We Have This Problem with the Hall Lockers

by

James E. Riley
Western Michigan University

One of the problems of problem solving is finding good problems to solve. A good problem is one that does not yield an obvious solution. A good problem can be modeled or solved by analogy. A good problem can be studied empirically. The solution of a good problem may be arrived at from several directions. A good problem will result in the solver gaining new mathematical insights. A good problem should be an enriching experience for students with wide ranges of mathematical maturity. A good problem is hard to find.

We all have our favorite problems. I don't recall where I first came across one of my favorite problems, but I've seen it in many forms. The form I like best is found in the Indiana materials (LeBlanc, Kerr, and Thompson, 1976). It concerns a fixture found in many schools in North America. You see, we have this problem with the hall lockers.

Imagine a school with 1000 hall lockers along one side of a hallway. All the locker doors are open. Imagine 1000 children coming in from recess approaching the open lockers. The first child in line, a devilish tyke, can not resist slamming the locker doors shut.

The second child in line wishes to be involved so he starts opening the locker doors. But he cannot open them as fast as they were closed. He is only able to open every other locker starting with the second locker.

The third child in line wants to get into the act. She does so by changing the state of every third locker starting with locker number three. That is, if a locker is open she closes it, and if a locker is closed she opens it.

The rest of the children pick up the pattern. The nth student will change the state of every nth locker. When the thousandth child has passed the thousandth locker, which ones will be open and which ones will be closed?

Far be it from me to deprive the reader of the joy of solving a problem or making a discovery. Therefore, this article will occasionally be interrupted by the symbol (*) to let the reader know that this is a good place to put down the monograph and pick up a pencil and try to solve a proposed problem.

*   *   *   *   *   *   *   *   *   *   *

The locker problem has been presented to classes of students ranging from fourth graders to college undergraduates. Those who were able to solve the problem did so by first modeling the problem and then looking for patterns in the modeled solution. A fourth-grade class in Sparta, Michigan lined 36 English textbooks along the chalk tray. The class then lined up, like the class coming in from recess, and walked past the books turning them to represent open or closed locker doors. A book with its cover facing front represented an open locker and a book with its back cover facing front represented a closed locker door. The following pattern emerged.

1	2	3	4	5	6	7	8	9	10	11	12
B	F	F	B	F	F	F	F	B	F	F	F
13	14	15	16	17	18	19	20	21	22	23	24
F	F	F	B	F	F	F	F	F	F	F	F
25	26	27	28	29	30	31	32	33	34	35	36
B	F	F	F	F	F	F	F	F	F	F	B

When the above sequence is studied, one of two (or maybe both) patterns become obvious to the solver. What are they?

*   *   *   *   *   *   *   *   *   *   *

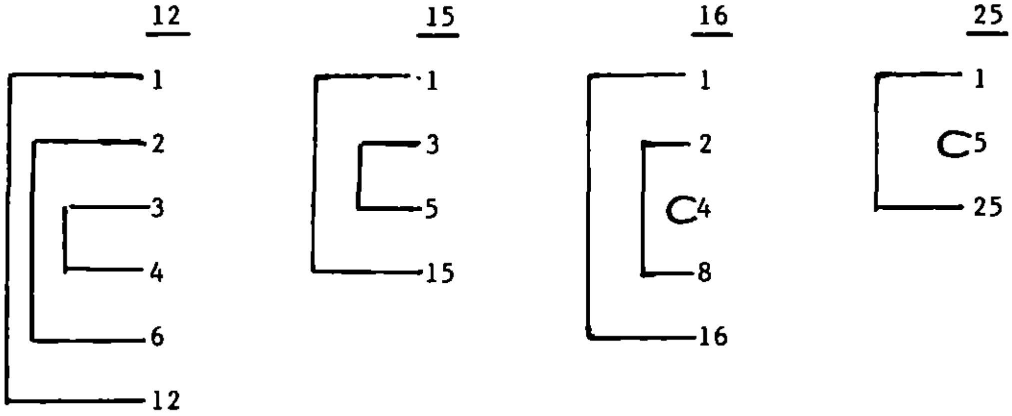
The more mathematically sophisticated solver recognizes that the closed numbered lockers are perfect squares; $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, etc. Younger children, because they are less at home with their multiplication facts, notice the following sequential pattern.

1 locker open	2 lockers closed
1 locker open	4 lockers closed
1 locker open	6 lockers closed
1 locker open	8 lockers closed
. . .	. . .

In either case, a solution to the locker problem has been found. But the solution is not mathematically satisfying. Why are the closed numbered lockers all perfect squares?

*   *   *   *   *   *   *   *   *   *   *

The numbered children who stop at any given numbered locker will be divisors of the locker number. The 1st, 2nd, 3rd, 4th, 6th, and 12th child will stop at locker 12. Notice that all numbers, <u>except perfect squares</u>, have even numbers of divisors. The divisors occur in pairs.



Any locker that has an even number of visitors will be left in the initial state because what one visitor does, the next will undo. Only those lockers with an odd number of visitors will be left in a changed state.

Generally fourth graders will stop at this point. However, the problem can be pursued a little further with fifth and sixth graders. Look again at the pattern created by the book model. Notice that the closed lockers (perfect squares) can be determined by the following sequence.

$$1 = 1$$

$$1 + 3 = 4$$

$$4 + 5 = 9$$

$$9 + 7 = 16$$

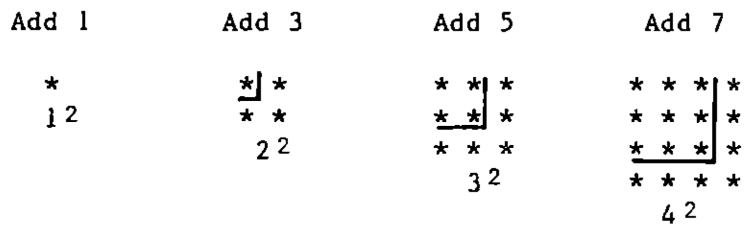
$$16 + 9 = 25$$

$$\ldots$$

Use a set of children's blocks (or a pencil and paper if your children's blocks are not at hand) to give a geometric interpretation to the above observation.

* * * * * * * * * * *

Later elementary school children have little difficulty showing that the addition of consecutive odd numbers of blocks will form a sequence of squares, with the length of a side one more than the preceding square.



We leave it to the ninth-grade algebra student to show that the succeeding terms of the above sequence can be algebraically expressed as

nth square + next odd number = (n + 1)st square

or

$$n^2 + (2n + 1) = (n + 1)^2$$

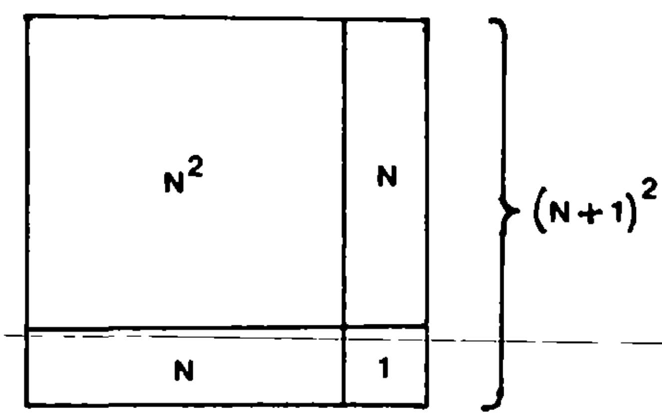
* * * * * * * * * * * *

A mathematical investigation that has fascinated students over the centuries is the finding of pythagorean triples. Pythagorean triples are positive integers (a, b, c) such that $a^2 + b^2 = c^2$. For example, 3, 4, and 5 make up a Pythagorean triple. The multiples of the triple (3,4,5) are also Pythagorian triples: (6,8,10), (9,12,15), etc. Pythagorean triples are said to be primative if a and b are relatively prime; i.e., if the greatest common divisor of a and b is 1.

Study the geometric succeeding-square model above and devise a scheme for finding infinitely many primative triples.

* * * * * * * * * * * *

A general form of the geometric model for the sequence of squares is the following.



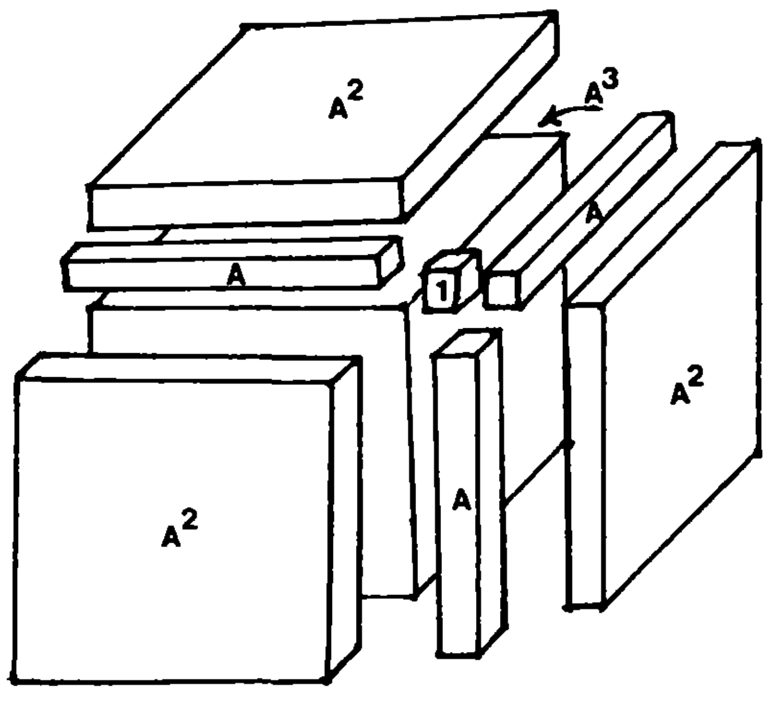
If in the formula $a^2 + b^2 = c^2$, we let $a = n$ and $c = (n + 1)$, then whenever $b = \sqrt{2n + 1}$ is a positive integer the triple (a, b, c) will be Pythagorean. Since $2n + 1$ will yield all odd numbers, it will also yield all odd perfect squares of which there are infinitely many. It can then be shown that n^2 and $2n + 1$ are relatively prime.

* * * * * * * * * * * *

Can the idea of Pythagorean triples be extended? For example, can we find triples (a, b, c) such that $a^3 + b^3 = c^3$? The Fermat conjecture states that such triples do not exist for $a^n + b^n = c^n$ where $n \geq 3$. The conjecture has been verified for all values of $n \leq 2500$ plus many more. The futility of the search can be demonstrated when one tries to extend the sequence model to the cube. Try it.

* * * * * * * * * * * *

To extend a^3 to $(a + 1)^3$, $3a^2 + 3a + 1$ must be added to a^3. This is easy to verify algebraically. The following figure shows the geometric interpretation of the extension.



If the extended Pythagorean triple is to hold for $n = 3$, then $3a^2 + 3a + 1$ must be a perfect cube. The following table shows the first ten perfect cubes and the values of $3n^2 + 3n + 1$ closest to the listed cube. The investigator will not be encouraged by what is shown.

n	$3n^2 + 3n + 1$	closest perfect cube
		1
1	7	8
2	19	27
3	37	
4	61	64
5	91	
6	127	125
7	169	
8	217	216
9	271	
10	331	343
11	397	
12	469	
13	547	512
14	631	
15	721	729
16	817	
17	919	
18	1027	1000

Thus we come to the end of a problem trail that started with some mischievous children and school hall lockers to an unsolved problem on the frontier of mathematics. Granted there were a number of side trails that could also be investigated such as the investigation of n-gon arrays and geometric numbers. Nonetheless, the trail we followed carried us through a number of problem-solving skills including modeling, emperical data collection, generalization, and logical thought.

References

Le Blanc, J. F., Kerr, D. R., and Thompson, M. Number theory. Reading, Massachusetts: Addison-Wesley, 1976.