Teaching the Solution of Arithmetic Story Problems as a True Problem-Solving Task

by

C. Mauritz Lindvall
University of Pittsburgh

When elementary school children solve arithmetic story problems, they should be engaged in a true problem-solving activity. By "true problem-solving" I mean that the pupils should be analyzing each story in terms of what is described and what they are asked to find and then using the results of this analysis to determine what operation(s) should be applied. The desired type of activity is not taking place when children are solving a series of stories by mechanically applying the one operation they are studying at that given time (for example, solving every story by merely multiplying the two numbers that are given because this is the operation that is the focus of study for this week). Nor are they having problem-solving experience if they are using some "key word" to help them guess what operation to apply. These latter approaches to the solution of story problems are not based on a real understanding of the problem, and permitting pupils to use such procedures will not result in their acquiring a real capability for problem solving.

Work on story problems can provide the occasion for students to have the type of problem-solving experiences which teach them how to apply their mathematical knowledge and skill to practical everyday problems. But this analytical type of problem solving is not an ability that children acquire as a natural by-product of instruction and drill on basic arithmetic operations. As is suggested in the recent set of recommendations prepared by the National Council of Teachers of Mathematics (1980), the solving of arithmetic story problems is an important and separate capability and one which must be as carefully taught as any other mathematical skill. This paper describes one approach to this difficult teaching task.

Steps to be Followed in Analyzing and Solving Arithmetic Story Problems

In attempting to make arithmetic story problem solving a systematic and thoughtful process, many teachers provide a list of steps that pupils are to follow in analyzing a story and deciding what arithmetic operation to employ. Table 1 provides an example of such a list. Is such a list of any value to the pupil? Does it describe a set of teachable skills that can provide guidance for instruction?

Table 1. One Example of a Series of Steps Pupils Could Be Taught to Follow in Solving Arithmetic Story Problems

1. Find out what is given.
2. Find the question to be answered.
3. Think about the operation that should be used.
4. Select an operation.
5. Carry out the operation and find the answer.
6. Re-read the story to see if your answer appears correct.

An examination of the sequence of steps shown in Table 1 would probably convince us that steps 1 and 2 should be relatively simple to teach. Problem-solving performance would be improved if we could get our pupils to attack stories by directing their thinking to the significant information provided in the story. Since pupils will, sooner or later, learn to carry out basic operations correctly, most teachers might also feel that steps 5 and 6 would cause no great problem with the majority of students. But what about steps 3 and 4? How can one teach children to "think about the operation that should be used"? Do we really know what to teach here? The purpose of this article is to examine steps 3 and 4 and to attempt to provide some guidance for teachers faced with the task of helping students learn to do the type of analytical thinking required in true problem solving.

Can We Teach Children to "Think About the Operation That Should be Used"?

Does the injunction to "think about the operation that should be used" represent a meaningful step for pupils to follow? At first glance this statement might appear to involve a side-stepping of responsibilities on the part of the teacher. It almost suggests that the teacher is saying, "I really don't know what to tell you to do exactly, but if you think about the story, you may be able to figure out what to do yourself." Certainly, "think about" is not a very specific bit of guidance. But, in a very real sense, isn't this exactly what the teacher must say? Isn't it really a recognition of the nature of "problem solving"? If there was a specific sequence of rather mechanical steps that students could follow to always identify the correct operation to apply to a story, this activity would not be "problem solving." It would merely be the application of a set of rules, or an algorithm. Problem solving must involve the step of "thinking about" what to do. Our step 3, then, is not a "cop-out" on the part of the teacher. It is a recognition of the fact that story problems should represent a true problem-solving activity for the pupil. The task of the teacher, then, is to help pupils learn to do some effective thinking in identifying the correct arithmetic operation to apply in the case of a given story. This teaching goal can be achieved if we give students tools for analyzing the story and representing its essential information in a form that suggests the needed operation.

How Should Children "Think About" Story Problems?

Obviously, the thinking that we wish to have children do in solving any story problem is thinking which reflects a clear understanding of the problem and that results in a correct answer. We would like each child to be able to respond to the request to "explain this story to me" by giving a clear description of what states or actions the story involves and how an answer is derived. We would not be satisfied with an explanation such as "whenever I see the word 'less', I always subtract." The type of description that we want is the type given quite often by kindergarten and first-grade children when they are presented with a simple story and a set of blocks and asked to "use these blocks to show me what this story means." Here, for example, if the story involves finding the number of marbles two boys had altogether when each had a given number, we might expect a typical student to count out sets of blocks to represent the number of marbles each boy had and then put these together in one set and count the total to find the answer. Students display their understanding of the story and its solution by translating it into a form which provides a clear presentation of the information needed to solve it (i.e., the number in the given sets and the set joining operation needed to get the answer). Note that these children must "think about" the story. They don't have command of any arithmetic operations that they can apply in some arbitrary manner.

Many kindergarten and first-grade children show this kind of understanding of simple stories before they have any formal instruction on arithmetic operations. It appears that it is only after children have learned something about formal arithmetic operations and are exposed to slightly more difficult stories, that they attempt to solve stories without really understanding them, without "thinking about" them. To obtain some useful insight into why this situation develops so frequently, it may be useful to review how story problem-solving ability first develops in a typical child.

How Do Children First Learn to Solve Story Problems?

Pre-school children, after they have learned to count, will frequently take part in a type of play with a parent or some older person in which they use this newly acquired ability to answer simple quantitative questions. The adult might say to the child, "Take 3 blocks for yourself. Now give me 2 blocks. How many blocks do you and I have altogether?" Another set of instructions might take the form of "Give yourself 7 cards. Now give me 3 of your cards. How many cards do you have left?"

Activities such as the above can be thought of as the child's first exposure to arithmetic story problems. After some minimum guidance and instruction, most children appear to have little difficulty in arriving at solutions to these simple quantitative problems. It should be noted that what children do here involves an "acting out" of the story. The characters in the story, "you" and "I", are actually present as are the objects described, blocks or cards. All that is required for solution is a correct counting of the set that represents the answer. Because of the specific and concrete nature of what is described and asked, most children quickly master such "story problems."

Shortly after showing some ability with stories of the above type, children are likely to be exposed to "pretend" stories. For example:

> Pretend that these blocks are pieces of candy. Suppose that you have 5 pieces of candy. Then, I give you 3 more pieces of candy. How many pieces of candy would you have then?

Or the story might be slightly less concrete and involve even more pretending.

> Pretend that you have 4 pieces of candy. Now pretend that your friend Sue is here and that she has 2 pieces of candy. How many pieces of candy do you and Sue have together?

In solving such a story children cannot fully act out the story in the same sense that they can act out stories involving "you" and "I". When they have mastered these "pretend" stories, they have acquired a slightly more sophisticated problem-solving capability, the ability to make use of simple abstractions. They realize that to solve problems about candy they do not have to count and sort pieces of candy. Since the key to solving problems about "how many" is focusing on number, or numerosity, the children can use any easily countable elements to represent the number of pieces of candy. They have abstracted the quality of numerosity as a key component that is needed for the solution of the story. Also, they recognize that it is not necessary to have "Sue" present in any concrete form. They can represent Sue's candy by building a set of two in some convenient location and identifying this set as "Sue's candy." In doing this they have abstracted the quality of "set identity" as another key component in story representation.

The foregoing analysis has suggested that very young children may solve arithmetic story problems at one of two levels. At the simplest, or beginning, level they really "act out" the story. At a slightly more advanced level, they develop a physical model or representation of the story and manipulate this model to solve the story. This can be represented as shown in Table 2.

Table 2. The Three Levels of Representation of a Story Problem (Arrows Show Desired Steps Followed by Student Having Mastery of This Capability)

	Representation of Problem	Method of Solution
Level 1	Problem in story form ↓	Act out the story
Level 2	Essential components represented in physical model ↓	Model is manipulated to determine answer
Level 3	Essential information and operation shown in math model (e.g., number sentence) - - - - - - →	Solution of math model

As has been described previously, the student's development of the physical model (e.g., using two sets of blocks to model a story that says, "Ann had 3 pieces of candy and Billy had 5 pieces of candy. How many did they have together?") involves abstracting from the story as originally given, that information which is essential for solving the problem. We shall see that this is a key phase in the child's "thinking about" the story.

Learning to Write Number Sentences for Stories

Skemp (1971) described the process of solving arithmetic story problems as one of making the necessary abstractions from the actual story in order to identify the exact information needed to carry out the steps needed for solution. The process of developing a physical model of a story situation, as presented in Table 2, has already been described as involving abstracting from an actual story, that information needed for developing the model and manipulating it to arrive at a solution. This means going from the actual characters, objects, and relationships or actions described in the story to a representation of these by means of, for example, sets of blocks appropriately arranged on a table. Going from such a physical model to the writing of an appropriate number sentence, or some other representation of an arithmetic operation, involves a further task of abstraction. This third level of abstraction involves representing the numerosity of a set of blocks through the use of the appropriate numeral and representing the correct set operation with the symbol for the corresponding arithmetic operation. That is, this stage involves the development of the number sentence, or "mathematical model," that can be used to solve the story.

In Table 2 the ultimate capability which we wish to have the pupil develop is indicated by the path represented by the dotted-line arrows; seeing the problem in story form, generating some type of physical representation, writing the number sentence for the story, and solving the number sentence to determine the answer to the story problem.

It is to be noted here that if this analysis is correct, children initially learn to write number sentences for story problems by first developing a general physical model of the story and then writing the number sentence for this model. That is, they make use of a skill previously acquired (developing and using a physical model to solve the story problem) as an intermediate step in developing the proper number sentence for the story.

When children demonstrate the ability to use a physical model to solve a story, there is little doubt that they "understand" the problem. They can tell you what different elements in the model represent as far as components of the story are concerned. They can also relate operations on the model to operations described in the story. The story and model have a one-to-one relationship with no mystery associated with it.

Writing the proper number sentence, then, is also a meaningful representation of the story because it is derived directly from a set operation. That is, children understand the story and the arithmetic operation that can be used to solve it because they understand the abstracted physical representation of the story that serves to identify the correct arithmetic operation. Although the example used in this paper is that of a very simple addition story, this translation of the verbal story into some type of intermediate representation (or series of representations) that can provide a meaningful link between the story itself and the appropriate arithmetic operation is what the problem solver must do in "thinking about" how to solve any story problem no matter how simple or complex.

It should be noted that the "physical model" referred to in Table 2 is some type of simplified representation that contains the information from the story which is essential for solving the problem. With the simple story that has been used as an example in the discussion to this point, this intermediate representation (intermediate between the actual story and the number sentence that can be used for solution) could well take the form of sets consisting of physical objects such as blocks, or sticks, or the child's fingers. However, with other stories the necessary model may take the form of a diagram on paper, of some version of a number line, of a data table, or any of a number of possible simplified representations of the essential information from the story.

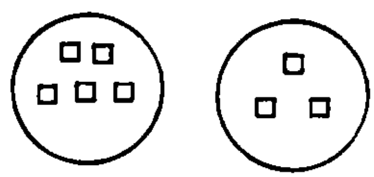
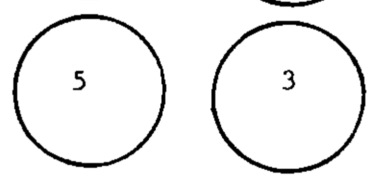
Suggested Procedures for "Thinking About" Story Problems

Table 3 gives an example of how a story problem might be represented and solved, showing representations that vary from an acting out of the actual story through increasing degrees of abstraction to the most abstract representation, the number sentence.

It is contended here that children can write a number sentence for a story, such as the one shown, with understanding, only if they first translate the story into some type of intermediate representation that makes totally clear what operation must be applied. This representation (which may be an almost instantaneous mental representation for the person who is highly proficient with the given type of story) in the case of the story shown in Table 3, must clearly show that solution of the problem requires the joining of two sets. Only when this is made clear to the problem solver can he or she proceed to write a number sentence with a full and correct understanding of why this particular mathematical operation is appropriate. That is, addition can be used to solve this story, not because it contains the word "altogether," but because the story, when correctly translated, describes the joining of two non-intersecting sets.

Of course, it can be pointed out that many pupils (and certainly most of us adults) do not have to go through any of these intermediate representations in order to solve this problem with complete understanding. Still, the evidence of our understanding must be shown by our ability, if called upon, to explain this particular story as one involving the joining of sets (or "putting groups of things together" or any comparable expression). That is, our understanding of this type of story and of the arithmetic operation needed to solve it is such that we mentally develop such intermediate representations and do this so quickly that we are not really aware of having taken this step. To achieve this type of understanding it is probably essential that all students actually be taught to use intermediate representations of stories and that they be given practice in using them at all the various points and

Table 3. Some Possible Stages that Children May Master as they Develop an Increasing Ability to Use Abstract Representations of a Story Problem in Arriving at a Solution

Representation of Problem	Method of Solution
Story	
Sally had 3. Jack had 4. How many altogether?	Student "acts out" the exact story as given
Some Intermediate Representations	
Act out, using dolls and actual objects	Story is acted out using this representation
Act out, using pictures of characters	Story is acted out using this representation
Model, using blocks (fingers, or any countable objects) but no pictures.	Model is manipulated to determine answer
Student draws sketches showing numerosity of sets and their separate identities	
Student draws diagrams to indicate identity of sets but uses numerals to indicate numerosity	
Mathematical Model	
Student writes number sentence	5 + 3 =

levels in the curriculum where new types of stories are presented, until it is obvious that they fully comprehend this process. Furthermore, instructional procedures that have the effect of causing the pupil to skip all intermediate representations of the problem will, more than likely, result in the pupil not fully comprehending the problem or its solution.

The various stages in the abstract representation of a story problem that are shown in Table 3 are presented only as examples of story representations, varying from the actual acting out of the story through intermediate representations that are increasingly abstract and provide an increasingly more direct basis for translation into a number sentence. Obviously, certain other intermediate representations might be substituted for those given here or might be inserted as additional steps in the sequence. Also, in working with any given student, only one or two of the intermediate stages might be needed to enable the student to grasp the basic structure of the problem and clearly understand what arithmetic operation to employ.

Diagnosing Pupil Difficulties in Solving Arithmetic Story Problems

The experience of most teachers provides evidence for the fact that there are large individual differences among students in their ability to solve story problems. Many students display an ability to grasp the meaning of a story quite quickly and have little difficulty in arriving at a solution. However, among the group of children who cannot solve such stories there appear to be great differences in the type of understanding (and lack of understanding) that they possess. With such students a diagnosis of specific difficulties would appear to be useful and probably essential.

It is suggested here that a sequence of stages in story representation as shown in Table 3 can provide the basis for a meaningful and useful diagnosis of pupil difficulties. It provides a means for determining the types of abstraction that a pupil can use in representing the essential meaning of a story. As an example, let's assume that Billy, one of our students, could not write the number sentence appropriate for the story shown in Table 3. We might start our diagnosis of his difficulties by seeing if he could model the story and its solution by using blocks. If he could do this, we would know that he had a good understanding of the story but needed further work on writing a number sentence for such a set operation. If he could not model the story with blocks, then we would proceed in the opposite direction and determine his ability to provide less abstract representations of the story. Could he act out the story if we provided dolls or pictures to represent the characters? If not, can he act out a similar story if we phrase it in terms of "you" and "I" and ask him to carry out the transactions described in the story? When we find the level at which Billy is able to operate, then we can build on that ability and proceed up through stages involving a greater degree of abstraction, making certain that he has ample time to master each stage in turn.

Teaching Problem Solving

The approach to the solution of arithmetic story problems that is outlined in this paper emphasizes the development of a complete understanding of each problem on the part of the pupil. It assumes that the goal of work on story problems at all grade levels is to teach general problem-solving abilities. For example, when we teach first-grade pupils to solve simple addition and subtraction stories, we are not merely teaching them "when to add" and "when to subtract." We are teaching them how to study and analyze problem situations so that they fully understand them and then, on the basis of this understanding, are able to determine what arithmetic operation(s) to apply. If this goal is to be achieved, teaching activities must be planned and carried out in a manner which emphasizes the importance of analysis and understanding. The following are some suggestions for conducting this type of teaching:

1. Classroom instruction and pupil assignments on story problems should give at least as much attention to the clear representation of the problem as to the calculation of the answer. For example, pupil written work on story problems should probably require an answer consisting of two components: (a) some type of diagram, drawing, or table that provides an abstracted representation of the essential problem components and relationships and (b) the mathematical computation used to obtain the answer. Both components should be graded.

2. Children should be taught, quite deliberately and specifically, how to develop paper-and-pencil representations of various types of stories. For example, this could include sketches of problems involving the combining of groups of things, the partitioning of groups, the combining of lengths and distances, the study of differences and relationships, and the comparison of sets of things. Continuing and frequent instruction on how to represent stories in this way should be a major feature of the teaching of problem solving.

3. The diagramatic representation of story problems must always include some type of representation of the quantities involved. This may take the form of countable elements in pictures of sets, of units on some scale of length, or of numerals.

4. It should be remembered that the arithmetic operations that will ultimately be used to solve most problems are basically efficient methods for determining quantities that would otherwise have to be found by counting. Having a clear perception, as obtained from the diagrammatic representation, of what would have to be counted to obtain the answer to a story problem is, then, essential for a correct identification of the proper operation to use.

5. Speed in arriving at a correct answer should not be emphasized in work on the solving of story problems. If importance is attached to speed of solution, this may only encourage students to make a guess concerning the operation to use and then proceed with computation. The emphasis should be placed on the careful analysis and clear representation of the problem.

Summary

Research on how adults proceed in their solution of complex problems that require a quantitative answer indicates that the most effective problem solvers do not go directly from the given problem situation to some type of equation or an arithmetic operation that immediately provides the solution. Rather they go through an intermediate stage in which they use sketches, diagrams, or other simplified representations of the problem to clarify the situation and to help them identify the needed operation(s). Research on how elementary school children solve arithmetic story problems also suggests that the successful problem solvers can make use of some type of intermediate representation to clarify the meaning of such stories. The present paper has attempted to outline how the development of such intermediate representations of story problems can be used in teaching pupils how to "think about" such stories and arrive at solutions.

Specifically, the approach advocated here suggests that understanding of a problem is gained by abstracting the essential information from the story and representing this in the form of a "model" or some type of "intermediate representation" which simplifies the problem situation. In the simple example of a problem used in this paper, the intermediate representation was presented in the form of sets of blocks or of pencil sketches of the sets involved. Such representations can be useful with problems involving groups of things that are to be joined or separated or operated upon in some way. Of course, with other types of problems other representations will be useful. For example, sketches of number lines or other indications of distances and a variety of types of charts and diagrams suggest themselves. As seen in Table 3, such diagrams may incorporate the use of numerals to indicate set size or the length of certain distances. This will be particularly necessary when amounts involved become at all large.

The paper also suggests that assessing the ability of individual students in terms of how proficient they are in developing such representations of stories can be a useful step in the diagnosis of difficulties and in identifying needed steps in instruction. It would appear that if elementary school children can acquire the ability to analyze arithmetic story problems through the type of intermediate representation proposed here, they can both become more effective in their present problem-solving tasks and acquire a basic skill in carrying out a general problem-solving procedure that will prepare them to be able to solve much more complex problems encountered in later years.

References

National Council of Teachers of Mathematics. An agenda for action: Recommendations for school mathematics of the 1980's. Reston, VA.: NCTM, 1980.

Skemp, R. R. The Psychology of Learning Mathematics. Middlesex, England: Penguin, 1971.