Visualization: A Problem-Solving Approach

by

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The teaching of mathematical problem solving is as complex as the problem-solving process itself. A myriad of questions faces the mathematics educator attempting to improve the problem-solving performance of students of all ages: How do I motivate the topic? How do I organize the material? Are there certain strategies which are of primary importance? Should problem solving be a group activity? What is unique about problem solving that makes it more difficult to teach and learn?

Much research has been completed concerning the analysis of the nature of problem solving. Although researchers adopt their own terminology, there appears to be some consensus concerning the stages in the problem-solving process. Krutetskii (1976) offered the following synopsis of this complex process:

> Apparently, three links or stages can always be traced in the solution of any problem (from the elementary to the very complicated). The solution of any problem seems to begin with the acquisition of initial facts, initial information about the problem, with thorough reflection, attempts to understand, and mastery. Then comes the solution proper, as a stage of processing or transforming the facts acquired for the purpose of obtaining the desired result. And finally, both the process and the result of the solution always leave some trace in the memory, somehow enriching a person's experience. (p. 183)

In other words, the three stages of the problem-solving process involve understanding the problem, planning an attack and carrying out that plan, and finally looking back and assimilating the knowledge gained. Spelling out this process points out the difficulties which arise when trying to teach mathematical problem solving. How do you teach a student how to understand a problem? How do you help a student plan an efficient attack? How do you encourage a student to learn from his mistakes, to generalize the solution, and to commit the solution to memory? Obviously, the task is not easy.

Helping a student understand a problem involves more than helping the student understand the individual words. In one study of fifth grade students' understanding of mathematical problems, the Cloze procedure was used to judge readability of a collection of ten problems. The problems were judged to be quite readable (at the fifth-grade level). Individual interviews were conducted with fifth-grade students, asking them to explain certain words. There were no glaring errors in the comprehension of the words. Yet students still had difficulty understanding the problems. If a student understands each word of a problem, what can the teacher do to help the student understand the problem?

Helping a student plan an attack is no less complicated. One of the unique characteristics of problem solving is that there is no algorithmic, step-by-step procedure for finding the most efficient solution. One fifth-grade student was feverishly working on a problem that had been assigned to the whole class. A hint for attacking the problem was given to the class, and part of the class benefited from the hint. However, this particular student seemed even more confused after having received the hint. "I don't see how that will help me get the answer," he fumed. He ignored his classmates and continued on with his own ideas. Suddenly, five minutes later, he raced up to the front of the room and asked for a point of clarification about the problem. "I got it," he whispered. The initial hint had not helped him because it was leading to a plan of attack that was not consistent with his view of the problem. How does a teacher help 30 students with different perspectives plan efficient strategies?

The final stage, generalizing the solution and assimilating the knowledge, appears to be a very individualized process, yet we expect the teacher to encourage this behavior. What types of actions occur in this third stage of the problem-solving process? How can a teacher help 30 students to assimilate this new knowledge when the previous knowledge structure for each individual student is so remarkably different?

This brief look at the three stages of the problem-solving process points out the fact that problem-solving behavior is very individualized. Each student approaches a given problem situation with his unique background, knowledge structure, inclinations, and cognitive styles. The student then reads the problem and attacks it based on his perspective of the problem. Multi-digit addition exercises, division of fraction exercises, subtraction with renaming exercises, and so on, can all be solved by step-by-step mechanical procedures; problem-solving situations cannot. The uniqueness of problem solving as a mathematical activity is that it is so very dependent on the problem solver. The major similarity between problem solving and other mathematical activities is that continual practice improves performance. Polya (1957) recognized the importance of practice and suggested that the way to improve problem-solving performance was by doing problems. He concluded:

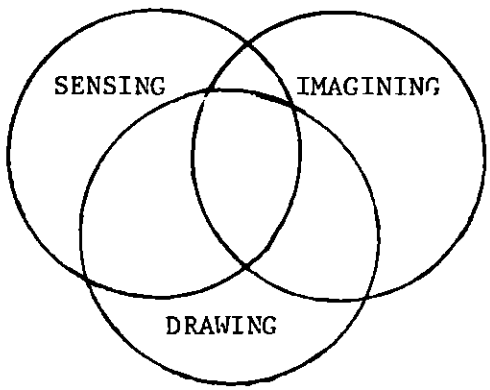
> Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation and practice... Trying to solve problems, you have to observe and to imitate what other people do when solving problems and, finally, <u>you learn to do problems by doing them</u>. (p. 4-5)

Working through the three stages of the problem-solving process in problem situation after problem situation does indeed improve problem-solving performance. The more one practises, the more adept one becomes at seeing hidden clues, gaining new perspectives and recognizing useless attempts to solve the problem. But is there some general technique, a way of thinking, that will induce good problem-solving behavior and encourage students not to feel frustrated and give up?

Visualization: A way of thinking

Visualization is a way of thinking and not merely a problem-solving strategy. It can be made to play an important role in each of the three stages of the problem-solving process. It is NOT a panacea for the classroom teacher, but it IS a useful mechanism for students to better cope with difficult problem situations.

Visual thinking is involved in numerous activities, such as when the gardener tries to imagine the garden before it blooms, when the newly-married couple rearranges furniture to make the little apartment appear spacious, when the outfielder in a baseball game knows exactly where to stand to catch the ball, or when the chemist has some insight into the molecular structure of some newly-discovered item. It involves sensing, imagining and drawing:



It involves dreaming, sketching diagrams, sculpting, manipulating concrete objects, and closing one's eyes and mentally manipulating objects. Everyone does visual thinking to some degree; creative problem-solving performance could be improved by encouraging more visual thinking in classroom activities.

In order to achieve the creativity and the flexibility that are required of problem solving, it will be a strong advantage to feel comfortable in a visual, imaginative mode of thinking.

Visualizing in the classroom

Before proceeding to suggestions for visualizing in mathematical problem-solving situations, it should be noted that visualization is a way of thinking and thus should be encouraged in all subject areas. When reading a passage from some source of literature, students should be encouraged to conjure up images of the scene. They should be asked to describe it with words or with pictures or with three-dimensional models. They should be asked to act it out. If the main character were to be placed in a given, new situation, how would she react?

In studying some historical period, students should again be encouraged to let visual imagery dominate their thoughts. How did the people of the period dress? How did they feel about the events of the period? How would these people react if they were living in today's society?

Teachers should create situations and foster imaginal thinking. For example, stimulate thinking by the following situation: Suppose you were walking all over a cube. Describe your feelings in words or in a two-dimensional picture. How would this experience be different from walking on a sphere?

With practice in all phases of the curriculum, visualization can become an integral part of all thinking. Asking "what if..." questions encourages students to become more creative, more flexible, and more aware of different perspectives in a given situation -- all essential characteristics of the problem-solving process.

Visualization in the problem-solving process

The encouragement of visualization skills can aid in all three stages of the problem-solving process. First of all, students can better understand a problem once they get a mental image of the problem situation. It may help if they restate the problem with words of their own choosing. It may help to act out the problem or to draw the situation or to construct some conrete model. In each case, the problem-solvers have translated the problem via some visual vehicle to suit their own perspective.

To organize a plan of attack and carry out that plan one may need to focus in on pictures and diagrams. Seeing is believing, and oftentimes seeing a pictorial representation of the information in a problem helps one to plan an attack and points out previously held misconceptions. Jim, the student mentioned previously who seemed bothered by the hint given in class, described what had happened when he finally came upon a solution. "The numbers just didn't add up," he said. "But then I drew a picture and saw that if zero was a number, I knew how to get the answer. And zero is a number, right?" The picture had helped Jim to see the problem, to understand it, to recognize his misconceptions of it, and to hit upon a solution. Due to his own, incorrect perception of the problem, the given hint did not aid him. He needed to re-structure his thinking, and this was accomplished most efficiently by letting Jim create a visual image of the problem situation. It should be noted that it was Jim's creation and not a representation given by the teacher. Thus, it fit into Jim's cognitive structure quite easily.

Finally, it is probably apparent that visual imagery is very important in the generalization of the problem and the assimilation of the knowledge gained. Having imagined the given situation in one's mind, variations of the situation seem to flow easily. For example, the following problem was posed to a group of in-service teachers:

> *What is the maximum number of pieces into which a pizza can be cut with five straight cuts?*

An obvious extension is to look at the problem for n straight cuts, where n is some integral value. But a more interesting extension was posed by one teacher who suggested looking at a three-dimensional cake, where cuts were permitted in more than one plane. This problem is much more difficult, but there was a strong correlation between individuals who had used visual imagery in the solution of the original problem and those who were successful at the problem extension.

Thinking in this visual mode appears to aid the typical problem solver at all age levels. It is not a strategy as such, but a multi-sensory approach to grasping the problem. By considering a couple of specific examples, a better understanding of this approach may be gained.

<u>Some specific examples</u>

Consider the following problem situation:

> *A fireman stood on the middle step of a ladder*, directing water into a burning building. As the smoke got less, he climbed up three steps and continued his work. The fire got worse so he had to go down five steps. Later, he climbed up the last six steps and was at the top of the ladder. How many steps were there?

Without any clues and without any training in visualization, the typical fifth-grade student will read this problem, understand each word, see three numbers, and proceed to the incorrect solution of adding the three numbers.

Training in the use of visual imagery encourages the student not to jump to the second stage of the problem-solving process prematurely. Understanding the problem is a prerequisite for organizing the plan of attack. In the visual mode, the student will first feel the fire, experience its heat, draw it on paper, act it out. The student will differentiate between climbing up the steps and going down. An entire visual image will be constructed. Perhaps a drawing of the ladder will be exhibited by the students or perhaps the mental image will be sufficient. In either case, the student will realize that climbing up the ladder is a motion in one direction and going down is a motion in the opposite direction. The numbers should not all be positive.

Extensions are up to the individual. What if the ladder were longer? What if there were an even number of steps?

Now consider a second problem:

> Sally had a new bike which she takes to school every day. On some days she rides the bike to school and walks home; on the other days, she walks to school and rides the bike home. The round trip takes one hour. If she were to ride the bike both ways, it would only take 1/2 hour. How long would it take if she walked both ways?

The typical solution involved subtracting the two given lengths of time: 1 hour - 1/2 hour = 1/2 hour. The solution is not reasonable, but this would not be noticed unless some visual imagery was evoked.

The student should again be encouraged to think and visualize the problem situation. Imagine riding a bike to school; compare it to walking to school. Which is more enjoyable? Which gets you there quicker? Draw a picture or act out the scene. If it takes 1/2 hour to ride the bike both ways, how long does it take to ride the bike one way? Why?

What other modes of transportation are there? How might you change this problem to include other possibilities?

Conclusion

The teaching of problem solving is a very complex operation. To teach a student three or four strategies, such as finding a pattern, constructing a table, or establishing a subgoal, may prove beneficial once the student has reached the second stage of the problem-solving process.

A more general pedagogical idea is to consider emphasizing a different mode of thinking -- visualization. The effects of this approach are simple, but profound. As Simon (1976) points out:

> An important component of problem-solving skills lies in being able to recognize salient problem features rapidly, and to associate with those features promising solution steps. Much current instruction probably gives inadequate attention to explicit training of these perceptual skills, and the kind of understanding that is associated with them. (p. 21)

Visualization can be promoted in the classroom. Students can enhance their perceptual skills. Problem-solving performance can be improved.

References

Krutetskii, V.A. The psychology of mathematical abilities in school children. Chicago: The University of Chicago Press, 1976.

Polya, G. How to solve it. Garden City, New York: Doubleday & Company, Inc., 1957.

Simon, H. Learning with understanding. Columbus, Ohio: ERIC Information Analysis Center for Science, Mathematics, and Environmental Education, 1976. (ERIC no. ED 113 206)