Problem Solving for the High School Mathematics Student

by

Cheryl Kantecki and Lee E. Yunker
West Chicago Community High School

An effective method for capturing the interest of students is to involve them in problem solving. This requires reaching beyond the realm of the textbook to expose students to a variety of problem-solving situations. Aside from capturing student interest, problem solving provides opportunities to develop mathematical skills, leads to new mathematical ideas, and motivates students to research mathematics. More importantly, problem solving stimulates imagination and allows students to exercise creativity.

The National Council of Supervisors of Mathematics Position Paper on *Basic Skills defines problem solving as the process of applying previously* acquired knowledge to new and unfamiliar situations. The word "process" implies that there are many facets to problem solving, and indeed there are. Problem solving entails more than merely arriving at a conclusion; problem solving entails the entire process of analyzing a problem, synthesizing, and evaluating.

The intent of this article is threefold. First, to identify problem-solving strategies and illustrate their uses with specific examples. Second, to suggest motivational techniques to involve students in problem solving. And, third, to provide the reader with a list of interesting and challenging problems along with their answers.

Major Problem-Solving Strategies

There is a wide variety of problem-solving strategies that could be mentioned as being important. This article will confine itself to six strategies that have wider application and can be easily employed by high school students. The six strategies are: elimination, modeling, reducing to a simpler case, using tables, guess and check, and patterns.

Elimination:

The elimination strategy is basically one of looking at all the possible solutions and eliminating, one by one, those that are not possible. Logic problems provide examples for the use of the elimination method.

The WHO'S WHO Problem:

Four married couples belong to a golf club. The wives' names are Kay, Sally, Joan, and Ann. Their husbands are Don, Bill, Gene, and Fred. Examine the following clues. They should help you decide who is married to whom.

- Bill is Joan's brother.
- Joan and Fred were once engaged, but "broke up" when Fred met his present wife.
- Ann has two brothers, but her husband is an only child.
- Kay is married to Gene.

Solution:

A chart like the one below lists the possible solutions. The clues help eliminate, one by one, the false solutions.

	KAY	SALLY	JOAN	ANN
DON	X	X	YES	X
BILL	X	YES	X	X
GENE	YES	X	X	X
FRED	X	X	X	YES

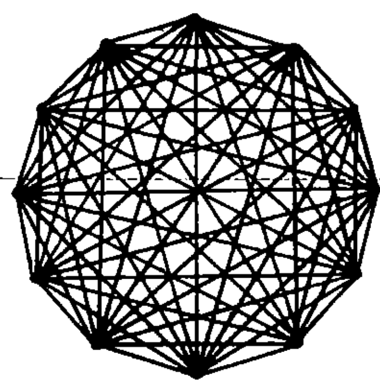
Modeling:

The modeling strategy is one of creating a model of the problem to be solved. This model may be an actual physical model or just a diagram on paper. In any event, the model helps the student see his way through to the solution.

THE HANDSHAKE Problem:

There are 12 people at a party. If everyone shakes hands with everyone else at the party, how many hand shakes take place?

Solution:



The hand shakes can be represented by the sides and diagonals of the dodecagon.

Number of Sides = 12 n
Number of Diagonals = $\underline{54}$ $\dfrac{n\,(n-3)}{2}$

Number of Hand Shakes = 66

Reducing To A Simpler Case:

This strategy is employed when the problem appears to be too large to comprehend or when trying a few of the cases may hold a hint to the larger solution.

THE LOCKER Problem:

This problem is about a high school and that favorite storage area, the high school locker.

At Gauss High there are 1000 students and 1000 lockers (numbered 1-1000). At the beginning of our story all the lockers are closed. The first student comes by and opens every locker. Following the first student, the second student goes along and closes every second locker. The third student changes the state (if the locker is open, he closes it; if the locker is closed, he opens it) of every third locker. The fourth student changes the state of every fourth locker, etc. Finally, the thousandth student changes the state of the thousandth locker. Which lockers will remain open after the thousandth student changes the state of the thousandth locker? [From: Columbus Project ESEA, Columbus, Montana.]

Solution:

For our purpose we shall investigate the state of the first sixteen lockers as students 1 through 16, open or close them. In the table below, let C and O represent closed and open lockers respectively. Student #1 begins by opening every locker. Thus in Row 1 of the table an "O" is placed beneath each locker number. Student #2 then closes every second locker. Thus in Row 2, "C" is placed beneath lockers 2, 4, 6, 8, ...16. Next, student #3 changes the state of every third locker; he closes locker number 3, opens number 6, closes number 9, opens number 12 and closes number 15. Hence in Row 3, "C" is placed beneath locker 3, 9, and 15 and "O" beneath 6 and 12. This process continues until student #16 changes the state of locker 16. Finally, from the table it is evident that lockers 1, 4, 9 and 16 are left open. Each of these locker numbers are perfect squares. Hence the solution: the lockers whose numbers are perfect squares are open.

Locker #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Student #																
1	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O
2		C		C		C		C		C		C		C		C

Student # \ Locker #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3			C			O			C			O			C	
4				O				O				C				O
5					C					O					O	
6						C						O				
7							C							O		
8								C								C
9									O							
10										C						
11											C					
12												C				
13													C			
14														C		
15															C	
16																O

Using Tables:

Tables are useful devices for organizing and keeping track of information.

THE HONEST BROTHERS Problem:

One of five brothers had broken a window. John said, "It was Henry or Thomas." Henry said, "Neither Earnest nor I did it." Thomas said, "You are both lying." David said, "No, one of them is speaking the truth, but not the other." Earnest said, "No, David, that is not true." Three of the brothers always tell the truth, but the other two cannot be relied on. Who broke the window?

Solution:

In the table, the headings at the top indicate the assumed guilt of each brother. The headings to the left indicate the truth or falseness of each statment. T and F represent true and false respectively. For instance, if John were guilty (Column 1) John's *statement* would be false, Henry's true, Thomas' false, David's true and Ernest's false. Columns 2 through 5 are completed in the same way. Upon completion, the only column which indicates that three brothers were telling the truth is column 3. Therefore, Thomas broke the window.

	John	Henry	Thomas	Ernest	David
John	F	T	T	F	F
Henry	T	F	T	F	T
Thomas	F	F	F	T	F
David	T	T	F	F	T
Ernest	F	F	T	T	F

Guess and Check:

Guess and check is a problem-solving strategy in which the problem solver actually guesses a solution and then checks to see if the solution satisfies the conditions of the problem. The experienced problem solver uses all of the information at hand to arrive at a reasonable solution.

THE ORDERED DIGITS Problem:

0	1	2	3	4	5	6	7	8	9

In the ten cells above inscribe a ten digit number such that the digit in the first cell indicates the total number of zeros in the entire number; the digit in the cell marked "1" indicates the total number of 1's in the number, and so on to the last cell, whose digit indicates the total number of 9's in the number. Zero is a digit, of course. The answer is unique.

Solution:

In the list below, an initial guess is made. Then a series of gradual changes are made until the correct solution is obtained.

	0	1	2	3	4	5	6	7	8	9
	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	1
	8	0	0	0	0	0	0	0	1	0
	8	1	0	0	0	0	0	0	1	0
	8	2	1	0	0	0	0	0	1	0
Answer	6	2	1	0	0	0	1	0	0	0

Pattern:

Searching for patterns can be an extremely helpful problem-solving strategy. Amazingly, problems which appear to be relatively difficult can in fact be quite simple when patterns are recognized.

THE CHINESE DINNER Problem:

Every 3 guests used a dish of rice between them, every 4 a dish of broth, and every 2 a dish of meat. There were 65 dishes in all. Can you figure how many guests there were?

Solution:

In order to "meat" the conditions of the problem the number of guests must be divisible by 12. We create the following table in which R, B, M represent the total number of required dishes of rice, broth, and meat, respectively.

# of Guests	R	B	M	Total # of Dishes	
12	4	3	6	13	increase of 13
24	8	6	12	26	

Since each increase of 12 guests will increase the total number of dishes served by 13 and there were 65 dishes served, there must have been 60 guests.

Motivations for Problem Solving

Often students need to be motivated to extend themselves in doing problems which are not going to be covered on their next test even though the techniques learned may be helpful in solving those test questions. The following suggestions have proved helpful in getting students involved in learning and using these problem-solving strategies.

The Weekly Challenge Problem. A problem may be presented on Monday with the teacher accepting solutions until Friday. The teacher keeps a weekly record of each student's correct solutions on a large chart on the bulletin board. Each week that a correct solution is turned in, the student gets a star behind his name. This chart is always there for all students to see and this chart soon motivates competition among the students in the class, each trying to outdo the other.

The Weekly Extra-Credit Problem. Each week a problem may be presented to the students for extra credit (something many students ask for). Then the teacher can use the extra-credit points at the end of each grading period to adjust the student's grade in whatever way he deems appropriate.

As a result of these efforts, the parents even get involved. Students will say, "Can I have an extra copy for my father or mother?" Or, the parent will see you and say, "Where do you get those problems?" This then gives you a great opportunity to explain to parents about the new emphasis in our profession on teaching problem solving.

Problems to Motivate and Challenge

The following problems are ones that we have used with our students. These are problems that we feel offer students good problem-solving situations in which they can exercise creative thinking and on which they can use one or more of the strategies mentioned above. The original sources for these problems are unknown to us but we thank their authors, whoever they are, for offering such fine problems for us to share with others.

Problem-Solving Exercises:

1. Miss Young has her 18 students seated in a circle. They are evenly spaced and numbered in order. Which student is directly opposite... a. student number 1? b. student number 5? c. student number 18?

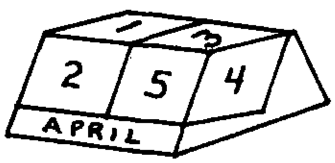
2. Mr. Evans seated his students in the same way as Miss Young's. Student number 5 is directly opposite number 26. How many students are in Mr. Evan's class?

3. Mrs. White teaches Phys. Ed. She had her students space themselves evenly around a circle and then count off. Student number 16 is directly opposite number 47. How many students are in Mrs. White's class?

4. EXTENSION: A huge number of boys are standing in a circle and are evenly spaced. The 7th boy is directly opposite the 791st. How many boys are there altogether?

5. Steve, Jim, and Calvin are married to Beth, Donna, and Jane, not necessarily in that order. Four of them are playing bridge. Steve's wife and Donna's husband are partners. Jane's husband and Beth are partners also. No married couples are partners. Jim does not play bridge. Who is married to whom?

6.

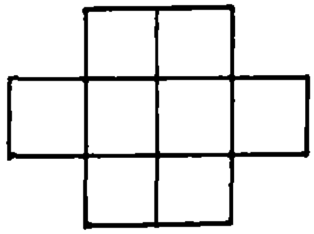


On the desk calendar above, the day can be indicated by arranging the two cubes so that their front faces give the date. The face of each cube has a single digit, 0 through 9. If the cubes can be arranged so that their front faces indicate a date 01, 02, 03, ... 31, find the four digits that cannot be seen on the left cube and the three on the right cube.

7. Find the smallest number which divided by each of the integers 2, 3, 4, 5, 6, 7, 8, 9, and 10, will give, in each case, a remainder which is 1 less than the divisor.

8. Divide a circle into four equal areas using three fences of equal length. Do not use your fences around the perimeter or on top of each other.

9. Fill in the following figure with the digits 1-8 in such a way that no two consecutive numbers are in boxes which touch at a point or side.



10. An exam has five true-false questions.
    a. There are more true than false answers.
    b. No three consecutive questions have the same answer.
    c. The students know the correct answer to problem number 2.
    d. Questions number 1 and number 5 have opposite answers. From the information above, the student was able to determine all the correct answers. What are they?

11. A man went into a hardware store and asked the clerk how much 1 cost. The clerk said 25 cents. He asked how much 10 would cost, and the clerk said 50 cents. "Good," he replied, "I'll take 1025." He then paid the clerk $1.00. What did he buy?

12. If it cost a nickel each time you cut and weld a link, what is the minimum cost to make a chain out of 5 links?

13. A goat is tied at the corner of a 20m x 40m barn with a 50m rope. If it can graze at any spot outside of the barn to which its rope can reach, what is the size of its grazing area?

14. Jim has a collection of records. When he puts them in piles of two, he has one left over. He also has 1 left over when he puts them in piles of 3 or piles of 4. He has none left over when he puts them in piles of 7. What is the least number of records he may have?

15. Golden Chain Problem: A Chinese prince who was forced to flee his kingdom by his traitorous brother sought refuge in the hut of a poor man. The prince had no money, but he did have a very valuable golden chain with seven links. The poor man agreed to hide the prince, but because he was poor and because he risked considerable danger should the prince be found, he asked that the prince pay him one link of the gold chain for each day of hiding. Since the prince might have to flee at any time, he did not want to give the poor man the entire chain, and since it was so valuable, he did not want to open more links than absolutely necessary. What is the smallest number of links that the prince must open in order to be certain that the poor man has one link on the first day, two links on the second day, etc.?

16. A man's age at death was 1/29th of the year of his birth. He was alive in 1900. How old was he in 1900.

17. Slow Horse Race: Two knights seek the hand of Princess Priscilla in marriage. Each boasts that he owns the fastest horse in all the land. So the king arranges a horse race. The king, however, is not eager to have his little girl marry, and he is especially unimpressed with her two suitors, so he decrees that the winner of the race, who will receive the princess's hand, will be the knight whose horse crosses the finish line last. It would seem that the race would never get under way; neither horseman would want to ride out ahead of the other. But Princess Priscilla, eager for marriage, thinks of a way to outwit her over-protective father. She whispers instructions to the two knights that ensure that the race will be run and that it will be fair. What did she tell the two knights?

18. Nines: Using only six nines, write a number that equals 100.

19. It is traditional in many families at Christmas time for each family member to give a gift to each of the other members. How many gifts would be given if there were 10 family members? How about for your family which has ___ members?

20. Sally has some change in her purse. She has no silver dollars. She cannot make change for a nickel, a dime, a quarter, a half dollar, or a dollar. What is the greatest amount of money she can have?

21. If a clock strikes six times in five seconds, how many times will it strike in ten seconds.

22. How much will it cost to cut a log into eight equal segments, if cutting it into four equal segments costs 60 cents?

23. Mervin was Calvin's best friend and the executor of Calvin's will when he passed away. Mervin rode his horse over to Calvin's ranch to settle the estate. Calvin had 17 horses that were to be divided among his family in the following way. Calvin's wife was to receive 1/2 the estate, his son 1/3, and his daughter 1/9. This posed a problem for Mervin. He did not want to kill any of the horses and yet he must divide the estate according to the will. How did he accomplish this task?

24. Herb the Hobo was attempting to cross a railroad bridge. When he was 3/7 of the way across he heard a train coming behind him. He ran to the far end and hopped off just as the train got to him. Later he calculated that he could have run to the other end of the bridge and still have survived. If the train was going 35 kph, how fast did Herb run?

25. In a survey of 25 college students at the University of Calgary, it was found that of the 3 newspapers, <u>Calgary Herald</u>, <u>Calgary Sun</u>, and the <u>Globe and Mail</u>, 12 read the <u>Herald</u>, 11 read the <u>Sun</u>, 10 read the <u>Globe and Mail</u>, 4 read the <u>Herald and the Sun</u>, 3 read the <u>Herald</u> and the <u>Globe and Mail</u>, 3 read the <u>Sun</u> and the <u>Globe and Mail</u>, and 1 person reads all 3.

    a. How many read none of the newspapers?
    b. How many read the <u>Calgary Herald</u> alone?

c. How many read the Calgary Sun alone?
d. How many read the Globe and Mail alone?
e. How many read neither the Calgary Herald nor the Calgary Sun?
f. How many read the Calgary Herald or the Calgary Sun or both?

26. A well is 10 feet deep. A frog climbs up 5 feet during the day but falls back 4 feet during the night. Assuming that the frog starts at the bottom of the well, on which day does he get to the top?

27. Gina and Tom raise cats and birds. They counted all the heads and got 10. They counted all the feet and got 34. How many birds and cats do they have?

28. The security guard at a bank caught a bank robber. The robber, the teller, and a witness were arguing when the police arrived. This was what the police learned in the confusion.
a. The names of the 3 men were Brown, Jones, and Smith.
b. Brown was the oldest of the three.
c. The teller and Jones had been friends for many years.
d. Brown was the brother-in-law of the witness.
e. Smith graduated from high school 5 years earlier than the robber.
Who was the robber? Who was the teller? Who was the witness?

29. How can 12 matches be arranged to make 6 regions of equal area?

30. The Editor of the Harvey School annual, The Harvey Hijinx, knows that 2985 digits were used to print the page numbers of the annual. How many pages were in the book?

31. In the Calgary Herald, the sports writing staff picked the winners for the first weekend of play in the Canadian Football League's 1982 football season. The picks are as follows:

Sportswriter "A"	Sportswriter "B"	Sportswriter "C"
Edmonton	Ottawa	Calgary
Montreal	B.C.	Edmonton
Calgary	Edmonton	Winnipeg
Saskatchewan	Montreal	Ottawa

No one picked Toronto to win. Who plays on the first scheduled weekend?

32. A certain highway was being repaired, so it was necessary for the traffic to use a detour. At a certain time, a car and a truck met in this detour which was so narrow that neither the truck nor the car was able to pass. Now, the car had gone three times as far into the detour route as the truck had gone, but the truck would take three times as long to reach the point where the car was. If both the car and the truck can move backward at one third of their forward speed, which of these two vehicles should back up in order to permit both to travel through the detour in the minimum amount of time?

33. The shuttle service has a train going from Washington to New York City and from New York City to Washington every hour on the hour. The trip from one city to the other takes 4 and 1/2 hours and all trains travel

at the same speed. How many trains will pass you in going from Washington to New York City?

34. What do the following words have in common: Deft, First, Calmness, Canopy, Laughing, Stupid, Crabcake, Hijack.

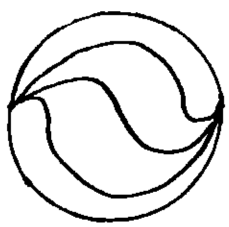
35. Supply a digit for each letter so that the equation is correct. A given letter always represents the same digit.

$$\begin{array}{r} A\ B\ C\ D\ E \\ \underline{\times \qquad\quad 4} \\ E\ D\ C\ B\ A \end{array}$$

36. A man travelled 5000 kilometres in a car with one spare tire. He rotated tires at intervals so that when the trip ended each tire had been used for the same number of kilometres. How many kilometres was each tire used?

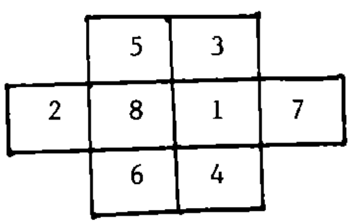
A N S W E R S

1. a. 10 b. 14 c. 9
(Note: the difference between the numbers of directly opposite persons is always the same.)
2. 16-5=11. Thus 11x2=22, the number of pupils in Mr. Evan's class.
3. 62 pupils.
4. 1568 boys.
5. Steve and Jane; Jim and Beth; Calvin and Donna.
6. Left cube: 0, 7, 8, and 6 or 9. Right cube: 0, 1, 2.
7. 2519
8.



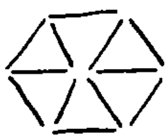
Or answers may vary.

9.



10. T, F, T, T, F (Kay is information in part C).
11. House Numbers.
12. 10 cents.
13. 2115π Sq. m.
14. 49
15. 1 Link (The 3rd link forms either end).
16. 15 or 44
17. Priscilla said, "switch horses."

18. $99 + \frac{99}{99}$
19. 90, second answer varies.
20. $1.19 (4 pennies, 4 dimes, 1 quarter and 1 half-dollar).
21. 11 times.
22. $1.40.
23. Mervin donates his horse to the estate. Then the wife gets 9 horses, the son 6 horses and the daughter 2 horses (9 + 6 + 2 = 17). So Mervin then takes his horse back and all are happy.
24. 5 kph.
25. a. 1 b. 6 c. 5 d. 5 e. 6 f. 19.
26. On day 6.
27. 7 cats, 3 birds.
28. Robber is Jones; Teller is Brown; Witness is Smith.
29.



or

30. 1023 pages.
31. Montreal - Winnipeg, Saskatchewan - Ottawa, Toronto - Edmonton, B.C. - Calgary.
32. The car.
33. 9.
34. Three consecutive letters of the alphabet.
35. A=2; B=1; C=90; D=7; E=8.
36. 4000 kilometres.