, 2, 4.

Now instead of focusing on the set of natural numbers, look at E = 2, 4, 6, 8, 10, 12 . . . . Using the same definition of prime as in N, list the primes in E.

(2) Amy Lowell (the poetess of human liberation of her day) goes out to buy herself some cigars. She has a bunch of change in her pants pocket. Reaching inside, she feels around and finds that:

-- she has nickels, dimes and quarters
-- there are 25 coins all together
-- there are three more nickels than dimes
-- the total amount of money is $7.15.

How many coins of each kind does she have?

We invite you to think about these two problems before reading on, without considering your level of mathematical sophistication as a particular hindrance or a help in working them out.

Recently, I discussed problem (1) with Zvi, a mathematics teacher (Brown, 1978). Below is a rough replay of our dialogue:

Zvi: The only prime in E is 2.
Me: Why?
Zvi: Because 2 is *the only* even prime.
Me: Why isn't 6 prime in E?
Zvi: It can't be.
Me: Why?
Zvi: Because 6 is divisible not only by 1 and 6 but by 2 and 3 as well.
Me: Is it?
Zvi: Yeah.
Me: How do you know?
Zvi: Just do it.
Me: Can we forget about E for a minute and look back only at N?
Zvi: Sure.
Me: I think that 5 is not prime in N.
Zvi: Why?
Me: 'Cause 5 is divisible by 2.
Zvi: No way!
Me: Why not?
Zvi: 'Cause you get 2 1/2 and you can't get "1/2's" when you divide.
Me: Why can't *you get* halves?
Zvi: You can't because when you divide the answer has to be "even" -- no fractions.
Me: What's wrong with fractions?
Zvi: They're not allowed when you try to divide in the natural numbers.
Me: Why not?
Zvi: They're just not. When you divide in the natural numbers, things have to go "evenly."
Me: Can we look again at E?
Zvi: Sure.
Me: Does 2 divide 6 in E?
Zvi: Yes, and so 6 is not a prime as I said before.
Me: Can you think of a way of conceiving of "divides" that would make the statement "2 divides 6" false in E?
Zvi: No! 2 does divide 6.
Me: But does it do it in E?
Zvi: Yes.
Me: *How do you know that 2 divides 6?*
Zvi: *Because* 3 x 2 = 6.
Me: But 3 doesn't belong to E.
Zvi: So?
Me: Why wouldn't you let me say that 2 divides 5 in N?
Zvi: 'Cause then you'd get a fraction for an answer.
Me: What's wrong with that again?
Zvi: I told you already. They're not allowed when you try to divide in the natural numbers.
Me: Can you give me a reason for why they're not allowed?

Zvi: They're just not. If you divide in N, it has to go evenly.

Me: Can you look at the situation in E again and find a way of excluding 3 as an answer when you try to see if 2 divides 6?

Zvi: No.

Me: Well, suppose we think about 2 1/2 as not being permissible as an "answer" when you try to divide 5 by 2 not because things don't go "evenly" but because 2 1/2 isn't a member of N!

Zvi: That's not really why. But so?

Me: Suppose you use that reasoning in E. Then 2 does not divide 6 because the only candidate 3, that could make it true does not belong to E! Therefore 6 is prime in E.

Zvi: You can't do that.

Me: Why not?

Zvi: Prime makes sense only in N, and it's only because 2 does not divide 5 "evenly" that 5 remains a prime in N.

Me: What does "evenly" mean again?

Zvi: No remainders when you try to divide!

I apologize for the long dialogue, but I hope the interchange is beginning to raise some questions about the nature of understanding. Before discussing things, let us turn to the second problem.

I have given a modified version of the Amy Lowell problem to many people over the past few years, and I have met with astounding results. Those people who have studied a great deal of mathematics almost always begin with something like:

Let d = number of dimes
d + 3 = number of nickels
25 - (d + d + 3) = number of quarters.

They then set up an equation taking into consideration the fact that the total amount of money is $7.15. In attempting to solve the equation, they frequently end up with a negative, fractional value for d. What do they do? Most sophisticated mathematicians then either look over the calculation to see where they may have made an error or they take out a new sheet of paper and do the same thing over again -- once more ending up with a fractional negative answer for d. I have seen this type of behavior continue for a half hour resulting in an even greater sense of frustration than when Pavlov presented an ellipse to a dog after training it to expect food if the event is preceded by a straight line and punishment if by a circle!

Let us explore the Amy Lowell example a little bit first. After some initial frustration, perhaps, it is possible to explore this problem intelligently and no amount of repeated equation solving will in itself reveal an intelligent approach to the problem. One intelligent approach would be to attempt to see the larger picture instead of immediately committing oneself to setting up an equation. In trying to relate the pieces to each other, it is possible to solve the problem by observing that if you had 25 coins and even if all of them were quarters, then it would be impossible to have $7.15. There is no way to relate 25 nickels, dimes and quarters so as to yield $7.15!

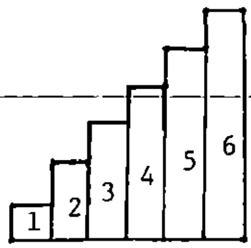
Well, what are we getting at here other than demonstrating an insightful vs. a "plodding" approach to a problem? The difference between the two approaches goes much deeper than that. In one case a primarily linear approach is taken to solve a problem -- an approach in which information is added bit by bit without regard for the large picture, and more importantly without any serious attempt to have intelligence prevail. In the other approach, an effort is made to view the pieces in relationship to the whole and to other pieces and to see how an intelligent reformulation of the problem reduces it to one of mental arithmetic rather than algebra.

Though we see this distinction (linear vs. holistic approaches) clearly in the more creative act of trying to solve the Amy Lowell problem, it is also present in more subtle form in Zvi's inability to "merely understand" what I was driving at in the dialogue. His problem was apparently not only that he would not allow us to extend the use of the word "prime" to an unfamiliar context, but that the concept of prime number could not be extended because he was unable to view failure of divisibility in N in any terms other than whether or not the "answer comes out even." He was incapable of seeing "coming out even" as only a partial view of what divisibility in N might mean and more importantly he was not capable of seeing that the concept of prime was not a concept in isolation but rather one that made sense in a context. That is, he had conceived of "prime" in such a way that it lacked "hinges" to the broad context of domain.

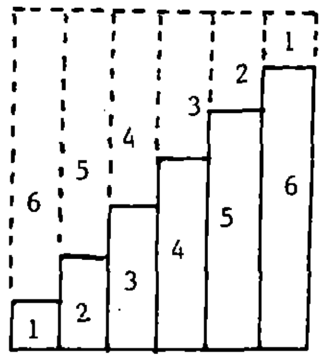
All of this from a mathematics teacher who could follow and teach any number of procedures involving primes in N -- getting prime factors of a number, adding fractions, reducing to lowest terms and so forth! He could do everything expected of him -- except perhaps understand the concept of prime.

The problem as we have identified it so far is one that Wertheimer (1945) addressed over a quarter of a century ago. Concerned with gestalt psychology and its ability to point out what distinguishes productive from non-productive thinking, he chose many mathematical examples to illustrate the point. As a matter of fact, he made the famous mathematician Gauss an almost popular hero by exploring in gestalt terms an alleged story of him as a youngster faced with the task of finding the sum of the natural numbers from 1 to 100.

It is by looking at 1 + 2 + 3 + 4, + ... + 97, + 98 + 99 + 100 in gestalt terms that we can begin to see how a shortcut might have emerged historically. Anyone who thinks of adding up the pieces in terms of a geometric staircase model (below) might readily see how the pieces could be viewed as part of a whole.



It is possible to view the above structure as only half a configuration embedded in a rectangle as in the figure below:



Compare this illuminating approach with the following found in many texts:

Start with: 1 + 2 + 3 + 4 + 5 + ... + 97 + 98 + 100. Now count backwards and arrange so that we have the following:

1	+	2	+	3	+	...	+	98	+	99	+	100
100	+	99	+	98	+	...	+	3	+	2	+	1

If we now add vertical pairs, we end up with:

$$101 + 101 + 101 + \dots + 101 + 101 + 101$$

Instead of finding the desired sum, we thus have twice the desired sum. How many times do we have 101 as a term in the sum? It is obviously 100 times. But then $100 \cdot 101$ gives us the value of <u>twice</u> the sum of the numbers from 1 to 100. Since we want the sum only once, the answer is $1/2 \cdot 100 \cdot 101 = 5050$.

It is not all clear how one might have thought up this scheme for finding the answer by examining twice the sum and writing one of them "backwards" unless one has seen a geometric type scheme as above.

Explaining how gestalt thinking works, Wertheimer comments,

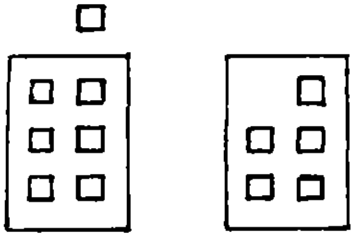
> The aim of discovering the inner relation between structure and task leads to regrouping, to structural reunderstanding. The steps and operations do not in the least appear to be a fortuitous, arbitrary sequence; rather they come into existence as parts of the whole process in one line of thinking. They are performed in view of the whole situation, of the functional need for them, not by blind accident nor as thoughtless repetition of an old rule-of-thumb connection.

A Second Look at Part/Whole Thinking

One reading of cognitive gestalt psychology is that its focus on the relationship of the part to the whole is essentially an inner state of mind. This is especially so if one reviews the experiments in perception and pays attention to references such as "flashes of insight" and the like in the literature. This is certainly suggested when Wertheimer claims:

> Often it is not even necessary to assign a task for sensible response to appear: it grows out of the inner dynamics of the situation. (p. 108)

And he illustrates his point with figures such as:



Apparently, it is a sign of gestalt thinking to place the "lonely" square from the top of the left figure to the inside of the right one.

Though the gestalt metaphor is valuable, we can find much of value in encouraging people to relate parts and wholes in ways that go beyond the purely cognitive "inner state" construct.

That is, a concerted effort on the part of educators to explore with youngsters the many different ways in which parts and wholes do or should relate to each other would seem to have enormous payoff. So much of our educational experience places us in the position of having or being parts of a whole, and yet we are given almost no encouragement to reflect upon that experience. In the previous subsection we focused on part/whole relationships from the perspective of specific problems, and we pointed out shortcomings that result from an inability to attempt to see how parts and wholes relate to each other. But this inability exists not only with regard to a problem and its components. It is also an issue with regard to a course in the context of one's mathematical experience and with regard to one's mathematical experience in relationship to other experiences.

Schools are notorious for encouraging a "piece-meal" approach to virtually everything. Youngsters are given very little opportunity to reflect upon how the pieces fit together. Frequently, there is no rationale, and if there is one, it may be frightening -- dealing more with conformity and authority than with the fostering of intelligence. That is, as we have come to understand dimensions of the "hidden curriculum," we see that much of what passes for education is not necessarily in the best interests of the children, nor of their teachers. Learning to wait and to be obedient are hardly designed to serve the intellectual interests of children, though they serve an important rite of passage in a technological society that has internalized these values.

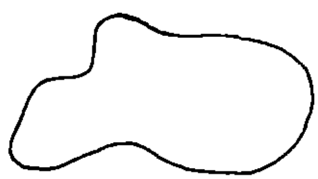
The problem is poignantly expressed by Matthew Lipman, Ann Sharp, and Frederick Oscanyan (1977) in their program of teaching philosophy for children.

> One of the major problems in the practice of education today is the lack of unification of the child's educational experience. What the child encounters is a series of disconnected, specialized presentations. If it is language arts that follows mathematics in the morning program, the child can see no connection between them, nor can he or she see a connection between language arts and the social studies that follow, or a connection between social studies and physical sciences.
>
> This splintering of the school day reflects the general fragmentation of experience, whether in school or out, which characterizes modern life...The result is that each discipline tends to become self-contained, and loses track of its connectedness with the totality of human knowledge...(p.6)

How can we as educators help students at all levels make better sense out of their fragmented lives and ours?

Consider for example the issue of relating parts to a whole not with regard to a specific mathematics problem, but with regard to an entire course. The basic assumption that students are not wise enough to see a whole picture until they have experienced completely all the pieces and thus that pieces are perceived temporally prior to wholes is at best a mischievous assumption and one that is responsible for a great deal of student malaise, animosity, and rejection.

One of my most educationally worthwhile teaching experiences occurred when I had the courage to begin a calculus course not by defining derivatives and definite integrals as I had done for a number of years, but by giving each student in the class a shape like:



I spent three weeks having them try (on their own and in collaboration with others) to find out an area for that region. A great deal of frustration ensued. Some very brilliant investigations took place. Beyond a number of individual differences, however, what emerged was an almost "instantaneous" (3 weeks compared to an academic year) appreciation for what calculus was getting at.

Halmos (1975), a first generation student of R. L. Moore, captures the essential elements of this experience in the following remarks:

> For almost every course one can find a small set of questions...questions that can be stated with the minimum of technical language, that are sufficiently striking to capture interest, that do not have trivial answers, and that manage to embody in their answers, all the important ideas of the subject. The existence of such questions is what one means when one says that mathematics is really all about solving problems, and my emphasis on problem solving (as opposed to lecture attending and book reading) is motivated by them. (p. 467)

Having begun to explore the part/whole phenomenon as an essential and poorly appreciated ingredient in understanding -- even in the mild sense of following an argument -- let us now turn to another dimension that challenges a passive interpretation of what is involved in coming to understand: problem generation.

Problem Generation

We begin once more with a small anecdote. First consider the problem below:

> The ten's digit of a two digit number is one half the unit's digit. Four times the sum of the digits equals the number. Find the number.

This problem was shown to me by a beginning mathematics teacher who was distressed upon discovering it in a text for one of her high school classes. She worked it out and based upon the solution decided that it was a mistake and that she would not assign it to her students. Why? If you tried to work this out algebraically, you most likely arrived at something like:

Let t = ten's digit
u = unit's digit

Then $t = 1/2u$
$4(t + u) = 10t + u$
.
.
.

you probably then ended up with something like:

$$6u = 6u$$

Therefore, any u should work and the only restriction on t would be that $t = 1/2u$. Her point is that unlike all digit problems she had done before, this one seemed highly irregular in that it implied many solutions. For example, 12, 24, 36, 48 at least would work. Thus:

$$4(3 + 6) = 36!$$

Her method of handling this irregularity was to dismiss it (though she did privately make inquiries).

How might one try to make sense out of this anomaly? In addition to asking "why?" directly, one reasonable way of proceeding would be to "probe" the phenomenon by asking any number of questions such as:

(1) Are there any other problems like this digit problem for which a similar phenomenon results? For example, when can I get the same results if the ten's digit is three times the unit's digit?
(2) To what extent is the result a consequence of the base selected? Would I get the same result in a base other than ten?
(3) What kind of problem for a three digit number would yield similar anomalies?

Lest we lose sight of the larger picture here, let us consider what is behind "probes" of the kind we are suggesting. At bottom is an inclination to generate problems. Though problem solving has become an explicit area of concern of mathematics educators at all levels, we seem to have lost sight of the fact that problem solving is rooted in a much more fundamental activity: problem generation.

Students who understand that it is legitimate to expect them to solve problems do not believe that it is similarly reasonable to expect them to pose problems. The irony of it all is that no one ever is capable of solving a problem (not just doing an exercise) without formulating some new problem along the way. The fact that students are disinclined to see mathematics (or perhaps any school activity) as a problem-posing enterprise first occurred to Marion Walter and me a number of years ago at which time we were team teaching a course to Harvard Master of Arts in Teaching students. Having a definite "lesson plan" in mind, we began by asking the students to give us some answers to:

$$x^2 + y^2 = z^2$$

We got dutiful responses like 3, 4, 5; 5, 12, 13, and even a few "wiseacre" ones like: 1, 1, $\sqrt{2}$; -1, 2, $\sqrt{5}$.

It occurred to us afterwards that the students were answering a non-question. No one (including us at the time) had realized that: $x^2 + y^2 = z^2$ is not a question, but an open form about which many questions could be asked or problems posed. For example, find x, y, z so that the Pythagorean equality misses by 1; or find three bona fide fractions that satisfy the equality; or give a geometric interpretation of the equality; and so forth.

If people are disinclined to generate problems even when the context is a "natural" one -- that is inspired by anomaly, surprise, doubt -- then how much more are they reluctant to do so when they are just being asked to "follow" or to "understand" someone else's presentation? Let us return again to the problem of Zvi and primes.

Zvi had learned very well what a prime number is according to the definition he was given. However, because he viewed "understanding" as a passive affair, it never occurred to him to go beyond the conception which he was "given": A number is prime if the only things that "go evenly" into the number are 1 and the number itself.

What else might he have done -- even if he were asked to accept that definition? If he had been inclined to see the world in less authoritarian and more "elastic terms," he might have asked, for example:

1. What's so special about numbers that have only two divisors? Can numbers have 3 divisors? (4 and 9 being two examples).
2. Can numbers have four divisors?
3. How many numbers are there with 5 divisors?
4. I wonder if there is some way to visualize prime numbers.
5. What is the biggest prime number?
6. Why are we focusing on divisibility? Is there something like primes with subtractions?
7. Are there any fractions that are prime?

These questions could be expanded at will, and we perhaps should be cautious in criticizing Zvi for never generating such a list. After all, we all have a finite time to invest in any activity and this was one that Zvi chose to "accept." The problem is, however, that believing that "mere understanding" is what Poincare depicted it to be -- a passive activity or achievement in which one keeps his "nose clean" -- Zvi had acquired very little understanding of any aspect of mathematics.

If you accept that in some sense one must create knowledge (as implied by the criticism that Zvi never asked any of these questions) in order to understand anything, then you might reasonably ask why a teacher (as opposed to the students) could not generate these questions to initiate understanding. The problem at least is that each of us comes to any experience with a highly idiosyncratic view of the world. The kinds of questions that make sense to me in terms of solidifying understanding are very different from those that make sense to you. Some of the questions I have asked above imply a need for visualization which others do not; some are asking for a very large context and some for a smaller one. Some are open to many alternative conceptions and others to a limited number.

It is not the disinclination to view any one phenomenon as "elastic" and "probe-able" that limits one's ability to understand so much as world view that conceives of understanding in such an inert way.

Summary

~~Using anecdotes and reflecting upon personal educational experiences, I~~ have attempted to suggest that a behavioristically rooted model of "understanding" has grave limitations. Referring both to the issue of part/whole and of problem generation, I have tried to illustrate how it is that understanding is a personal and aggressive construct in the sense that no one is capable of doing your understanding for you. There is perhaps a sense of human liberation captured by a stance which makes the "it" in the most "cool" of all subjects less mechanistic and more of a private phenomenon. In what ways, however, is this "mushier" conception of the "it" capable of shedding light on the self as part of an educational experience? We turn now to that question.

III. Towards An Integrated Notion of Self

In 1913, Dewey (1975) produced an extremely important work that has been a well-kept secret in educational circles, Interest and Effort in Education. He focused on a question of fundamental importance to practitioners and one which dividends the advocates of "free," "open," and "traditional" education. He asked whether teachers ought to take major responsibility for "interesting" children in the perhaps dull substance of their education, or should they expect youngsters to exert "effort" on their own in order to master material even (or especially) if it is "uninteresting" to them? All of us have, perhaps, heard or made arguments that support these two conflicting points of view. Opinions about the benefit or harm of "sugar coating" content frequently falls back upon disagreements with regard to these two poles.

Dewey begins his book by making a plausible case for each point of view, and then proceeds to point out what he conceives of as a basic fallacy in both of them.

> The common assumption is that of the externality of the object, idea or end to be mastered to the self. Because the object or end is assumed to be outside self it has to be made interesting; to be surrounded with artificial stimuli and with fictitious inducements to attention. (p. 7)

Having linked the need to make things interesting to the erroneous notion of separation of self and object, he finds the same fallacy in "effort" as a fundamental obligation of the student.

> Or, because the object lies outside the sphere of self, the sheer power of "will," the putting forth of effort without interest has to be appealed to. (p. 7)

He sees a resolution of the dilemma to be in the direction of unification of object and self.

> The genuine principle of interest is the principle of the recognized identity of the fact to be learned or the action proposed with the growing self; that it lies in the direction of the agent's own growth, and is, therefore, imperiously demanded, if the agent is to be himself. Let this condition of identification once be secured, and we have neither to appeal to sheer strength of will, not to occupy ourselves with making things interesting. (p. 7)

In further blurring the sharp distinction between "self" and "object," Dewey reveals himself as the unacknowledged originator of the new popular concept of "hidden curriculum" in education. He comments:

> The question of educative training has not been touched until we know what the child has been internally occupied with, what the predominating direction of his attention, his feelings, his disposition has been while he has been engaged upon this task. <u>If the task appeals to him merely as a task, it is as certain psychologically as is the law of action and reaction physically, that the child is simply engaged in acquiring the habit of divided attention; that he is getting the ability to direct eye and ear, lips and mouth to what is present before him so as to impress those things upon his memory, while at the same time he is setting his thoughts free to work upon matters of real importance to him.</u> [emphasis my own] (p. 8,9)

If there is any portion of the curriculum that has become the hallmark of separation of object and self, it is mathematics. What kind of thinking is needed in order to provide a different conception of their relationship? We shall in the following subsections provide possible directions for integrating the two, without attending to any detailed scheme of implementation.

<u>Part/Whole Thinking A Third Time</u>

We ought, perhaps, to be more cautious in making such harsh judgments of Zvi and "blind" efforts on the Amy Lowell problem. How might we expand some of our criticism in the subsection entitled "A Second Look at Part/Whole Thinking" so as to focus not primarily upon "making objective sense," but upon greater self understanding?

Consider those people who approach the Amy Lowell problem in an algorithmic way. Now, it is possible for them to justify their approach. After all, the problem did resemble ones they had done before and there is certainly considerable efficiency involved in placing similar problems in an already well-worked-through mold. Such an argument would then select the existing algebraic structure as a "whole" within which this small problem is a part. Those who decide to view the problem in such a way as to relate the parts to the whole <u>within the problem itself</u> (rather than to the whole of an algebraic structure) could justify their procedure on other grounds. They might argue, for example, that though efficiency may be a virture -- all other things being equal -- <u>this</u> case appeared to be <u>different</u> enough from others they explored to warrant a reconsideration.

Well, why did they consider it different? Why were the algorithmic thinkers willing to run the risk of missing the <u>uniqueness</u> of the Amy Lowell problem for the sake of efficiency? To what extent were they out to see each experience in mathematics as part of a more general phenomenon, and thus easily incorporated into existing structures, and to what extent were they desirous of viewing new phenomena in a unique way? To say that something is to be viewed uniquely does <u>not</u> imply that it is not to be seen as a part of something larger -- but only that the something larger need not necessarily be an already well-established structure.

Now a great deal of deliberate mathematics education does err in the direction of diminishing novelty. In fact the search for order, for isomorphic structures and substructures, for harmony where apparent disharmony exists, is frequently taken as the hallmark of mathematics. Whether or not this ought to be the case is an interesting and important question, but it is perhaps desirable for us to transform this philosophy of mathematics question into an educational one.

An educational transformation would have us provide many opportunities for students to approach problems and to view solutions either as unique experiences or as something fitting into existing structures. To what extent and under what circumstances do they feel comfortable with the uniqueness of a particular mathematical experience? Why? How does that reflect upon the desirability of finding uniqueness in non-mathematical circumstances as well?

It is quite conceivable that by understanding their stance towards the value of uniqueness or the unexpected in a mathematical context, students of mathematics may begin to understand how they value uniqueness and novelty in other areas as well.

An effort to relate in a personal way the role of the unique and the unexpected in attempting to assimilate and accommodate new worldly input may move us in the direction of self-understanding.

Problem Generation Revisited

Earlier we suggested how understanding mathematics *per se* requires a form of problem generation. Here, we turn to relationships between problem generation and self-understanding.

There is an important sense in which we are known to others as well as to ourselves by the kinds of questions we ask and the problems we generate. Such activity is frequently more courageous and involves considerably more risk than appears on the surface. The asking of questions and the generation of problems when done in a spirit of inquiry not only reveals an initial state of ignorance and a desire to know, but also has embedded within a set of assumptions. Such activity tells the world something about the specifics of what we believe and in addition has the ability to inform others of the intensity of these beliefs.

Are we willing to propose "foolish sounding" questions and under what circumstances? For what purposes? Earlier we discussed how far several hundred years investigating the parallel postulate revealed basic assumptions which were in fact incorrect (i.e., that the parallel postulate can be proven from the rest of Euclidean geometry). What is less obvious, however, is the enormous courage required to even ask the question in what has become a twentieth century spirit (showing logical independence of propositions).

In 1822, Johann Bolyai wrote a letter to his father Wolfgang in which he disclosed his new and daring form of the parallel postulate question. Johann did so with intellectual curiosity, but also with great fear that he would be perceived as risking his sanity by even asking the question. How could it make sense to even conceive of a world in which there were no lines parallel to a given line through a given outside point? How indeed! It was only after the dust was cleared from an intellectual revolution from Copernicus through Darwin through Freud through Einstein that we could say that sanity had prevailed.

How much do we and our students risk when we ask questions that have embedded the potential for even minor revolutions? Especially if our question verges on foundational issues, we run a thin line between meaninglessness and revolutionary finds.

Consider the following incident that occurred in a number theory course of mine. We were trying to show that if a perfect square is even, then the square root of that number is also even. (For example, 16 and 36 are even, and so are 4 and 6.) An indirect proof led us after several stages to the following assertion:

$$2n + 1 = 2m.$$

That is, an odd number would have to equal an even number. Just as we were about to "cap" the proof by a reducto ad absurdum claim, someone shyly asked:

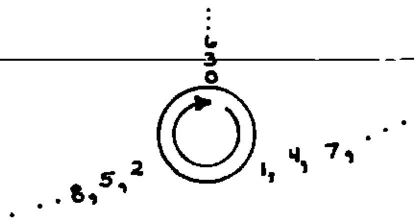
"Why can't an odd number equal an even one?"

Why indeed! Using any number of experiential arguments in the set of natural numbers, we can come to believe that it is impossible for an odd number to equal an even one.

Despite all that, we tried to push the logic further. If $2n + 1 = 2m$, then simplifying we'd get $1 = 2(m - n) = 2 \cdot x$.

So now, we are led to conclude that twice an integer must equal 1. All our experience rebels against the conclusion, but where do we go from here? A natural inclination would be to try to prove that $1 = 2x$ has no solution in the set of natural numbers or integers. We had just begun the course and no one had adequate machinery to pursue that issue at the time, so we tried another tack. Instead of trying to prove the equality false as we know it to be in the set of natural numbers, we began to explore where it might be true. The equation $2 \cdot x = 1$ obviously has a solution in the set of fractions, but that system appears to be different in so many ways from the natural numbers that the find was unrevealing.

After some highly creative exploration, we found a system that "felt" closer to the natural numbers but within which $2 \cdot x = 1$ has a solution: Clock arithmetic.



Starting with zero and moving clockwise through 1 and 2, and then circling back to the zero for 3 and 1 for 4, and so forth, we can "wrap" all the integers around the circle forming 3 separate classes. Choosing to define addition and multiplication in a "natural" way, we find that there is a number x so that $2 \cdot x = 1$; the equivalence class generated by two works.

But what does that say about odds and evens in clock arithmetic? And how does clock arithmetic compare with the natural numbers? What properties does one have that the other lacks which enables us to find numbers that are both odd and even in one system but not in the other?

What have we done here? By shifting the context slightly (from natural numbers to clock arithmetic) a foolish question emerges as the starting point for some deep exploration -- including the opportunity to re-explore the question in the original context with greater insight! On a minor scale, we too have performed a "Bolyai." We took a very foolish-sounding question seriously and found a home within which we emerged a hero!

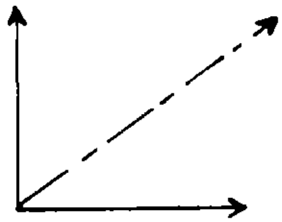
As with Bolyai, pushing the question challenged every bit of experience, and finding a non-trivial home for the question was a testimony to our ability momentarily, at least, to suspend logic in favor of a creative leap (keeping in mind the level of experience of the class at the time).

That the exploration was mathematically rewarding and successful in some sense should not blind us to the potential for interplay between logical and creative thinking in mathematics. We tend to stress the former as the hallmark of mathematical thought so much that we lose sight of the fact that problems are generated by human beings and that such generation makes use of the mind not as a logic machine alone but as an instrument for poetic thought as well.

We are capable of generating not only by modifying the attributes of a given problem (as we suggested in the Zvi example) and not only by refuting experience and logic, but also by making use of extralogical tools of thought such as imagery, metaphor, and the like.

Unfortunately, so much of mathematical training at all levels unnecessarily constricts rather than liberates us by focusing on the narrowly conceived end product of following a <u>proof</u> that we lose both the ability and the inclination to generate ideas through the use of these tools.

I recall as a junior taking my first graduate level mathematics course. It was finite dimensional vector spaces offered by a world-famous mathematician. The first day we were told that the only things that count are the axioms and definitions together with rules of logic, and that it was solely that apparatus to which we ought to appeal in the doing of mathematics. Anything else was to be interpreted as a bastardization of the discipline. He proceeded to list the axioms of a vector space, and as sometimes happens under such circumstances, he got stuck. He stood before us, mumbled a few words and then turning his back to the class, and blocking the blackboard with a stomach that was adequate for the purpose, he sketched a diagram that looked something like:



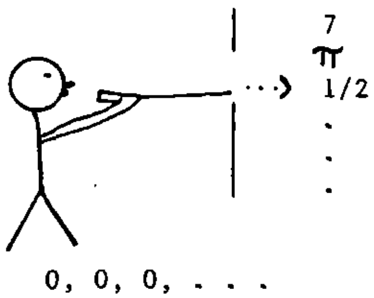
In an attempt then to be consistent with his original advice, he quickly erased his sketch and proceeded to list a few more axioms and to prove a few theorems "based solely upon definitions, axioms and logic."

If there is one thing I look back on proudly with regard to that experience, it is that I dropped the course immediately, and took it the following semester from a mathematician who, though less world-renowned, was more in tune at the very least with his own style of learning.

Now, this is an extreme case of confusing generation and verification, but if we are warned against using even isomorphic type diagrams in this extreme case, how much more of a heinous crime to use imagery of a looser nature!

All kinds of images and metaphors direct my activity not only at problem generation but at problem solving and in just plain recalling as well. This machinery is apparently the most well-guarded secret when it comes to mathematical thought.

For me, "zero" is not the midpoint of an infinite line, nor is it primarily the identity element under addition. Instead, it is the following "fellow" from multiplication.



He holds a machine gun, looks through a peep-hole and as each of the numbers marches before the wall he annihilates them and collects them as little images of himself.

It seems to me than an important part of a humanistic education and experience is disclosing sharing, and understanding the significance of the images that direct our thinking. If that can be done well within the context of mathematical thinking, where can it not be done?

In addition to imagery, use of metaphor is a powerful problem generator. Two brief personal illustrations will make my point. One day I was "doodling" with the following multiplication facts:

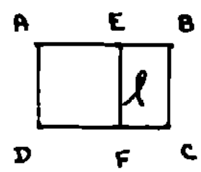
$$
\begin{aligned}
1x3 &= 3\\
2x4 &= 8\\
3x5 &= 15\\
4x6 &= 24\\
5x7 &= 35
\end{aligned}
$$

I was wondering what sense to make out of that when the metaphor of "striving" popped into my mind. I saw each number to the right of the equation in an existential sense as "striving" to become something it had not yet become. Instead of what I had there, I saw:

1 x 3 is almost 4
2 x 4 is almost 9
3 x 5 is almost 16
4 x 6 is almost 25

The right hand side formed perfect squares, and what started out as a metaphor ended up as an exploration that led to a totally new algorithm for doing multiplication (Brown, 1974).

At another time I was learning about the golden rectangle:



ABCD is a golden rectangle if I can construct a line ℓ parallel to a side so that a square (AEFD) is created together with another rectangle (EBCF) similar to the original. I saw this phenomenon not as squares and rectangles, but as organisms giving birth to another generation yielding the most ideal form possible and one offspring that is a miniature version of the original. Again this metaphor led to the generation of many different problems that had never been dreamed of before (Brown, 1976).

IV. Summary

Where are we now? What do we do with these various observations? We are suggesting that if for purposes of understanding mathematics an important part of the curriculum is part/whole thinking and problem generation then for purposes of understanding self a reflection on those same dimensions is needed. What is there that encourages or inhibits each of us from generating problems? To what extent do we make use of (and perhaps hide) images, metaphor and fantasy in generating problems? What kinds of risks do we personally take in the questions we ask and the problems we generate?

To what extent are we influenced not only by the machinery of logic, anti-logic and poetry as we attempt to generate, but by the presence of a role