Problem Solving: Some Means and Ends

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Problem solving is a form of mental activity that is characteristically creative and requiring of ingenuity in conception or reflection. Problem solving goals in school mathematics are both the most important ends of the mathematics curriculum and means to the ends of concept development and skill development as well.

A problem is a difficulty that has some novel aspect; there is no readily available solution so the creativity of the problem solver is called upon to produce a solution or to resolve a difficulty. The presentation of problems in a mathematics classroom has potential of motivating and exciting reluctant learners. The sense of novelty and the challenge of a difficulty that summons forth ingenuity set a positive tone for a classroom. In essence, students can feel that their minds are being developed, that the initiator of the problem-solving activity values their ingenuity!

Why, then, do students rebel at problem solving? Could it be that what appears in problem-solving sections of textbooks is another form of skill rather than problem solving? Examining the discussion of textbooks suggests that this is exactly the case. The prescribed rules preceding a "problem-solving" section tend to indicate a lock-step, algorithmic approach that is antithetical to the goals of problem solving. As Polya (1957, 1962, 1965) describes the "one rule under the nose" sequencing of school mathematics (including current textbook "problem solving" sections) the creativity and ingenuity requirements are not present; the element of novelty in problems and a sense of reality of content and context of problems are also missing.

SYSTEM FOR DEVELOPING PROBLEM-SOLVING ABILITY

Problem solving is a vital factor in the growth and development of mathematical knowledge and know-how. As such it requires systematic effort and total integration into the mathematics curriculum at all levels. The very concept of mathematics, the view of what the subject is and of what a mathematician does is more accurately portrayed with problem solving than with any other mathematical activity. It is the heart of mathematics and should, likewise, be at the heart of the mathematics curriculum. The thought patterns that can be learned by observation, experience and participation in problem-solving activity are, at the least, as valuable throughout life as are the "basic skills" of arithmetic. Furthermore, these thought patterns strengthen skills and conceptual understanding, thereby producing mental structure and organization that aid retention and generality of these skills and concepts.

These patterns of thought break the dependence and "show me how first" attitude in students and replace it with more confidence and independence. Granted, systematic, long-term instruction and experience in problem solving are necessary to move a student to this independence and confidence in mathematical thought. Furthermore, concurrent development of content cannot be neglected. It is as Kantowski (1980) suggests that planned instruction and experience over a long period of time are necessary for the development of problem-solving ability. Such problem-solving instruction and experience are necessary, nevertheless, for students to move from the point of "following" what is shown to them and being reproducers to a more productive, creative use of mathematics.

PROCESSES THAT AID CONCEPT DEVELOPMENT

There are a number of problem-solving processes that can aid in concept development. For example, "reformulation" is a heuristic process that recognizes the "novel" characteristic of a problem and suggests acting upon this characteristic. Consider the following problem:

> What is the least number that leaves a remainder of 3 when divided by 5, a remainder of 2 when divided by 4, a remainder of 1 when divided by 3, and a remainder of 0 when divided by 2?

A reformulation of the problem is as follows:

> What is the least number that is 2 less than a multiple of 5, 2 less than a multiple of 4, 2 less than a multiple of 3, and "2 less" than a multiple of 2?

Frequently, the feature that makes a problem interesting is the fact that the concept must be "thought of" in an unusual way such as remainder in division being considered as excess or deficiency in multiplication (or visa versa). This strengthening of the ties between pairs of "opposite operations" provides a greater intuition for the structure of mathematics. Both the concept of least common multiple and the deep mathematical relationship of opposite operations give new perspectives to those who attempt problems like that above, particularly if approached in the problem-solving spirit.

"Working backward" is another powerful problem-solving process (a heuristic) that also can enhance concept and skill development. Consider the problem:

> A San Francisco streetcar turns curves at the bottom of hills very rapidly. Thus a driver must take care in making the turns. One day a careless driver turned the first of two curves so rapidly that he

"threw off" 1/2 of the passengers plus half a passenger and at the second sharp downhill curve "threw off" 1/2 of the remaining passengers plus half a passenger. Despite all of this, all passengers remained "whole" (except for a few minor scrapes and bruises) and the number of passengers who remained aboard the streetcar at the end of the ride was 20. How many started this treacherous streetcar ride?

Again, working through the solution of this problem has the potential for further development of the concept of "opposite operations." Consider the events of the streetcar ride. Let's make a list of these events.

Several passengers begin the streetcar ride.
Passengers are riding down the hill to the first major curve.
Half of the passengers are "thrown off."
Half of another passenger is "thrown off."
Passengers are riding down the hill to the second major curve.
Half of the <u>remaining</u> passengers are "thrown off."
Half of another passenger is "thrown off."
Twenty passengers remain aboard the streetcar.

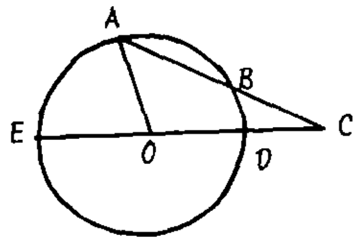
What <u>preceded</u> the "twenty passengers remained aboard the streetcar" condition? Consider visualizing the situation as a motion picture in slow motion running in <u>reverse</u>. The twenty passengers, then the alleged 1/2 passenger are "doubled" to get back to the condition of "riding back up the hill from the second major curve." Continuing back up the list of events, we express them mathematically and in reverse sequence.

20	:	Twenty passengers remain.
20 + 1/2	:	Twenty passengers plus half a passenger are riding.
2(20 + 1/2)	:	Riding "back up the hill" from the second curve.
41 + 1/2	:	Half of the passengers are now "back on" the streetcar.
2(41 + 1/2)	:	Riding "back up the hill" from the first curve.
83	:	"All" passengers are back on the streetcar.

The depth of understanding of the concept of "opposite operations," such as dividing by 2 and multiplying by 2, subtracting 1/2 and adding 1/2, as well as the reversal of the order of operations develop during such problem-solving experiences. Further, the structure of mathematical ideas and strategies for solution of mathematical problems accrue through systematic experiences of this sort.

As a third example of processes that can aid in skill development and development of conceptual understanding, we now consider the heuristic process referred to as "decomposition." Let us again do this within the context of a problem.

Chord $\overline{AB}$ of circle O is extended to meet the extension of diameter $\overline{ED}$ at C. $\overline{AO}$ is drawn. If $\overline{BC} \cong \overline{AO}$, what is the relationship between angle AOE and angle ACE?



The key idea of "decomposition" is that of breaking the problem into subproblems, the solution of which taken together is a solution to the original problem. To this end, consider the alternate goal of finding the measure of angle CAO. If this quantity were known, the problem would be solved, since angle EOA is an exterior angle of triangle AOC and as such angle EOA is equal in measure to the sum of angles CAO and C. Now, how could we get angle CAO? If we could determine the measure of angle ABO (since $\overline{AO}$ and $\overline{BO}$ are radii of the same circle), we would know angle CAO. This we are able to do since $\overline{BC} \cong \overline{AO}$ (and hence $\overline{BO}$) was given.

While examining the decomposition heuristic, a new view of isosceles triangles and skill in the use of the theorem "the exterior angle of a triangle is equal in measure to the sum of the measures of the remote interior angles" emerge. This blending of means of problem solving and ends of concept development is efficient; each complements the other; each is strengthened by the analysis of the other.

As a final example of heuristic processes that provide useful attack-mechanisms for problem solving while simultaneously serving to develop concepts and skills, let us consider the heuristic "use of definition." One may ask, "in what sense is 'use of definition' heuristic; that is, how does 'use of definition' serve to aid discovery?" Consider the problem:

On the first of 20 laps in a stock car race, JG averaged 120 km per hour, while on the next 20 laps, his average speed was 110 km per hour. What was his average speed for the 40 laps?

It almost seems as if the rate should be 115 km/h. But no, that is too obvious and also assumes equal times for the two sections of 20 laps (which was not the case). How can we then approach the problem? Make use of definition. To use definition, we must introduce both total distance and total time. To this end, let:

D = distance for 20 laps.

D/120 = time for the first 20 laps.

$$D/110 = \text{time for the next 20 laps.}$$

$$2D = \text{total distance.}$$

$$D/120 + D/110 = \text{total time.}$$

Then the average rate is given by:

$$\text{Average Rate} = 2D \,/\, (D/120 + D/110) = 2 \,/\, (1/120 + 1/110)$$

$$= 114.78 \text{ km/h.}$$

While strengthening the concept of average speed, this use of definition has led to the discovery of harmonic mean, a deep concept that provides a greater understanding of inverse relationships.

These examples of heuristic processes suggest the general idea that teaching for problem solving can produce greater conceptual understanding and more complete and lasting skills. This is true of a wide variety of heuristic processes and is surely not limited to those used above to illustrate the idea. The spiral development of these and other heuristic processes can bring confidence to the activity of problem solving, can strengthen basic skills and conceptual understanding, and most of all can provide students a more honest view of mathematics and mathematical thought.

ORGANIZING FOR PROBLEM-SOLVING INSTRUCTION

Selecting Problems

Many heuristics useful to solving problems are general and are thus useful across the subfields of mathematics. For example, a guess and test heuristic is quite useful in geometry as well as in arithmetic, number theory, and algebra. Thus selecting problems to develop the knowledge of such heuristics and how to use them is subject independent, at least to an extent worthy of development. Selecting problems appropriate to a group of students that allow the use and discussion of guess and test, decomposition and recombination, "use of definition," symmetry, working backward, solving simpler related problems, and so on is a beginning point for teaching for problem solving at all levels of mathematics instruction.

Secondly, exercises taken out of context are frequently useful as problems. For example, carefully chosen examples from chapters further ahead in a textbook can easily provide novel, nonroutine problem-solving experiences. Their solution provides readiness activity for the upcoming concepts as well as a chance for the development of such modes of attack on situations for which a ready-made plan for solution has not been expositorily presented. Consider, for the sake of illustration, the following pair of related problems:

> How many numbers are there that leave a remainder of 1 when divided into 59?

What is the greatest integer that will divide into 85 and 141 and leave the same remainder?

Working these in problem-solving sessions with discussions of heuristic processes useful to their solution and with a post hoc examination of the solutions can go far in setting the stage for the development of the concept of greatest common divisor as well as the skill of finding the greatest common divisor.

Some textbooks are so structured as to foreshadow upcoming content. The problem-solving approach, in such cases, can be quite beneficial in the development of a student's power of mathematical reasoning and a perspective of what mathematics is really like.

Finally, introducing problems from previously encountered sections of a textbook sometime after new content has been discussed may provide interesting new insights into problem-solving processes and concepts. Consider, for example, the "distance-rate-time" problem mentioned by Krutetskii (1976, p. 126).

A cyclist is supposed to be at a destination at a definite time. It is known that if he travels at a rate of 15 km per hour, he will arrive an hour early, and if his speed is 10 km per hour he will be an hour late. At what speed should he travel in order to arrive on time?

This would naturally fit as an exercise following a section in an algebra book on simultaneous equations. Suppose, however, that the problem were reintroduced after a discussion of "least common multiple" (for example, in a discussion of operations on rational algebraic expressions just after the idea of getting a least common denominator of two fractions). An alert student might observe, under the circumstances, that the solution to the problem required a number that is divisible by 10 and by 15, as well as three consecutive integers T-1, T, and T+1. Searching the multiples of 30 would lead to the multiple 60, which is divisible by the three consecutive integers 4, 5, and 6 and thus to the solution of 12, since

$$4 \times 15 = 60$$

$$5 \times ? = 60, \quad (? \text{ is } 12, \text{ the solution}), \text{ and}$$

$$6 \times 10 = 60.$$

The structure of this problem becomes clearer through the concept of harmonic means. The solution path becomes rather direct as follows:

$$\text{Rate} = \frac{2}{1/10 + 1/15}$$

The justification of this as a solution is left as an exercise, or as a problem as the case may be, for the reader.

CREATING RELATED PROBLEMS

As conceptual understanding and problem-solving ability grow concurrently the development of sets of related problems that bring out structures become essential. Problem-solving strategies are instilled through insight into structure. The development of this insight arises over time and through the abstraction of similarities in otherwise quite different problems. Consider now the following set of problems.

> Among the first 50 natural numbers, there are 25 that are divisible by 2, and 16 that are divisible by 3. How many of the first 50 natural numbers are divisible by either 2 or 3?
>
> How many numbers less than 1000 are there that are divisible by both 3 and 5?
>
> What is the probability that a number chosen at random from the first 25 perfect squares is also a perfect cube?
>
> How many numbers less than 1500 are there that are divisible by neither 3 nor 5?
>
> The Rainbow Ice Cream Parlor, which is open seven days per week, has unusual schedules for its employees. For example, Jane works every 7th day, Tim works every 3rd day and Rick works every 2nd day. If Jane, Tim and Rick were all working on June 1, during how many days in June were none of the three working (June has 30 days)?

These problems, if experienced in problem-solving sessions distributed across a portion of the school year (that is, hopefully not all at the same time), can provide a variety of useful ideas. First, a view of "counting" arises (999 ÷ 3 "counts" the number of multiples of 3 less than 1000, etc.) that may give new insight into the number system. Secondly, set relationships and a new perspective on set union and intersection may appear. All in all, the structure of a problem and a sense of the importance of comparing and contrasting <u>related problems</u> should emerge through such experiences.

TOWARD UNDERSTANDING STRUCTURE AND USING RELATEDNESS

The beginnings of the importance of the activity of mathematical problem solving as a means of developing intuition about mathematics comes from understanding problem structure and using relatedness. As the previous sections have suggested, sets of related problems, when taken together, help develop this idea of structure. Further, approaching related problems from different vantage points will provide a unity to the problem set and the problem-solving experience that will develop strategies of problem solving in students.

Let's consider two other problems, each of which is related to a previously encountered problem in our discussion.

In rehearsal of a Broadway opening a director estimated a certain number of hours of practice were essential. She calculated that by rehearsing 60 hours per week the show would be ready 1 week earlier than the opening date, but by practising 40 hours per week it would not be ready until 1 week after the opening date. How many hours per week of rehearsal would be necessary for the show to be ready on time?

If Coca Cola is 30 cents per 12 ounce can, we will be able to buy one less can than we need; but, if it sells for 20 cents per can we can buy one can more than we need. How much money have we for buying Coca Cola? How many cans do we need?

These problems, though very different in context and content, are structurally quite similar. A meaningful activity that helps a student develop insight into the structure of a problem is that of creating a problem that is "like" a given problem.

Experiencing problem solving, learning the thought processes of mathematics (the heuristics of Polya) and working toward the concepts of related problems and the structure of a problem are a sequence of problem-solving activities that are valuable goals of mathematics. Not only do they describe important ideas of a mathematics curriculum, but they provide new vehicles for accomplishing traditional goals of skills and conceptual understanding.

References

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