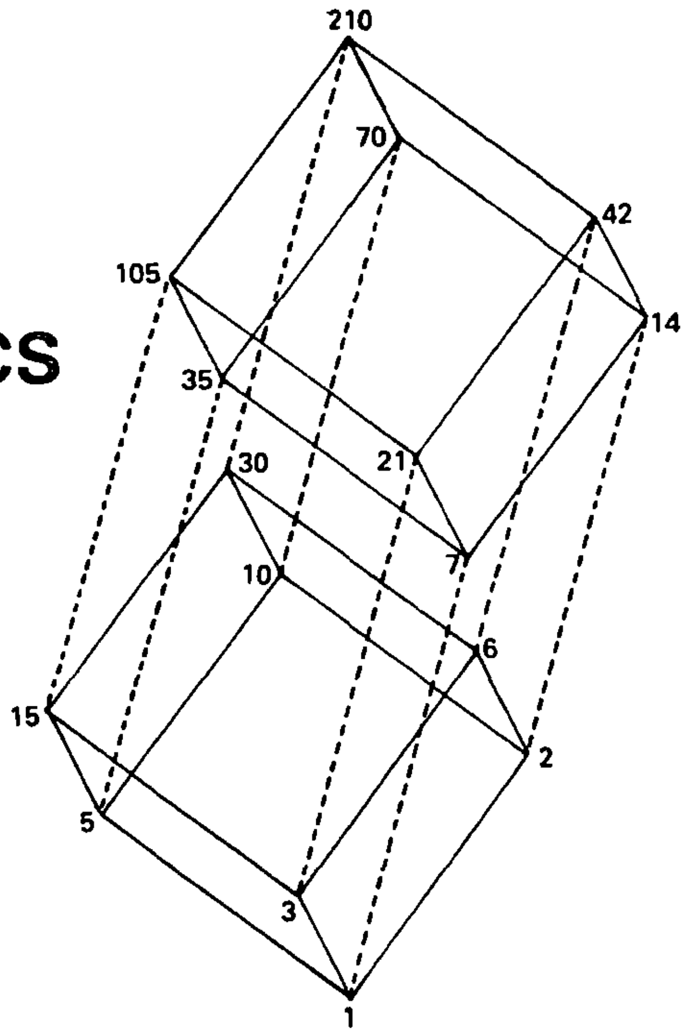
MATH Monograph No. 7

Editor: Sid Rachlin Technical Editor: Judy McDonald April 1982

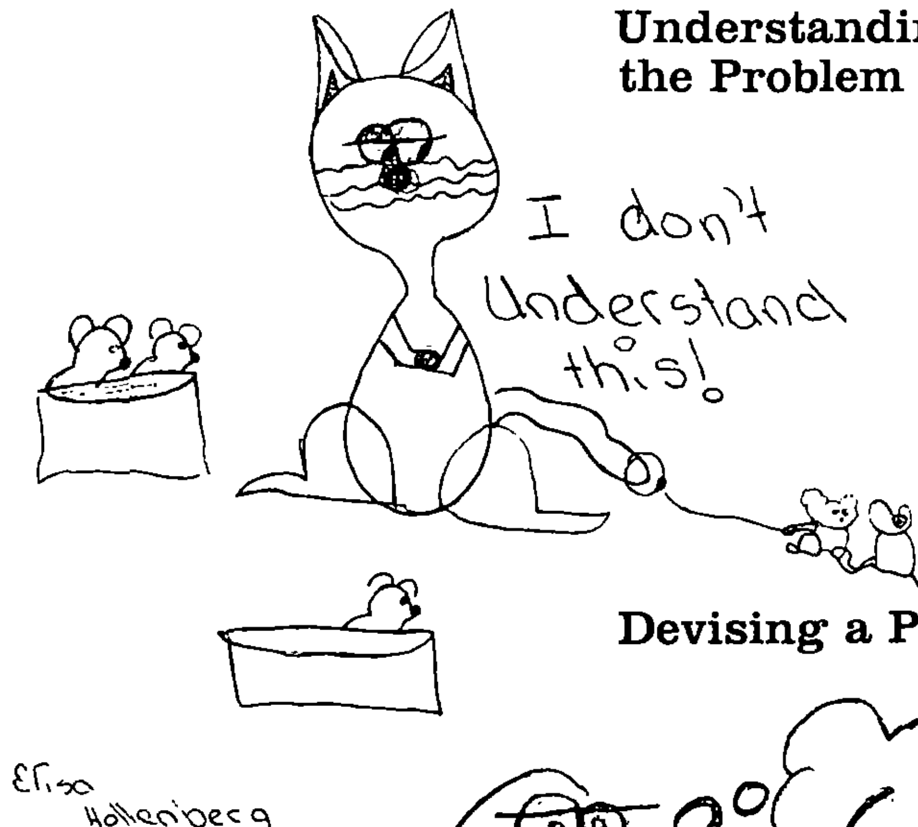
Problem Solving in the Mathematics Classroom



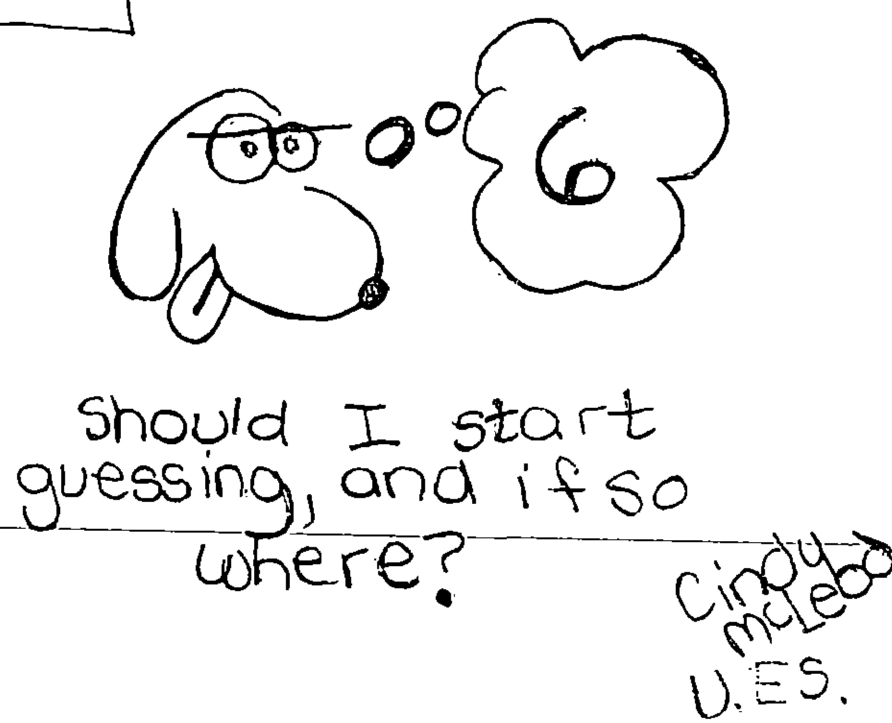
MCATA Publication of the Mathematics Council of The Alberta Teachers' Association

Problem-Solving Strategies

Understanding the Problem



Devising a Plan



MATH

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Contents

Math Monograph No. 7 is published by The Alberta Teachers' Association for the Mathematics Council. Editor: Sid Rachlin, Faculty of Education, University of Calgary, Calgary, Alberta T2N 1N4. Editorial and Production Services: Communications Staff, ATA. Address all correspondence regarding the contents of this publication to the editor. Opinions of the writers are not necessarily those of the Mathematics Council or of The Alberta Teachers' Association. Members of the Mathematics Council receive this publication free of charge. Non-members wishing a copy may obtain one by forwarding $6 to The Alberta Teachers' Association, 11010 - 142 Street, Edmonton T5N 2R1.

MATHEMATICS COUNCIL, ATA
Executive, 1981-82

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Contributors

David L. Albig
Department of Math/Stat/CPSC
Radford University
Radford, Virgina 24142

Dr. A. Anderson
25 Hayword Court N.E.
Medicine Hat, Alta.
T1C 1E2

Dr. J. Barnett
Department of Mathematics
Fort Hays State University
Hays, Kansas 67601

Elliott Bird
C.W. Post Center
Long Island University
Greenvale, New York 11548

Stephen I. Brown
State University of New York at Buffalo
Buffalo, New York 14260

Dr. Randall I. Charles
Board of Education
County of Wood
1210 - 13th Street
Parkersburg, W. Virginia

Dr. John Firkins
Department of Mathematics
Gonzaga University
Spokane, Washington 99258

Dr. S. Heck
Ohio State University
Faculty of Early and Middle Childhood Education
1945 North High Street
Columbus, Ohio 43210

Bruce Hedderick
John Ware Junior High
10020-19th St. S.W.
Calgary, Alberta

Karen L. Jones
Round Rock Independent School District
Round Rock, Texas 78664

Dr. Cheryl Kantecki
West Chicago High School
West Chicago, Illinois 60185

Richard King
Annie Foote Elementary School
Calgary Board of Education
515 Macleod Trail S.E.
Calgary, Alberta

Dr. S. Krulik
College of Education
Temple University
Philadelphia, Pa. 19122

Charles E. Lamb
The University of Texas at Austin
Austin, Texas 78712

Dr. Kil S. Lee
Department of Curriculum and Instruction
University of New Orleans
Lake Front
New Orleans, Louisiana 70122

Dr. Frank K. Lester, Jr.
School of Education
3rd & Jordon
Mathematics Education #101
Indiana University
Bloomington, Indiana 47405

Dr. C. Mauritz Lindvall
University of Pittsburgh
LRDC Building
3939 O'Hara Street
Pittsburgh, PA 15760

C. Edwin McClintock
School of Education
Florida International University
Miami, Florida 33199

Dr. Barbara Moses
Bowling Green State University
Department of Mathematics
Bowling Green, Ohio 43403

Sid Rachlin
614 Ed Tower
The University of Calgary
Calgary, Alberta
T2N 1N4

H. Lawrence Ridge
Faculty of Education
University of Toronto
371 Bloor Street West
Toronto, Ontario
M5S 2R7

Dr. James E. Riley
Department of Mathematics
Western University of Michigan
Kalamazoo, Michigan 49008

Dr. J. Rudnick
College of Education
Temple University
Philadelphia, PA 19122

Dr. A. H. Schoenfeld
Mathematics Department
Hamilton College
Clinton, New York 13323

Dr. Verena Sharkey
3606 Concorde Pike
Wilmington, Delaware 19805

Fredrick L. Silverman
Louisiana State University
at Shreveport
Shreveport, Louisiana 71115

Dr. Raymond Spaulding
Department of Math/Stat/CPSC
Radford University
Radford, Virginia 24142

C. Ray Williams
Ohio State University
Faculty of Early and Middle
Childhood Education
1945 North High Street
Columbus, Ohio 43210

Dr. Lee E. Yunker
West Chicago High School
West Chicago, Illinois 60185

Editor's Comments

by

Sid Rachlin
The University of Calgary

The development of the ability to solve problems has long been recognized as one of the major goals of mathematics education. With the recommendation by the National Council of Teachers of Mathematics (NCTM) that "Problem Solving Must be the Focus of School Mathematics in the 1980's," the goal of teaching problem solving in the mathematics classroom has taken on the fervor of a campaign slogan. Still, there is only one group of people capable of attaining this goal: the classroom teachers.

Interest in problem solving is not new to Alberta's educators. Perhaps the finest work on the teaching of problem solving in the early childhood years was published by Doyal Nelson and Joan Kirkpatrick Worth of the University of Alberta in the 37th Yearbook of the NCTN. Long before the problem-solving bandwagon began to roll, Alberta Education noted in its 1977 Curriculum Guide for Elementary Mathematics that problem solving was a "unifying theme which permeates all the strands of the elementary school mathematics program." Representative of the new impetus for the teaching of problem solving are the changes suggested in the 1981 revision of Alberta Education's Curriculum Guide for Elementary Mathematics. The revised curriculum guide includes the learning of problem-solving skills as a separate strand. The authors of the guide are quick to point out that the inclusion of a problem-solving strand "is not intended to portray problem solving as a topic unto itself." Rather than a change in the curriculum, the inclusion of the problem-solving strand represents an effort to make teachers more aware of how they might teach for the development of problem-solving abilities. In order to help facilitate the teaching of the new scope and sequence of problem-solving skills, Alberta Education has produced a manual entitled Let Problem Solving Be The Focus in the 1980's. In addition to a scope and sequence chart of problem-solving skills for elementary mathematics categorized by Polya's four phases for solving problems (understanding the problem, devising a plan, carrying out the plan, and looking back), the manual provides an array of suitable problems for students of varying developmental abilities.

This edition of the Math Monograph provides teachers with a wide range of articles on the teaching of problem solving in the mathematics classroom. The monograph is separated into four sections: Understanding the Problem, Devising a Plan, Carrying Out the Plan, and Looking Back. The first section includes articles which provide a sense of the "problem" of teaching for problem solving: What is given? What is our goal? In general terms, how might the goal be attained? The second section provides suggested plans for attacking related classes of problems. The articles in part three focus on the solution to specific problems. Finally, the last section includes articles which reflect on the past, present, and future of teaching for problem solving. At times the placement of an article into a section was arbitrary: i.e., several articles could be placed comfortably within any one of the sections.

This monograph presents the thoughts of a diverse group of authors, representing seventeen states and provinces. Despite the diversity, two common threads run through the articles. The first is a common notion for the meaning of the word "problem." A problem is defined as a "task" which an individual attempts to resolve, given that this resolution is within the person's ability and it is not resolved by the person's immediate application of some algorithm. Resolution of the task is taken as the individual's belief, stated or implied, that he has obtained the "actual" solution. Whether or not a task is a problem is dependent on the characteristics of the individual and his attempted paths to resolution. The second common thread woven in the articles is the belief that the actions of the teachers can affect the development of their students' abilities to solve problems.

The articles included in this monograph have been specifically written for this audience. They are not reprinted from other publications. Several people are responsible for the selection and editing of the articles for the monograph. In the spring of 1980 over 40 letters soliciting submissions for possible inclusion in the monograph were sent to individuals speaking on the teaching of problem solving at meetings of the National Council of Teachers of Mathematics in Calgary, Regina and Seattle. Letters were also sent to appropriate speakers at the Vancouver meeting of the Research Council on Diagnostic and Prescriptive Mathematics. Under the guidance of Lyle Pagnucco and Rich King, at least four teachers reviewed each submission. Articles were accepted based on the recommendations of the reviewers and edited to avoid redundancy and to ensure that the examples provided were phrased appropriately for the intended audience. Later the articles were re-edited in an attempt to provide the greatest possible amount of information in a limited amount of space. For example, only those bibliographic entries specifically referred to in an article were included with the list of references at the end of each article. Readers interested in detailed bibliographies of mathematical problem solving are directed to Sarah Mason's annotated bibliography in the 1980 Yearbook of the NCTM and Frank Lester's chapter on problem-solving research in NCTM's Research in Mathematics Education.

Finally, it is with sincere appreciation that I acknowledge the skillful and dedicated efforts of Judy McDonald. It is through her technical skill that you are now able to share this edition of the MATH Monograph with your fellow teachers.

Part I

Understanding the Problem – The Given and the Goal

Look at your problem from all angels.

Problem Solving: Some Means and Ends

by

C. Edwin McClintock
Florida International University

Problem solving is a form of mental activity that is characteristically creative and requiring of ingenuity in conception or reflection. Problem solving goals in school mathematics are both the most important ends of the mathematics curriculum and means to the ends of concept development and skill development as well.

A problem is a difficulty that has some novel aspect; there is no readily available solution so the creativity of the problem solver is called upon to produce a solution or to resolve a difficulty. The presentation of problems in a mathematics classroom has potential of motivating and exciting reluctant learners. The sense of novelty and the challenge of a difficulty that summons forth ingenuity set a positive tone for a classroom. In essence, students can feel that their minds are being developed, that the initiator of the problem-solving activity values their ingenuity!

Why, then, do students rebel at problem solving? Could it be that what appears in problem-solving sections of textbooks is another form of skill rather than problem solving? Examining the discussion of textbooks suggests that this is exactly the case. The prescribed rules preceding a "problem-solving" section tend to indicate a lock-step, algorithmic approach that is antithetical to the goals of problem solving. As Polya (1957, 1962, 1965) describes the "one rule under the nose" sequencing of school mathematics (including current textbook "problem solving" sections) the creativity and ingenuity requirements are not present; the element of novelty in problems and a sense of reality of content and context of problems are also missing.

SYSTEM FOR DEVELOPING PROBLEM-SOLVING ABILITY

Problem solving is a vital factor in the growth and development of mathematical knowledge and know-how. As such it requires systematic effort and total integration into the mathematics curriculum at all levels. The very concept of mathematics, the view of what the subject is and of what a mathematician does is more accurately portrayed with problem solving than with any other mathematical activity. It is the heart of mathematics and should, likewise, be at the heart of the mathematics curriculum. The thought patterns that can be learned by observation, experience and participation in problem-solving activity are, at the least, as valuable throughout life as are the "basic skills" of arithmetic. Furthermore, these thought patterns strengthen skills and conceptual understanding, thereby producing mental structure and organization that aid retention and generality of these skills and concepts.

These patterns of thought break the dependence and "show me how first" attitude in students and replace it with more confidence and independence. Granted, systematic, long-term instruction and experience in problem solving are necessary to move a student to this independence and confidence in mathematical thought. Furthermore, concurrent development of content cannot be neglected. It is as Kantowski (1980) suggests that planned instruction and experience over a long period of time are necessary for the development of problem-solving ability. Such problem-solving instruction and experience are necessary, nevertheless, for students to move from the point of "following" what is shown to them and being reproducers to a more productive, creative use of mathematics.

PROCESSES THAT AID CONCEPT DEVELOPMENT

There are a number of problem-solving processes that can aid in concept development. For example, "reformulation" is a heuristic process that recognizes the "novel" characteristic of a problem and suggests acting upon this characteristic. Consider the following problem:

> What is the least number that leaves a remainder of 3 when divided by 5, a remainder of 2 when divided by 4, a remainder of 1 when divided by 3, and a remainder of 0 when divided by 2?

A reformulation of the problem is as follows:

> What is the least number that is 2 less than a multiple of 5, 2 less than a multiple of 4, 2 less than a multiple of 3, and "2 less" than a multiple of 2?

Frequently, the feature that makes a problem interesting is the fact that the concept must be "thought of" in an unusual way such as remainder in division being considered as excess or deficiency in multiplication (or visa versa). This strengthening of the ties between pairs of "opposite operations" provides a greater intuition for the structure of mathematics. Both the concept of least common multiple and the deep mathematical relationship of opposite operations give new perspectives to those who attempt problems like that above, particularly if approached in the problem-solving spirit.

"Working backward" is another powerful problem-solving process (a heuristic) that also can enhance concept and skill development. Consider the problem:

> A San Francisco streetcar turns curves at the bottom of hills very rapidly. Thus a driver must take care in making the turns. One day a careless driver turned the first of two curves so rapidly that he

"threw off" 1/2 of the passengers plus half a passenger and at the second sharp downhill curve "threw off" 1/2 of the remaining passengers plus half a passenger. Despite all of this, all passengers remained "whole" (except for a few minor scrapes and bruises) and the number of passengers who remained aboard the streetcar at the end of the ride was 20. How many started this treacherous streetcar ride?

Again, working through the solution of this problem has the potential for further development of the concept of "opposite operations." Consider the events of the streetcar ride. Let's make a list of these events.

Several passengers begin the streetcar ride.
Passengers are riding down the hill to the first major curve.
Half of the passengers are "thrown off."
Half of another passenger is "thrown off."
Passengers are riding down the hill to the second major curve.
Half of the remaining passengers are "thrown off."
Half of another passenger is "thrown off."
Twenty passengers remain aboard the streetcar.

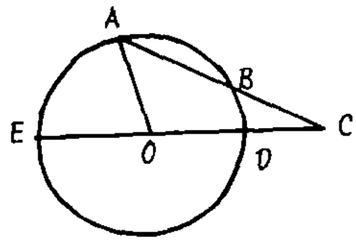
What preceded the "twenty passengers remained aboard the streetcar" condition? Consider visualizing the situation as a motion picture in slow motion running in reverse. The twenty passengers, then the alleged 1/2 passenger are "doubled" to get back to the condition of "riding back up the hill from the second major curve." Continuing back up the list of events, we express them mathematically and in reverse sequence.

20	:	Twenty passengers remain.
20 + 1/2	:	Twenty passengers plus half a passenger are riding.
2(20 + 1/2)	:	Riding "back up the hill" from the second curve.
41 + 1/2	:	Half of the passengers are now "back on" the streetcar.
2(41 + 1/2)	:	Riding "back up the hill" from the first curve.
83	:	"All" passengers are back on the streetcar.

The depth of understanding of the concept of "opposite operations," such as dividing by 2 and multiplying by 2, subtracting 1/2 and adding 1/2, as well as the reversal of the order of operations develop during such problem-solving experiences. Further, the structure of mathematical ideas and strategies for solution of mathematical problems accrue through systematic experiences of this sort.

As a third example of processes that can aid in skill development and development of conceptual understanding, we now consider the heuristic process referred to as "decomposition." Let us again do this within the context of a problem.

Chord $\overline{AB}$ of circle O is extended to meet the extension of diameter $\overline{ED}$ at C. $\overline{AO}$ is drawn. If $\overline{BC} \cong \overline{AO}$, what is the relationship between angle AOE and angle ACE?



The key idea of "decomposition" is that of breaking the problem into subproblems, the solution of which taken together is a solution to the original problem. To this end, consider the alternate goal of finding the measure of angle CAO. If this quantity were known, the problem would be solved, since angle EOA is an exterior angle of triangle AOC and as such angle EOA is equal in measure to the sum of angles CAO and C. Now, how could we get angle CAO? If we could determine the measure of angle ABO (since $\overline{AO}$ and $\overline{BO}$ are radii of the same circle), we would know angle CAO. This we are able to do since $\overline{BC} \cong \overline{AO}$ (and hence $\overline{BO}$) was given.

While examining the decomposition heuristic, a new view of isosceles triangles and skill in the use of the theorem "the exterior angle of a triangle is equal in measure to the sum of the measures of the remote interior angles" emerge. This blending of means of problem solving and ends of concept development is efficient; each complements the other; each is strengthened by the analysis of the other.

As a final example of heuristic processes that provide useful attack-mechanisms for problem solving while simultaneously serving to develop concepts and skills, let us consider the heuristic "use of definition." One may ask, "in what sense is 'use of definition' heuristic; that is, how does 'use of definition' serve to aid discovery?" Consider the problem:

On the first of 20 laps in a stock car race, JG averaged 120 km per hour, while on the next 20 laps, his average speed was 110 km per hour. What was his average speed for the 40 laps?

It almost seems as if the rate should be 115 km/h. But no, that is too obvious and also assumes equal times for the two sections of 20 laps (which was not the case). How can we then approach the problem? Make use of definition. To use definition, we must introduce both total distance and total time. To this end, let:

D = distance for 20 laps.

D/120 = time for the first 20 laps.

$D/110$ = time for the next 20 laps.

$2D$ = total distance.

$D/120 + D/110$ = total time.

Then the average rate is given by:

$$\text{Average Rate} = 2D\ /\ (D/120 + D/110) = 2\ /\ (1/120 + 1/110)$$

$$= 114.78 \text{ km/h}.$$

While strengthening the concept of average speed, this use of definition has led to the discovery of harmonic mean, a deep concept that provides a greater understanding of inverse relationships.

These examples of heuristic processes suggest the general idea that teaching for problem solving can produce greater conceptual understanding and more complete and lasting skills. This is true of a wide variety of heuristic processes and is surely not limited to those used above to illustrate the idea. The spiral development of these and other heuristic processes can bring confidence to the activity of problem solving, can strengthen basic skills and conceptual understanding, and most of all can provide students a more honest view of mathematics and mathematical thought.

ORGANIZING FOR PROBLEM-SOLVING INSTRUCTION

Selecting Problems

Many heuristics useful to solving problems are general and are thus useful across the subfields of mathematics. For example, a guess and test heuristic is quite useful in geometry as well as in arithmetic, number theory, and algebra. Thus selecting problems to develop the knowledge of such heuristics and how to use them is subject independent, at least to an extent worthy of development. Selecting problems appropriate to a group of students that allow the use and discussion of guess and test, decomposition and recombination, "use of definition," symmetry, working backward, solving simpler related problems, and so on is a beginning point for teaching for problem solving at all levels of mathematics instruction.

Secondly, exercises taken out of context are frequently useful as problems. For example, carefully chosen examples from chapters further ahead in a textbook can easily provide novel, nonroutine problem-solving experiences. Their solution provides readiness activity for the upcoming concepts as well as a chance for the development of such modes of attack on situations for which a ready-made plan for solution has not been expositorily presented. Consider, for the sake of illustration, the following pair of related problems:

> How many numbers are there that leave a remainder of 1 when divided into 59?

What is the greatest integer that will divide into 85 and 141 and leave the same remainder?

Working these in problem-solving sessions with discussions of heuristic processes useful to their solution <u>and</u> with a post hoc examination of the solutions can go far in setting the stage for the development of the concept of greatest common divisor as well as the skill of finding the greatest common divisor.

Some textbooks are so structured as to foreshadow upcoming content. The problem-solving approach, in such cases, can be quite beneficial in the development of a student's power of mathematical reasoning and a perspective of what mathematics is really like.

Finally, introducing problems from previously encountered sections of a textbook sometime after new content has been discussed may provide interesting new insights into problem-solving processes and concepts. Consider, for example, the "distance-rate-time" problem mentioned by Krutetskii (1976, p. 126).

A cyclist is supposed to be at a destination at a definite time. It is known that if he travels at a rate of 15 km per hour, he will arrive an hour early, and if his speed is 10 km per hour he will be an hour late. At what speed should he travel in order to arrive on time?

This would naturally fit as an exercise following a section in an algebra book on simultaneous equations. Suppose, however, that the problem were reintroduced after a discussion of "least common multiple" (for example, in a discussion of operations on rational algebraic expressions just after the idea of getting a least common denominator of two fractions). An alert student might observe, under the circumstances, that the solution to the problem required a number that is divisible by 10 and by 15, as well as three consecutive integers T-1, T, and T+1. Searching the multiples of 30 would lead to the multiple 60, which is divisible by the three consecutive integers 4, 5, and 6 and thus to the solution of 12, since

$$4 \times 15 = 60$$

$$5 \times ? = 60, \quad (? \text{ is } 12, \text{ the solution}), \text{ and}$$

$$6 \times 10 = 60.$$

The structure of this problem becomes clearer through the concept of harmonic means. The solution path becomes rather direct as follows:

$$\text{Rate} = \frac{2}{1/10 + 1/15}$$

The justification of this as a solution is left as an exercise, or as a problem as the case may be, for the reader.

CREATING RELATED PROBLEMS

As conceptual understanding and problem-solving ability grow concurrently the development of sets of related problems that bring out structures become essential. Problem-solving strategies are instilled through insight into structure. The development of this insight arises over time and through the abstraction of similarities in otherwise quite different problems. Consider now the following set of problems.

> Among the first 50 natural numbers, there are 25 that are divisible by 2, and 16 that are divisible by 3. How many of the first 50 natural numbers are divisible by either 2 or 3?
>
> How many numbers less than 1000 are there that are divisible by both 3 and 5?
>
> What is the probability that a number chosen at random from the first 25 perfect squares is also a perfect cube?
>
> How many numbers less than 1500 are there that are divisible by neither 3 nor 5?
>
> The Rainbow Ice Cream Parlor, which is open seven days per week, has unusual schedules for its employees. For example, Jane works every 7th day, Tim works every 3rd day and Rick works every 2nd day. If Jane, Tim and Rick were all working on June 1, during how many days in June were none of the three working (June has 30 days)?

These problems, if experienced in problem-solving sessions distributed across a portion of the school year (that is, hopefully not all at the same time), can provide a variety of useful ideas. First, a view of "counting" arises (999 ÷ 3 "counts" the number of multiples of 3 less than 1000, etc.) that may give new insight into the number system. Secondly, set relationships and a new perspective on set union and intersection may appear. All in all, the structure of a problem and a sense of the importance of comparing and contrasting related problems should emerge through such experiences.

TOWARD UNDERSTANDING STRUCTURE AND USING RELATEDNESS

The beginnings of the importance of the activity of mathematical problem solving as a means of developing intuition about mathematics comes from understanding problem structure and using relatedness. As the previous sections have suggested, sets of related problems, when taken together, help develop this idea of structure. Further, approaching related problems from different vantage points will provide a unity to the problem set and the problem-solving experience that will develop strategies of problem solving in students.

Let's consider two other problems, each of which is related to a previously encountered problem in our discussion.

In rehearsal of a Broadway opening a director estimated a certain number of hours of practice were essential. She calculated that by rehearsing 60 hours per week the show would be ready 1 week earlier than the opening date, but by practising 40 hours per week it would not be ready until 1 week after the opening date. How many hours per week of rehearsal would be necessary for the show to be ready on time?

If Coca Cola is 30 cents per 12 ounce can, we will be able to buy one less can than we need; but, if it sells for 20 cents per can we can buy one can more than we need. How much money have we for buying Coca Cola? How many cans do we need?

These problems, though very different in context and content, are structurally quite similar. A meaningful activity that helps a student develop insight into the structure of a problem is that of creating a problem that is "like" a given problem.

Experiencing problem solving, learning the thought processes of mathematics (the heuristics of Polya) and working toward the concepts of related problems and the structure of a problem are a sequence of problem-solving activities that are valuable goals of mathematics. Not only do they describe important ideas of a mathematics curriculum, but they provide new vehicles for accomplishing traditional goals of skills and conceptual understanding.

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An Instructional System for Mathematical Problem Solving

by

Randall I. Charles
Woods County Schools

Considerable attention has been given to the topic of mathematical problem solving. Indeed, most teachers now know that one of the most important goals of mathematics education is to develop in each child the ability to solve mathematical problems. Although most teachers are aware of the important role problem-solving experiences play in mathematics education, teachers still have a variety of questions concerning how to develop and implement a mathematical problem-solving program.

Most of the questions teachers have regarding the development and implementation of a problem-solving program are concerned with two issues.

- How should one organize a mathematics program to include problem-solving experiences?
- What specifically should be done to develop students' mathematical problem-solving abilities?

The purpose of this paper is to describe an instructional system for mathematical problem solving that was designed to answer these questions. The instructional system described here was designed for and used with average and high-achieving students at the junior high school level. At the end of the paper some ideas are provided concerning ways in which the instructional system might be modified for low-achieving junior high school students.

Components of the Instructional System

There are two components to the instructional system described in this paper: organizing for instruction and teaching strategy. The organizing for instruction component provides answers to five questions.

(1) What types of problems should be used?
(2) What should be the minimum time allotment for problem solving?
(3) What grouping patterns should be used for problem-solving activities?
(4) What material is needed for problem solving?
(5) How should students be evaluated?

The teaching strategy component identifies specific behaviors one can use in the classroom to help develop students' attitudes and abilities related to mathematical problem solving.

Organizing for Instruction

Problem-solving experiences are not presently an integral part of most instructional programs. Therefore, teachers interested in providing students with problem-solving experiences must make several difficult decisions concerning the structure of a problem-solving program. This section provides one answer to each of the five questions given above related to "organizing for instruction." Although it is clear other answers could be given to each of these questions, it is also clear that each teacher must at some time provide answers to at least these questions if problem solving is to become a significant part of one's instructional program.

Question #1: What types of problems should be used?

There are many different types of mathematical problems that could be used in a problem-solving program. However, before one begins to select particular types of problems it is imperative that a position be established regarding the nature of a mathematical problem and problem solving. In this paper, a mathematical problem is considered to be a mathematical situation in which an individual or a group is called upon to perform a task for which that individual or group has no readily accessible procedure for determining a solution. Problem solving, as used in this paper, refers to the coordination of previous experience, knowledge, and intuition in an effort to determine an outcome to a situation for which a procedure for determining a solution is not known (see Lester, 1978).

Two types of mathematical problems were selected for this instructional system: process problems and translation problems. Below are examples of each:

Examples - Process Problems

The tennis club was planning a tournament for its club with 8 members. Each member was to play every other member. How many matches need to be scheduled?

It takes 1,140 pieces of type to number the pages of a book. Each piece of type is used only once. How many pages are in the book?

Examples - Translation Problems

Fourteen bears each ate 3.4 kg of meat. After all the bears had finished eating, 7.4 kg were left over. How much meat was there in all?

A record costs $5.98. How much do four records plus a sales tax of 3% total?

Process problems emphasize a three-step process: (a) understanding the problem, (b) developing and carrying out a solution strategy, and (c) evaluating the solution. Translation problems, frequently called "textbook word problems," emphasize translating from a real world situation to a mathematical sentence.

Some have argued that translation problems do not belong in a "good" problem-solving program. However, others have argued that translation problems may serve a purpose when they are used in particular ways (e.g., see Charles, 1981). Most textbook word problems are matched to the concepts and skills involved in their solution. That is, if a textbook problem is located at the end of a lesson or chapter, the concepts and skills involved in the solution process are generally developed in the same lesson or chapter. When textbook problems are presented in <u>matched</u> situations, there is little difficulty identifying the concepts embodied by the story situations.

Typical textbook problems are included in the system described in this paper. However, in this system these problems are only used in <u>non-matched</u> situations, meaning that the concepts and skills involved in the solution process are ones the students have not worked with for at least two weeks. Experience with non-matched problems may promote the types of behaviors identified in the definition of problem solving given earlier. Furthermore, experiences with non-matched problems may also promote greater understanding of the concepts embodied by story situations.

Question #2: What should be the minimum time allotment for problem solving?

Regardless of the amount of time one has available for mathematics instruction, a commitment must be made establishing problem solving as a significant part of the curriculum. Table 1 suggests possible <u>minimum</u> time guidelines for a junior high school program. These guidelines were selected based on a 60 minutes per day allotment for mathematics and they suggest that, as a minimum, approximately one-third of one's time teaching mathematics should be devoted to problem-solving experiences. These guidelines also suggest that experiences with process problems should dominate a problem-solving program. Furthermore, these guidelines are minimum and extensions of the time devoted to problem solving should be given to process problems, not translation problems. Finally, experience using these guidelines shows that allotting one-third of one's program to problem solving is not excessive. In fact it is quite possible that considerably more time could be given to problem-solving activities, particularly as one gets further along in the school year, without detrimental effects on the quantity and quality of other instruction.

Table 1

Possible Minimum Time Allotments for Problem Solving

Period	Content	Frequency	Length
Sept. (1 week)	translation problems	4 days/wk.	5-10 min./day

Sept. (3 wks.)	process problems	1 day/wk.	30-40 min./day
	translation problems	3 days/wk.	5-10 min./day
Oct.-June (32 wks.)	process problems	2 days/wk.	30-40 min./day
	translation problems	2 days/wk.	5-10 min./day

Question #3: What grouping patterns should be used for problem-solving activities?

An instructional program for mathematical problem solving should include individual, small-group, and whole-class experiences. Each of these grouping patterns emphasizes particular problem-solving behaviors not emphasized by the others. For example, one of the behaviors required of a student in a small group situation that is not involved in individual work is the need to comprehend, evaluate, and act upon ideas and questions raised by others. The process of dealing with the ideas and questions of other students, influences and may facilitate one's own thinking and progress toward the solution of a problem.

Following are four guidelines for selecting grouping patterns.

- use small groups (3-4) for most in-class work with process problems
- use individual work for most in-class work with translation problems
- encourage individual work with process problems via homework.

  Some teachers have found it useful to give a process problem for homework on Thursday and discuss the students' work on that problem the following Tuesday. Furthermore, the homework problem is frequently an extension of the problem attempted on Thursday or a problem whose solution involves strategies similar to those used in the problem attempted on Thursday.

- include whole-class activities as part of your teaching strategy.

Question #4: What material is needed to teach problem solving?

A collection of "good" mathematical problems is a necessary ingredient for a problem-solving program. Of course, it is not an easy task to identify "good" mathematical problems. Although experience using a problem may be the best judge of quality, there are some characteristics one should attempt to manifest in the set of problems used for instruction. For process problems there are at least four desirable characteristics. They should:

1. interest students; problems may or may not be from the real world,

2. involve relatively little formal math, that is, the mathematical content needed to solve the problem should be familiar to students,

3. not be able to be solved solely by using a computational algorithm (at least not one known to the students), and

4. be able to be solved using more than one strategy.

For both translation and process problems, characteristics like the following should be considered when organizing sets of problems.

1. content. The problem set should reflect a variety of mathematical content (e.g., geometric as well as numeric).

2. logical structure. The logical structure of problems should be varied. Logical structure refers to factors such as extraneous or insufficient data, the number of conditions in the problem, and the number of steps to solution.

3. problem setting. Problems should be presented in a variety of settings. Two factors that should be considered are real world versus "pure" mathematical settings and the existence or nonexistence of pictures accompanying problem statements.

4. reading. Reading-related factors should be varied in the problem set. Two factors to be considered are the amount of reading in problem statements and the existence of special words and symbols (e.g., "two" versus 2).

In addition to good sets of problems, the teaching strategy one selects for problem solving can establish a need for particular instructional materials. For the teaching strategy described in this paper, the problem-solving bulletin board shown in Figure 1 is required. The ways in which this bulletin board are used for instruction are explained later.

Problem-Solving Strategies

Helping Strategies	General Strategies
1. Read the problem again.	1. Look for a pattern, generalize.

2. Look for key phrases.	2. Guess and check.
3. Write what you know.	3. Work backwards.
4. Make an organized list, table, or chart.	4. Write an equation.
5. Use a picture, objects, or graph.	5. Use deduction.
6. Experiment or act-out the problem.	
7. Use simpler numbers.	
8. Solve a simpler problem.	

Figure 1: Problem-solving strategies bulletin board.

Question #5: How should students be evaluated?

This question is the most difficult one to answer. One reason it is difficult is that the realities of the classroom and the goals for teaching problem solving are not always compatible. For most students, all of their experiences in mathematics have at some time been assessed through some form of a written test. In turn, scores on a collection of written tests are transformed into a final mark for the mathematics class. Thus, for many students the importance of an activity is determined by the contribution of that activity to one's mark in mathematics.

There are two essential goals for teaching mathematical problem solving:

1. to improve a student's willingness to attempt to solve mathematical problems and to persevere in those attempts when success is not immediate, and

2. to develop a student's ability to select and utilize problem-solving strategies.

Both of these goals are not presently and perhaps may never be subject to assessment through traditional written test formats. Thus, a conflict exists between the goals of a problem-solving program and the assessment expectations of many students.

The most desirable way to deal with this conflict is to change pupils' assessment expectations. Although this is difficult the results are worth the effort. One way pupils' assessment expectations may be changed is through the use of a teaching strategy that emphasizes the goals given above. A teaching strategy that promotes the attainment of these goals is one that enables each student to find some success in most problem-solving experiences. For many students, the enjoyment provided by successful problem-solving experiences is sufficient to allay achievement expectations. The teaching strategy developed in the next section of this paper has that potential.

Another technique useful for dealing with assessment expectations is to implement some type of an accountability system. For example, teachers who assign a process problem for homework frequently verify efforts to solve the problem by collecting students' "work." The assessment in this case focuses on whether a student did or did not show evidence of attempting to solve the problem. The use of an accountability system enables one to satisfy the assessment expectations of students while focusing on the attainment of the goals for teaching problem solving.

Finally, it is important to note that the nonexistence of written assessment instruments related to the goals for teaching problem solving does not mean that one should not attempt to assess these goals. Rather, some form of a subjective but systematic process for recording observations of students' growth should be established. Individual student interviews and analyses of written work can be combined to help assess progress toward the goals for teaching problem solving.

Teaching Strategy

Fundamental to the development of a teaching strategy are the goals one has for teaching mathematical problem solving. Two essential goals for a problem-solving program were identified above. The teaching strategy described in this paper was designed to promote the attainment of these goals.

There are two related parts to the teaching strategy described here: the classroom climate and teaching actions. The classroom climate component identifies behaviors a teacher should model to develop a classroom atmosphere conducive to mathematical problem solving. The teaching actions component identifies some specific behaviors to use to help develop a student's abilities to select and utilize problem-solving strategies.

The Classroom Climate

It is absolutely essential that the classroom atmosphere be conducive to mathematical problem solving. In fact, experience suggests that the classroom climate is so important in the development of a successful problem-solving program that establishing a conducive atmosphere for problem solving should be the most important goal of all problem-solving experiences at the beginning of the school year. This is particularly true if students have not had prior experience in a mathematical problem-solving program.

Most students will have had some experience with translation problems by the time they are in junior high school. Unfortunately, by this time many students have developed a strong dislike for typical textbook problems. One reason students develop a dislike for these problems may be the result of having too few experiences with them throughout the elementary school program. Many teachers have found that providing junior high school students with frequent non-threatening experiences solving non-matched problems is a useful strategy for changing attitudes toward textbook word problems.

Process problems dominate in the instructional system developed in this paper. Process problems are quite unlike translation problems and most students have not had any exposure to process problems. For these reasons, the remainder of this section will focus on ways to develop a classroom atmosphere conducive to work with process problems. However, most of the ideas are also applicable to work with non-matched translation problems.

One must anticipate and develop ways for dealing with two probable consequences when students are introduced to process problems. First, many students will be reluctant to pursue their ideas when they are not confident these ideas will lead to a correct solution. Students sometimes reveal this situation through a comment like "I don't know what to do" or by simply not doing anything. The second consequence is that many students may not be able to obtain a correct solution or any solution to particular problems. Students frequently reveal this situation by not wanting to share their solution attempts with others, including the teacher.

Both of these consequences need not be detrimental to a student's work with process problems. There are at least two behaviors a teacher can use to deal with these consequences.

1. Encourage students to explore <u>any</u> ideas (i.e., strategies) that may help them understand and/or solve a mathematical problem and do not censor ideas generated by students.

2. Recognize and reinforce different kinds of excellence.

Regrettably, most students "new" to a problem-solving program believe there is one and only one way to solve every mathematics problem. Students must realize this is not true and process problems are an excellent vehicle for doing this. In those initial experiences with process problems, when students seemingly don't know where to start, discussions with students and questions should be used to elicit any ideas that might be explored. One idea from a group is not enough. Continued discussions and questions should be used to illustrate that different ideas are acceptable and desirable in problem-solving situations. There are, of course, situations in which students really don't know how to start work on a problem. The teaching actions discussed shortly provide one way to deal with this.

Concomitant to encouraging and eliciting ideas from students is avoiding censorship of students' ideas. When working with process problems, it is quite likely that students will generate ideas that, with great certainty, will not lead to a correct solution. At these times, it is very important that students not be stopped from pursuing their ideas. There are two reasons why censorship should be avoided. First, it frequently happens that ideas that appear "unproductive" to others may indeed be productive to the user. Another and perhaps the most important reason is that a classroom atmosphere that is conducive to problem solving is one in which students are keenly aware of the freedom as well as the desirability of exploring any strategies for understanding and/or solving mathematical problems.

Most students "new" to a problem-solving program also believe <u>the</u> goal of all problem-solving experiences is to obtain a correct solution. Because of this, students are frequently frustrated when they do not obtain a solution to a process problem. For <u>all</u> experiences with process problems and particularly for those initial experiences, the emphasis of the problem-solving activities should be on behaviors other than obtaining a correct solution. The goals for teaching problem solving suggest at least three behaviors that should be continually recognized and reinforced in problem-solving situations: (a) a student's willingness to start work on a problem, (b) a student's perseverance in attempting to solve a problem, and (c) the selecting of a strategy for solving a problem, regardless of whether that strategy did or did not lead to a correct solution.

Teaching Actions

The teaching actions selected for problem solving must be consistent with one's view of how problem solving is learned (see Bourne, Ekstrand, and Dominowski (1971). The teaching actions described here are based on an information-processing point of view. In this view of problem solving, the task of the problem solver is to select from a variety of "alternatives" those that will move him/her toward a solution. The "alternatives" confronting a problem solver lie in areas such as the different problem-solving strategies one can use (e.g., drawing a picture, working backwards, etc.), the various intuitions one generates in the process of solving a problem, and the variety of previous experiences one brings to a problem-solving situation. The primary goal of the teaching actions described here is to develop the ability to search among and evaluate alternatives when solving mathematical problems. Concomitant to this goal are the needs to make students aware of strategies useful in solving mathematical problems and to develop students' abilities to utilize these strategies.

The teaching actions developed here are for work with process problems, not translation problems. Although some of the ideas in this section may be appropriate for translation problems, experience suggests that at the junior high school level average and high-achieving students need little "teaching" to develop their abilities related to translation problems.

Learning how to solve mathematical problems is quite different from other types of learning one encounters in the study of mathematics. For example, in concept learning there is a particular kind of subject matter one is concerned with, namely, concepts. Similarly, in skill learning, a mathematical skill is the object of instruction. Problem solving, on the other hand, is not concerned with a particular kind of subject matter but instead is concerned with a process. This difference between problem solving and other types of learning in mathematics has implications for the development and evaluation of effective teaching actions.

One implication of this difference is that teaching actions for problem solving should change over time, whereas teaching actions for particular kinds of subject matter should remain fixed. For example, if a teacher develops 20 concepts over some period of time, the teaching actions used to develop the first concept should be similar to the teaching actions used to develop the last concept. On the other hand, if the teaching actions used for initial problem-solving experiences are successful at developing students' abilities related to the problem-solving process, then the teaching actions should change as students' abilities related to that process improve. In the discussion that follows, particular attention is given to the ways in which teaching actions should change as students' abilities develop.

It was suggested earlier that small groups of 3 to 4 students be used for most in-class experiences with process problems. The time allotted for work with process problems (at least 30-40 minutes per session) can be divided into three sections according to the work in small groups. The three sections are simply BEFORE students form their groups and start work on a problem, DURING the time students are in small groups working on a problem, and AFTER students have completed work on a problem (for whatever reason) and have returned to a whole-class structure. In each of these time divisions there are particular teaching actions one should use. Table 2 shows the teaching actions one should use in the "middle" of a classes' development of their problem-solving abilities. The teaching actions shown in this table are discussed first, followed by a discussion of ways in which these actions should be modified for students' initial experiences with process problems and ways in which these actions should change as students' abilities to work with process problems continue to develop.

Table 2

Teaching Actions for Process Problems

BEFORE

1. Read the problem to the class or have a student read the problem.

2. Ask if there are words or phrases they do not understand; provide explanations as needed. (Note: Be careful that one's explanations do not suggest a solution strategy.)

DURING

1. Question students and observe their work to identify where the students are in the problem-solving process.

   a. They are trying to understand the problem.
   b. They are developing or carrying out a solution strategy.
   c. They have obtained an answer.

2. If necessary, refer students to the problem-solving strategies bulletin board and encourage them to select and implement a strategy or strategies.

3. If necessary, provide hints and questions.

4. For early finishers, give an extension to the problem or have the students make up an extension to the problem.

AFTER

1. Show and discuss strategies used by the class which did and did not lead to a correct solution.

2. Name the strategies used by the students and draw their attention to the names of the strategies on the problem-solving strategies bulletin board.

3. If possible, relate the problem to previous problems and discuss possible extensions of the problem.

4. Evaluate the strategies used by the class.

5. If appropriate, discuss special features of the problem.

BEFORE. The two teaching actions at this stage should be used to illustrate the importance of carefully reading mathematical problems and of focusing on the meanings of words and phrases that may have special interpretations in mathematics. Also, at this stage, it is very important that the problem statement be visible to every student. Preferably, all students would have a copy of the problem so they can write anything on the statement which may help them understand and solve the problem.

DURING. The first three teaching actions shown in Table 2 are the most critical at this stage. Teaching action 4 is a classroom-management strategy for meeting the needs of high-achieving students. However, in the AFTER segment of the teaching actions, "problem extensions" play an important role for all students.

There are two reasons why one should identify a group's "location" in the problem-solving process. First, categorizing a group with respect to the problem-solving process enables one to diagnose a group's strengths and weaknesses related to problem solving. For example, one group of students may frequently have difficulty understanding process problems while another may frequently have difficulty generating ideas for solution strategies. Explicitly identifying a group's location in the problem-solving process enables one to provide appropriate instructional emphasis throughout the problem-solving program. The second reason it is important to categorize groups with respect to the problem-solving process is to facilitate the implementation of teaching action 3. This teaching action is discussed shortly.

The most critical moment in teaching problem solving is the time when students indicate to the teacher that they are "stuck," that is, that they have come to a blockage in their solution of the problem and they don't know what to do or try next. Teaching action 2 suggests that the first time a group encounters a blockage, they should immediately be referred to the problem-solving strategies bulletin board and encouraged to select and implement some strategy or strategies for solving the problem. Although the implementation of this teaching action is quite easy, its importance should not be underestimated. There are at least two reasons why the use of the bulletin board is important.

1. The use of the bulletin board forces a group to self-select a problem-solving strategy without relying on teacher direction. This action is consistent with the goal of developing students' abilities to search and evaluate alternatives in the problem-solving process.

2. The bulletin board provides a "crutch" needed by most students. Figure 1 shows there are at least 13 strategies from which one must select a strategy or strategies for each problem situation. Experience shows it is unrealistic to expect students to keep all of these strategies in memory unless they have had considerable experience using them.

The use of the problem-solving strategies bulletin board is often sufficient to enable a group to continue work on a problem. However, at those times when the use of the problem-solving strategies bulletin board is unproductive, teaching action 3 should be used. The purpose for using hints and questions is to facilitate, not remove, students' decision-making. To do this, however, hints and questions must be very carefully selected so as not to identify completely the direction to solution. Also, it is important to realize that hints and questions, even carefully selected ones, will usually be received differently by different students. For some students, hints and questions will be no help in moving them toward a solution. For other students, hints and questions are confusing and may even suggest to the students that their approach will be unproductive if it appears different than the approach suggested by the hint. And, of course, for many students hints and questions are indeed useful. Often, hints and questions enable students to pursue a direction previously considered unproductive, to identify the inappropriateness of a particular approach, or to identify an idea to pursue when one was not apparent.

The hints and questions one wishes to use for a particular problem should be identified prior to the problem-solving session. When hints and questions are written in advance, the teacher is forced to "think through" a problem before, rather than with, the students. As a result, the teacher is better able to identify where students are in the process of solving the problem and is better prepared to select appropriate hints and questions during the problem-solving session. The hints and questions one prepares should be categorized according to the three steps in the problem-solving process: (a) understanding the problem; (b) developing and carrying out a solution strategy; and (c) evaluating the solution.

Although hints and questions should be categorized according to the problem-solving process, the particular hint or question one provides at a given moment may not be from the category at which the students are currently working. For example, students may have reached a blockage in the process of carrying out a solution strategy and to get them past that blockage the teacher may ask a question related to understanding the problem. The teacher must also be prepared for the situation in which his pre-selected hints and questions are not appropriate, and another comment is needed to help the group along.

Finally, it is important to realize that just as the three steps in the problem-solving process are not necessarily sequential and disjoint, teaching actions 1 through 3 at this stage are also not necessarily sequential and disjoint. Frequently, these teaching actions may be used in a "cyclic" fashion and may be repeated several times in solving one problem.

AFTER. Near the end of the time for small group work, at least two students, each from a different group, should be asked to place their solution efforts on the chalkboard. If possible, one of the solution efforts should provide a correct solution and one an incorrect or no solution at all. After the students have put their work on the board, a whole-class discussion should be used for the teaching actions at this stage.

Teaching actions 1 and 2 are straightforward. Their purpose is to focus on the selection and implementation of problem-solving strategies. Teaching action 3 is intended to demonstrate to students that problem-solving strategies are not problem specific and to help students recognize different kinds of situations in which particular strategies may be useful. Although it is acceptable and desirable that different solution strategies be sought for every problem, it is important that the different approaches that led to a solution be evaluated (see teaching action 4). The criteria for evaluating solution strategies are generally dependent on the problem being solved. For example, if pattern finding is the general strategy needed for solving a problem, the approaches used should be evaluated with respect to the degree to which each facilitates identifying a generalization of the pattern.

Finally at this stage, any special features of the problem just attempted should be discussed with the class. For example, some problem statements include a picture or a diagram. In these instances, the way(s) in which the picture or diagram influenced the students' ideas should be discussed.

Adjust instruction for initial experiences. The extent to which the teaching actions in Table 2 need to be modified for experiences with process problems at the beginning of the year depends on the amount of previous experience students have had with process problems. For the guidelines below the assumption is made that students have not had any prior experience with process problems.

Before - Teaching actions 1 and 2 in Table 2 should be used at this time. Also, an additional teaching action one should use is a whole-class discussion concerning (a) understanding the problem and (b) developing and carrying out a solution strategy. The hints and questions one prepares for a problem can be used as a basis for this discussion. For these initial experiences, the first stage of the problem-solving process should be completed as a whole-class activity. In other words, students should understand what they are being asked to find in a problem before they begin work in their small groups. For the second stage of the problem-solving process, developing and carrying out a solution strategy, the class should discuss but not implement possible solution strategies. The use of a whole-class discussion for these purposes facilitates success on the students' initial experiences with process problems.

During - There are four key points at this stage.

(a) The primary goal should be to establish a classroom climate that is conducive to mathematical problem solving.

(b) Most students will not have prior experience in small-group problem-solving situations. Therefore, there is a tendency for students not to share their ideas with others and for students to work individually on their initial attempts. In these initial experiences, attention must be given to establishing a group problem-solving effort. This can be facilitated by telling a group from the start that it is fine to use more than one approach; however, everyone in the group must understand what approaches are being tried. Related to this is the fact that within a group there are usually one or two students who "see the solution" before others. To promote small-group problem solving, one should insist that everyone in the group be able to explain how a solution was obtained. This not only promotes small-group problem solving but also serves as an instructional technique with respect to the selection and implementation of problem-solving strategies.

(c) The hints and questions used during initial experiences with process problems should be more directive than those used later. That is, hints and questions should, with great certainty, enable a group to continue work on a problem.

(d) The problem-solving strategies bulletin board should not be on the wall when school starts. Rather, the problems one uses at the beginning of the program should be selected so they elicit the strategies one wants to include on the bulletin board. As each problem is solved, the strategies used in the solution should be named and the name of the strategy should be added to the bulletin board. This approach enables one to "build-up" the strategies bulletin board over time. Obviously, teaching action 2 in Table 2 cannot be used for the first process problem attempted. After that, this teaching action should play a role in the problem-solving session.

After - The concept of a "problem-solving strategy" will be new to most students. As a result, direct instruction from the teacher concerning the implementation of particular strategies is needed at the beginning of the program. For example, the first time students are exposed to "using a table," they will no doubt need help in constructing a table in a way that facilitates solving the problem. Therefore, in those initial experiences considerable time must be given to teaching actions 1 and 2 in Table 2. The only other modification in Table 2 is that teaching action 3 at this stage will not play a large role since the students have not had prior experience with process problems.

Adjusting instruction for later experiences. There are some changes that should occur in the teaching actions at this stage and, in fact, naturally occur quite often as students' problem-solving abilities develop.

Before - Both of the teaching actions shown in Table 2 may be phased out. The more experience students have with process problems the more facile they become at reading mathematical problems.

During - One modification of the actions at this stage is that the hints and questions one uses should be more general than those used earlier; that is, they should be less directive with respect to productive solution strategies. Another change in the teaching actions at this stage is related to the bulletin board. One goal of the teaching actions in Table 2 is that students will eventually commit to memory all of the strategies on the bulletin board. It may be possible at some point in the program to "tear down" the problem-solving strategies bulletin board. The first time students reach a blockage in solving a problem they should still be encouraged to select and implement a strategy. However, now the "crutch" they had been using has been removed.

After - There are two modifications in the teaching actions at this stage. First, students should assume more responsibility for naming and explaining strategies used in problem solving. Also, relating a problem to similar ones and searching for extensions of a problem should play an important role in the problem-solving session.

Modifying the Instructional System for Low-Achieving Classes

The instructional system described above was designed for and used with average and high-achieving students at the junior high school level. However, with some modification the system developed here may also be appropriate for low-achieving classes.

In the organizing-for-instruction component there are two changes that seem important.

1. Problem-solving activities should be included in the program. Some examples of problem-solving activities are: (a) identifying extra information in problem statements; (b) writing a question for a story; and (c) telling how to solve a problem that does not involve numbers.

2. Problem solving should occur daily. Work with process problems may be delayed until after students have had considerable experience with problem-solving activities.

In the teaching actions component there are seven changes that seem important.

Before

1. Teaching actions 1 and 2 in Table 2 may be needed for a longer period of time and perhaps always.

2. Whole-class discussions with respect to understanding problems and developing and carrying out solution strategies should be continued for a longer period of time.

During

3. References to the problem-solving strategies bulletin board should be somewhat specific (e.g., "Try one of these two helping strategies to get you started.").

4. Hints and questions should be more directive.

5. The problem-solving strategies bulletin board should remain on the wall for the majority of the year.

6. Considerable attention must be given to establishing a classroom atmosphere conducive to mathematical problem solving.

After

7. Do not ask students to put their solution efforts on the chalkboard initially. Copy students' work on the board for them until they develop a willingness to do so themselves.

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Diagnosing Reading Difficulties in Verbal Problem Solving

by

Jeffrey C. Barnett
Fort Hays State University

Solving word problems has traditionally been one of the most difficult activities in the mathematics curriculum at all grade levels. This is not surprising, since word problems often require the higher levels of reasoning of analysis and synthesis. Students must not only know how to compute and work mathematical algorithms, but most importantly, they must know which algorithms to use and when.

In recent years, many models have been developed to help explain the problem-solving process. Although these models vary considerably in the number and type of stages, they all include reading and language processing in the early phases of the problem-solving process. There is, however, some disagreement as to the relative importance of reading ability in solving word problems in mathematics across age and ability levels. It seems safe to assume that the ability to read and interpret word problems with facility is a necessary, but not a sufficient condition for problem-solving success. Unfortunately, many students with poor language processing skills never really get into the analysis phase of solving word problems. Unfamiliar vocabulary and difficult syntax can distract the students' attention from the problem's structure, often resulting in confusion, frustration, and lack of confidence.

Fortunately, there is evidence that teachers can help their students become better problem solvers by devoting special attention to reading problems and language skills as they relate to mathematics. Perhaps a starting point in the development of reading instruction in mathematics is to convince both teachers and students that reading mathematical word problems and textual material is very different than reading regular English prose. These differences require the adjustment of reading habits and study skills.

In the following discussion, we shall consider a number of specific problems in the reading of mathematical word problems. For convenience, our discussion will be divided into four major areas of concern: semantics, syntax, context, and interpretation skills. Semantics refers to the <u>meanings</u> of words and phrases. Syntax refers to the <u>arrangements</u> of words and phrases, the <u>form</u> of words and symbols, and the <u>grammatical</u> structure of problem statements. Although the term "context" has several meanings, we shall restrict our attention to the setting of the problem, along the dimensions real-imaginative, concrete-abstract, and factual-hypothetical. Interpretation skills refers to the ability to understand graphs, charts, tables, etc. Within each of these areas, several problems or sources of difficulty will be identified, followed by suggestions for remediation and some follow-up activities.

Category I: Problems with Semantics

A. Problem: New vocabulary terms which a student may encounter in reading mathematical material and word problems may have no relevance to the student's everyday vocabulary. In this situation, these new words are often memorized without understanding, making recall in the appropriate problem-solving situations difficult. For example, words such as addend, secant, denominator, and hypotenuse have little use outside the context of mathematics.

Remediation: The teacher can help students understand and remember new vocabulary by writing each new word on the chalkboard or overhead, defining it, and providing examples. One effective technique is to provide several examples of the new term as well as several non-examples. Students can then be asked to try to develop their own definition, based on similarities and differences between the examples and non-examples. Students can also be asked to generate their own word problems that use the new vocabulary. These can be displayed on cards or on a bulletin board along with definitions and pictorial examples. One possible idea for a bulletin board is shown in Figure 1.

With some new vocabulary terms, teachers can use structural linguistics to help students learn and remember definitions. The meanings of prefixes, suffixes, and certain root words may already be familiar to students. Terms such as polygon, quadrilateral, isoceles, acute, obtuse, and pentagon lend themselves to structural analysis. For example, the term "obtuse" is related to the word "obese" which means fat or overweight. An "obtuse" angle is a "fat" angle; i.e., an angle whose measure is more than 90 degrees. Similarly, the term "acute" means sharp, as in an "acute pain." Therefore, an "acute" angle is a "sharp" angle or one whose measure is less than 90 degrees. With the help of a dictionary and thesaurus, many other aids to the memorization and comprehension of the terms can be found.

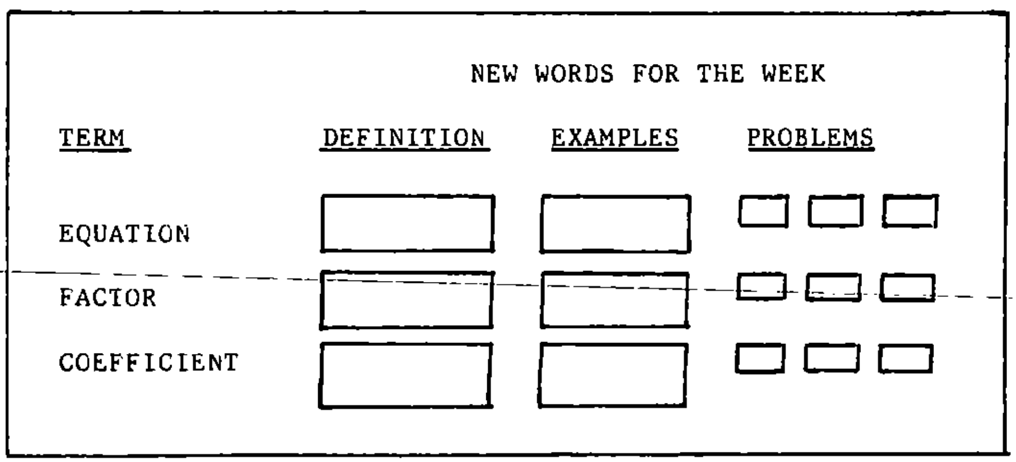


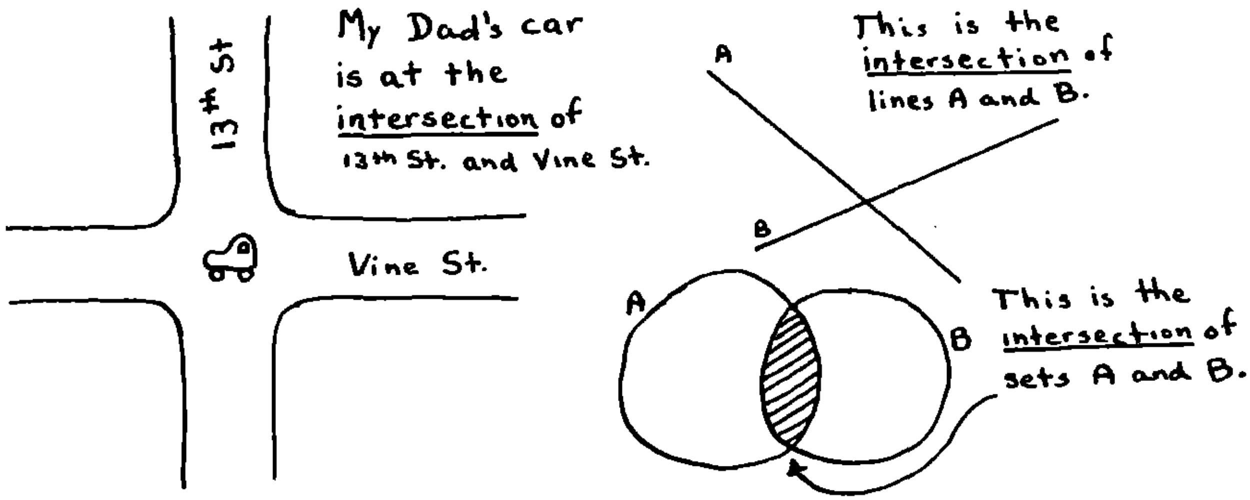
Figure 1

To ensure that new vocabulary has been learned meaningfully, continuous evaluation should be planned on quizzes, unit tests and on a day-to-day basis during class discussions. Brief, dittoed tutorials on the correct use of new vocabulary terms can be constructed by teachers to supplement classwork and the text, for those students who need additional help.

B. Problem: Some words have similar meanings in English prose and mathematical material, but students fail to perceive these similarities when the words are used in a mathematical context. Terms such as improper, union, disjoint, intersection, commutative, associative, and distributive often cause students difficulty in mathematics, even when they are familiar with these terms as part of their everyday vocabulary.

Remediation: Students will need some help from the teacher in interpreting familiar terms in a mathematical context. For example, the intersection of two streets is a familiar concept to most students. A child standing at the intersection of two streets is on both streets at the same time. Similarly, the intersection of two lines or two sets is the set of points that are in both sets or on both lines at the same time. The term "associate" means to group or to be grouped with, as in "She is associated with Girl Scouts." In the "associative" property of addition, the parentheses group or associate numbers to be added.

The parallel meanings of words such as those above should be discussed in detail when introduced for the first time. Students can be asked to write sentences which illustrate the use of the terms in both mathematical and non-mathematical contexts. As an assignment, students can draw pictures that illustrate the similarities in the use of the terms.



Since marked differences may exist between a child's familiarity with one word and a different form of the same word, special attention should be devoted to entire word families. For example, Kane, Byrne, and Hater (1974, pp. 75-90) found that 76.6 percent of seventh- and eighth-grade children were familiar with the term associative but that only 39.1 percent were familiar with the term associativity. Having children read word problems out loud is a useful technique for determining which words and phrases are causing the most difficulty.

C. Problem: Some vocabulary terms in mathematics have different meanings outside the context of mathematics.

Remediation: Special attention should be devoted to words that have the same spellings and pronunciations but different meanings in regular English prose. For example, words such as base, mean, root, times, prime, round, and right (and there are many others!) often cause confusion. In addition to pointing out these differences, teachers could have their classes make a list of words which have these different meanings and use them to make a bulletin board.

D. Problem: The teacher may not realize that some vocabulary terms are not familiar to all of the students, particularly in the beginning of the school year.

Remediation: A pretest of mathematical vocabulary can be given at the start of the year or unit of instruction, to determine which students are not familiar with the required terms. If the class members are significantly divided on their knowledge of background mathematical vocabulary, the class can be divided into groups for specific instruction on the meaning and use of the required terms.

E. Problem: Students may be unable to identify "key words" in the problem statement that provide clues as to which operation can be used to arrive at a solution.

Remediation: The problem of being able to determine how to "set up" a problem, that is, to determine which operations to use for a solution, is probably cited more often than any other difficulty that students have when solving word problems. The ability to recognize "key words" and to use them as clues to a problem's underlying structure is not easy to cultivate, but can be developed with practice over a period of time. One productive method for teaching the relationship between "key words" and problem structure is to have students underline the words they consider to be clues to the operations required. For example, in the problem below, the word "and" indicates the operation of addition, the word "of" indicates multiplication, and the word "left" indicates subtraction:

> David earned $2.50 for cutting grass after school on Friday, and $2.30 for weeding the garden on Saturday. He gave 1/3 of his earnings to his friend for helping. How much did he have left?

$$1/3 \underset{}{\overset{\text{of}}{\times}} (\$2.50 \overset{\text{and}}{+} \$2.30) = 1/3 \times \$4.80 = \$1.60$$

$$\$4.80 \overset{\text{left}}{-} \$1.60 = \$3.20$$

During the course of the year, the class can compile a list of "key words" that are associated with the four basic arithmetic operations and display them on a bulletin board along with several examples. As an activity, students could write problems using key words from the class list. Young children could be instructed to write simple problems based on given number facts in addition, subtraction, multiplication, and division. By reversing this procedure and using a calculator for the actual computations, students can be given extensive practice in translating "key words" in problem statements to mathematical operations.

Caldwell (1980) suggests the following set of problems as a basis for a class discussion on how the modification of a few "key words" can alter a problem's meaning and solution:

- John <u>has</u> five dollars. He <u>earns</u> three dollars. How much does he <u>have</u> now?

- John <u>has</u> five dollars. He <u>saves</u> three dollars. How much does he <u>have</u> now?

- John <u>has</u> five dollars. He <u>spends</u> three dollars. How much does he <u>have</u> now?

- John <u>needs</u> five dollars. He <u>has</u> three dollars. How much does he <u>need</u> now?

As a follow-up activity, Caldwell suggests writing problems with the key words deleted. Students can then experiment with varying the key words to produce problems with very different solutions. For example:

- Find the number that is the _____ of 8 and 4.

- Oscar _____ 15 cookies. He _____ three cookies. How many _____ cookies does he have ?

<u>F. Problem:</u> Students may not realize that the same term may not always indicate the same operation.

<u>Remediation:</u> Although some words such as sum, total, difference, decrease, and, left, more, and less are often indicators of required operations, students should be cautioned to look carefully at the context, for numerous exceptions do exist. For example, in the first of the three problems that follow, the word "a" does not imply addition. In the second, the word "left" does not imply subtraction. The word "sum" in the third problem does indicate addition, but the operation has already been performed and therefore is not required in the solution process.

- What is the product of 8 <u>and</u> 4?

- Jane ran 6 blocks north and turned <u>left</u>. She then walked 7 blocks west. How far was she from her original starting place?

- Bobbi found that the <u>sum</u> of her 4 quiz scores was 56. What was the average score?

Perhaps the best method is to have children look for potential key words as they read through the problem. During a second reading they can use contextual clues to help determine which of the identified words are actually operational or procedural indicators. Systematic exercises of this type can help them focus on semantic hints to discover which operations or procedures are required for a solution. As a reminder, the class could compile an additional list of potentially misleading key words with an asterisk.

Category II: Problems with Syntax

A. Problem: Position of the question sentence and sequence of important data in the problem statement may cause difficulty in determining the problem's mathematical structure.

Remediation: Children need extensive practice with sets of word problems that have similar mathematical structures, but vary considerably in their wording. Teachers can provide practice sets where the position of the question sentence is varied systematically. For example:

- In how many hours can Joe and Bill paint a garage working together, if Joe can do the entire job alone in 15 hours, and Bill can do the entire job alone in 11 hours?

- Joe can paint a garage in 15 hours. In how many hours can Joe and Bill paint the garage working together, if Bill can paint it alone in 11 hours?

- Joe can paint a garage in 15 hours. Bill could paint it in 11 hours. How long would it take them to paint the garage if they worked together?

Research indicates that children tend to have more difficulty with problems when the data are presented in an order that is different from that needed to solve the problem. Students can be asked to generate sets of word problems that use the same data and require the same answer, but which vary the order of the data in the problem statements. For example, a student might come up with the following set:

- A grocer bought 17 dozen pears for $14.65. If 5 dozen spoiled, at what price per dozen must he sell the remaining pears to make a profit equal to 3/5 of the total cost?

- A grocer wished to make a profit of 3/5 of the total cost of his fruit. If 17 dozen pears cost him $14.65, and 5 dozen spoiled, for how much per dozen must he sell the remaining pears to realize the desired profit?

- In a crate containing 17 dozen pears, a grocer finds that 5 dozen have spoiled. How much per dozen should he charge for the remaining pears to make a profit of 3/5 of his total cost, if the crate costs $14.65?

As an additional activity, students can be asked to construct several more problems which use the same data, but which ask different questions. One example is the following:

- In a crate that contains 17 dozen pears, a grocer finds that 5 dozen have spoiled. He sells the remaining pears for $23.44, which will give him a profit of 3/5 of the original cost of the crate. How much per dozen did the pears in the original crate cost the grocer?

By pooling the contributed sets of problems from the members of the class, the teacher can construct activity sheets which require students to identify which problems are syntax variations of each other.

B. Problem: Students may have difficulty interpreting signs, symbols and special mathematical notation.

Remediation: When reading word problems, students should be encouraged to use the words that provide the meaning of symbols and notation, rather than the symbols and notation themselves. Word problems which use English words instead of numerals, such as "two hundred twenty-five" instead of "225", can be rewritten using the numerals instead of their English counterparts. Teachers should frequently test knowledge of symbols and notation on tests, quizzes, and during class discussions.

C. Problem: Students may center their attention on the numbers present in a word problem too early, and perform random computations without thinking through which operations are required.

Remediation: To help students avoid premature centering on numbers, the teacher could have them rewrite the problem without numerals. For example:

With numbers: A jogger runs along the edge of a field going north for 12 minutes at 10 km/hr, and then runs east for 1/2 hour at 14 km/hr. If she runs in a straight line back to where she started, at what speed must she travel to reach home in 28 minutes?

Without numbers: A jogger runs north at one rate, and then east at another rate. If she runs in a straight line back to where she started, at what rate must she travel to reach home in a given time?

This second version is one of several ways of rewriting the problem without the numerals. The numberless version could clarify the overall situation and help students see this as a rate problem involving the Pythagorean theorem.

D. Problem: Word problems which contain many pronouns can cause confusion.

Remediation: During a second reading of a problem statement, students should be encouraged to substitute nouns for pronouns if they find the action of the problem confusing.

- Henry's guinea pig has a baby which ~~he~~ Henry named Sam. ~~He~~ Sam weighed .2 kg at birth. Henry observed that ~~he~~ Sam gained .2 kg every five weeks. At that rate, how many kilograms will ~~he~~ Sam weigh after six months?

E. Problem: Students may have difficulty in distinguishing between relevant and irrelevant facts in a problem statement. Problem statements which are particularly long often cause difficulty as well, in that students can get overwhelmed with surplus information.

Remediation: Students should be encouraged to focus their attention on the action of the problem, centering on important verbs. Listing facts in their proper relationships (particularly in the form of short, mathematical sentences which can later be combined into equations) is a very useful activity. Students can be given sets of word problems and asked to try to identify all extraneous information. In the following example, the student crossed out the excess information, in an effort to simplify the problem.

- ~~Bob's uncle~~ Bill uses five ~~3 oz.~~ bottles of concentrate to make 8, ~~1/2 litre~~ bottles of root beer. How many 1/2 litre bottles of root beer can he make with twelve ~~3 oz.~~ bottles of concentrate?

Having the student read the above problem statement out loud, without the crossed-out words, can make the problem's structure more apparent. Note that the students may have to reread the problem statement several times to be able to decide which words are really not needed.

Students should also be given practice in constructing their own problems which have the same mathematical structure, but contain different amounts of extraneous information, have different sentence structures, or vary in length.

Category III: Problems with Context

A. Problem: Students may have difficulty perceiving similarities in the mathematical structure of word problems which have different contextual embodiments.

Remediation: Students need to be shown how problem settings can be modified without changing the mathematical structure of the problems. Modifications in context can be across the dimensions concrete- abstract, factual-hypothetical, or real-imaginary. The following four problems (Caldwell and Goldin, 1979) illustrate variations of context using combinations of the concrete-abstract and factual-hypothetical dimensions.

- There is a certain given number. Three more than twice this given number is equal to 15. What is the value of the given number? (Abstract-factual)

- There is a certain number. If this number were 4 more than twice as large, it would be equal to 18. What is the number? (Abstract-hypothetical)

- Susan has some dolls. Jane has 5 more than twice as many, so she has 17 dolls. How many dolls does Susan have? (Concrete-factual)

- Susan has some dolls. If she had 4 more than twice as many, she would have 14 dolls. How many does Susan really have? (Concrete-hypothetical)

Problems can be modified along the real-imaginary dimension by having students construct problem sets with given themes or subjects, such as their income and expenditures for a week (real) or problems involving fanciful characters such as dragons.

Caldwell (1980) has suggested that students could be asked to fill in missing words in problem statements, so as to change the context. For example:

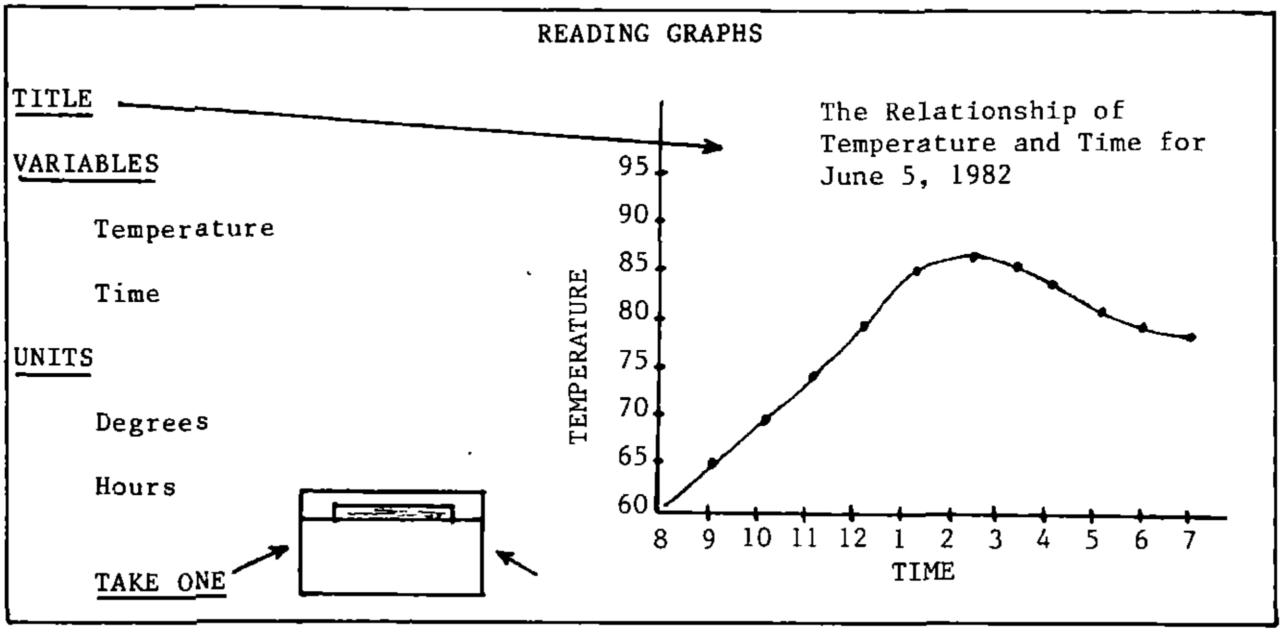
- Judy has 27 ____. She ___ 5 and then ____ 14. How many _____ does she have in all?

Activity sheets which require students to identify problems in different contexts with similar mathematical structures could provide a useful follow-up activity.

Category IV: Problems with Interpretation

A. Problem: Students may not understand processes used to read and interpret a graph.

Remediation: Students should be exposed to word problems which make use of a graph to summarize information. Conversely, students should be given practice problems which require a graph as part of the solution, or for which a graph may help conceptualize a solution. When a problem uses a graph, the student should read the title of the graph to determine the kind of information that the graph provides. The variables described on the axis or parts of the graph should be listed, with a brief description of how they are related. In some graphs, the units of measure may be spaced on different scales, so this should be checked carefully. How to read a graph is an excellent topic for a bulletin board. The diagram below makes use of a pocket of problems which make use of graphs. The teacher can change the problems weekly, or use student-generated problems.



B. Problem: Students may not know how to read tables.

Remediation: Although calculators have replaced the use of tables in many cases, there are still times when the ability to read required data from tables is essential. Students should be exposed to many types of tables and given practice finding required information. As an activity, students can be asked to write a description of how to use the table, as if their description was to be read by a younger student. The description should include information about the headings and entries, and at least one example. A bulletin board can be constructed with student-generated problems based on data available from tables collected from newspapers, magazines and old texts.

C. Problem: Students may not know where to find additional information to help them solve difficult problems.

Remediation: Students should be thoroughly familiar with the location and use of the various parts of their text, such as the index, appendix, and glossary (to look up definitions they may have forgotten). Reference texts, additional tables, study guides, etc. should also be made available.

Summary

As the above list of reading problems indicates, many students are poor problem solvers in mathematics due to the lack of language processing skills. When reading deficiencies are discovered, teachers may take steps toward remediation. However, the best plan is systematically to provide reading instruction in mathematics throughout a child's academic career. The activities suggested above are just some of the many ways that teachers can help students learn the importance of reading word problems slowly and carefully, with an attitude of aggressiveness and attention to detail. Once these adjustments in reading rate and purpose are made, students should be able to approach word problems in mathematics with increased confidence and ability.

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Leading Problem Solving in an Elementary School Classroom

by

Kil S. Lee
University of New Orleans

In order to develop a successful problem-solving program, identifying appropriate problems for students at different ability levels is important. If a question is too easy, it is not a problem for a student and if the question is too difficult, a student will usually not even attempt to solve it. A problem can be solved only after a problem solver is willing and able to mobilize his or her resources to solve the problem.

Identifying appropriate methods of solving problems for students at different ability levels, rather than different problems, is often a challenge for a teacher. For example, the following problem could be posed for an algebra student: The sum of Susan's age and Kenneth's age is 21 and Susan is five years older than Kenneth. How old is Susan and how old is Kenneth?

Algebra students would most likely solve this problem using simultaneous equations:

$$x + y = 21 \qquad\qquad \begin{array}{rr} & x + y = 21 \\ -) & x - y = 5 \\ \hline & 2y = 16 \\ & y = 8 \end{array}$$

$$x - y = 5$$

$$x - 8 = 5$$

$$x = 13$$

Susan is 13 years old and Kenneth is eight years old. But this problem can be taught to students who are at the concrete operational stage and without the methods of algebra. A way of helping elementary school children solve the problem will be presented in the following paragraphs.

Elementary school children experience anxious feelings when they are faced with some word problems. In order to reduce their anxiety, they often immediately write down numbers from the statement of the problem and perform an operation between the numbers "21 + 5 = 26, 26 years old"; "21 - 5 = 16 years old." One student wrote

$$\begin{array}{r} 21 \\ \underline{5} \\ \end{array}$$

and asked her teacher, "What do I do? Do I add?"

Thus it is important in problem solving to have some rules or methods which can help one work confidently toward a solution. In How to Solve It (1957), Polya described certain rules and methods (called heuristics) which are helpful in solving problems. But Polya's list of heuristics is not in an appropriate form for elementary school children because of the number, the language, and the complexity of the heuristics. The following list is adapted for elementary school children from Polya's list of heuristics.

Heuristics for Elementary School Children

I. Understanding the problem
   (A) What is involved in the problem?
   (B) What are the relationships among the involved items?
   (C) What are the questions to be answered?

II. Making a plan
   (A) Can drawing a picture help?
   (B) Can making a chart help to solve it?
   (C) Consider special cases and look for a pattern.
   (D) Consider one condition and then add another condition.
   (E) Have you solved a similar problem?

III. Carrying out the plan
   (A) Carry out the plan
   (B) Check each step

IV. Looking back
   (A) Is your answer reasonable?
   (B) Try to find another way to solve it.
   (C) Make a similar problem.

In the early stages of teaching problem solving, teacher demonstration of problem solving by conscientious use of heuristics is essential. "At first, we must understand the problem: What are we talking about?...The problem involves the ages of two persons, Susan and Kenneth. What are the relationships between the ages of Susan and Kenneth?... Susan is five years older than Kenneth and the sum of the ages of Susan and Kenneth is 21 years. Can you give an example where Susan is five years older than Kenneth?" If there are correct responses, it is fine but if there aren't any responses, the teacher can give an example: If Susan is six years old, how old is Kenneth? At this point, some students will say, "Kenneth must be one year old." The teacher should elicit more examples from the students. This is simple enough that everyone can give an example except perhaps those who seldom pay attention to the class. At this point, it is not wise to ask questions of the students who most likely cannot answer correctly. Rather, the teacher would ask for examples from the students who are capable of giving correct examples and would sit near the student who does not pay attention so that that student would feel that he might be the next one to be called upon. The teacher should give enough opportunities to other able students until the uninterested student could give an example. Then the teacher could ask: "You know that the problem says that Susan is five years older than Kenneth. Now, if Susan is six years old, how old should Kenneth be?" When the uninterested student answers correctly, the teacher should give a positive response which would encourage the student to be a part of the problem-solving activity.

Similarly, discuss the second condition of the problem: The sum of Susan's age and Kenneth's age is 21. This discussion will provide an opportunity for all the students to understand the problem clearly including the previously uninterested students. The students will now be ready to answer questions, regardless of whether they are sure they are correct.

"Now, what are we looking for?...*Susan's age and Kenneth's age*." One student remarked, "Gee, it is a hard problem. If the problem just said that Susan is five years older than Kenneth then it is easy." This remark shows that the student comprehends the problem now. The teacher can point out that "it is difficult because we must consider two conditions at the same time. So let's consider only one condition for awhile and then add the other condition later." We could start with the condition that Susan is five years older than Kenneth. The teacher then elicits examples and records them on the board.

Susan's age	Kenneth's age
6	1
5	0
10	5
18	13
12	7
11	6
13	8
15	10
17	12
.	.
.	.
.	.

Usually, the "good students" will begin to give examples, but the condition is simple enough and has been discussed in understanding the problem, so virtually everyone in the class will give examples. Hence the blackboard may be covered with examples for the one condition. Then the second condition should be considered: the sum of Susan's age and Kenneth's age is 21. The class begins to add Susan's age and Kenneth's age for each example on the board. Eventually, they will find the correct solution.

Susan's age	Kenneth's age	Sum
6	1	7
5	0	5
7	2	9
10	5	15
19	14	33
12	7	19
11	6	17
13	8	21
15	10	25
17	12	29
.	.	.
.	.	.
.	.	.

None of the students will be embarrassed because his or her predicted example is incorrect because there are so many examples on the board that no one would have noticed whose example is correct and whose example is incorrect.

After finding the solution, the teacher should point out the importance of understanding the problem which not only involves reading the statement carefully but also thinking about examples for each condition. The teacher can point out that "if you were solving this problem by yourself, it would be helpful if you would arrange your examples with one condition in an orderly way." For examples, you could write

Susan	Kenneth
.	.
.	.
.	.
9	4
10	5
11	6
12	7
13	8
14	9
15	10
16	11
.	.
.	.
.	.

and check a few of the sums of the ages

Susan	Kenneth	Sum
.	.	
.	.	
.	.	
9	4	13
10	5	15
11	6	

Susan	Kenneth	Sum
12	7	
13	8	21
14	9	23
15	10	
16	11	27
.	.	
.	.	
.	.	

Then you could find a way in which your examples would give you the correct answer without checking all the situations. However, in the author's experiences, children would often find the sums for all the cases even after they found the answer. The teacher can also present alternate ways of solving the problem so that students will recognize that there are many other ways of solving problems and they will modify learned methods to ways that are most comfortable for them. An example of a way some students solved the problem is as follows:

$$\begin{array}{r}10\\+5\\\hline 15\end{array}\times \quad \begin{array}{r}9\\+4\\\hline 13\end{array}\times \quad \begin{array}{r}12\\+7\\\hline 19\end{array}\times \quad \begin{array}{r}13\\+8\\\hline 21\end{array}\checkmark$$

These children recognized that the way to find the correct answer is by first checking two cases and then using the sums to lead to the solution.

In order to develop a successful problem-solving program, a teacher's first task is to identify problems which are not too easy and not too difficult, problems the students will be able to solve only after they mobilize their resources. Therefore the teacher must consider the students' mathematical background as well as their cognitive development levels. Second, a teacher must demonstrate, by placing himself or herself in the student's place, how conscientious use of heuristics can help to solve problems. Third, a teacher must provide many opportunities for students to engage in problem-solving activities so that each student will experience success in solving problems after hard work. The excitement of successful problem solving may be imprinted on some of the students' minds and it may help to develop a character of curiosity and inquisitiveness for their lifetime. These characteristics are common to many great thinkers.

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Problem Solving for the High School Mathematics Student

by

Cheryl Kantecki and Lee E. Yunker
West Chicago Community High School

An effective method for capturing the interest of students is to involve them in problem solving. This requires reaching beyond the realm of the textbook to expose students to a variety of problem-solving situations. Aside from capturing student interest, problem solving provides opportunities to develop mathematical skills, leads to new mathematical ideas, and motivates students to research mathematics. More importantly, problem solving stimulates imagination and allows students to exercise creativity.

The National Council of Supervisors of Mathematics Position Paper on *Basic Skills defines problem solving as the process of applying previously* acquired knowledge to new and unfamiliar situations. The word "process" implies that there are many facets to problem solving, and indeed there are. Problem solving entails more than merely arriving at a conclusion; problem solving entails the entire process of analyzing a problem, synthesizing, and evaluating.

The intent of this article is threefold. First, to identify problem-solving strategies and illustrate their uses with specific examples. Second, to suggest motivational techniques to involve students in problem solving. And, third, to provide the reader with a list of interesting and challenging problems along with their answers.

Major Problem-Solving Strategies

There is a wide variety of problem-solving strategies that could be mentioned as being important. This article will confine itself to six strategies that have wider application and can be easily employed by high school students. The six strategies are: elimination, modeling, reducing to a simpler case, using tables, guess and check, and patterns.

Elimination:

The elimination strategy is basically one of looking at all the possible solutions and eliminating, one by one, those that are not possible. Logic problems provide examples for the use of the elimination method.

The WHO'S WHO Problem:

Four married couples belong to a golf club. The wives' names are Kay, Sally, Joan, and Ann. Their husbands are Don, Bill, Gene, and Fred. Examine the following clues. They should help you decide who is married to whom.

- Bill is Joan's brother.
- Joan and Fred were once engaged, but "broke up" when Fred met his present wife.
- Ann has two brothers, but her husband is an only child.
- Kay is married to Gene.

Solution:

A chart like the one below lists the possible solutions. The clues help eliminate, one by one, the false solutions.

	KAY	SALLY	JOAN	ANN
DON	X	X	YES	X
BILL	X	YES	X	X
GENE	YES	X	X	X
FRED	X	X	X	YES

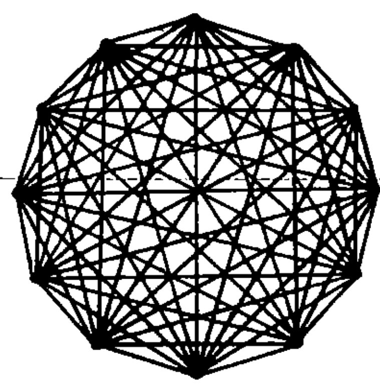
Modeling:

The modeling strategy is one of creating a model of the problem to be solved. This model may be an actual physical model or just a diagram on paper. In any event, the model helps the student see his way through to the solution.

THE HANDSHAKE Problem:

There are 12 people at a party. If everyone shakes hands with everyone else at the party, how many hand shakes take place?

Solution:



The hand shakes can be represented by the sides and diagonals of the dodecagon.

Number of Sides =	12	n
Number of Diagonals =	54	$\frac{n(n-3)}{2}$

Number of Hand Shakes = 66

Reducing To A Simpler Case:

This strategy is employed when the problem appears to be too large to comprehend or when trying a few of the cases may hold a hint to the larger solution.

THE LOCKER Problem:

This problem is about a high school and that favorite storage area, the high school locker.

At Gauss High there are 1000 students and 1000 lockers (numbered 1-1000). At the beginning of our story all the lockers are closed. The first student comes by and opens every locker. Following the first student, the second student goes along and closes every second locker. The third student changes the state (if the locker is open, he closes it; if the locker is closed, he opens it) of every third locker. The fourth student changes the state of every fourth locker, etc. Finally, the thousandth student changes the state of the thousandth locker. Which lockers will remain open after the thousandth student changes the state of the thousandth locker? [From: Columbus Project ESEA, Columbus, Montana.]

Solution:

For our purpose we shall investigate the state of the first sixteen lockers as students 1 through 16, open or close them. In the table below, let C and O represent closed and open lockers respectively. Student #1 begins by opening every locker. Thus in Row 1 of the table an "O" is placed beneath each locker number. Student #2 then closes every second locker. Thus in Row 2, "C" is placed beneath lockers 2, 4, 6, 8, ...16. Next, student #3 changes the state of every third locker; he closes locker number 3, opens number 6, closes number 9, opens number 12 and closes number 15. Hence in Row 3, "C" is placed beneath locker 3, 9, and 15 and "O" beneath 6 and 12. This process continues until student #16 changes the state of locker 16. Finally, from the table it is evident that lockers 1, 4, 9 and 16 are left open. Each of these locker numbers are perfect squares. Hence the solution: the lockers whose numbers are perfect squares are open.

Locker #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Student #																
1	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O
2		C		C		C		C		C		C		C		C

Student # \ Locker #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3			C			O			C			O			C	
4				O				O				C				O
5					C					O					O	
6						C						O				
7							C							O		
8								C								C
9									O							
10										C						
11											C					
12												C				
13													C			
14														C		
15															C	
16																O

Using Tables:

Tables are useful devices for organizing and keeping track of information.

THE HONEST BROTHERS Problem:

One of five brothers had broken a window. John said, "It was Henry or Thomas." Henry said, "Neither Earnest nor I did it." Thomas said, "You are both lying." David said, "No, one of them is speaking the truth, but not the other." Earnest said, "No, David, that is not true." Three of the brothers always tell the truth, but the other two cannot be relied on. Who broke the window?

Solution:

In the table, the headings at the top indicate the assumed guilt of each brother. The headings to the left indicate the truth or falseness of each statment. T and F represent true and false respectively. For instance, if John were guilty (Column 1) John's *statement* would be false, Henry's true, Thomas' false, David's true and Ernest's false. Columns 2 through 5 are completed in the same way. Upon completion, the only column which indicates that three brothers were telling the truth is column 3. Therefore, Thomas broke the window.

	John	Henry	Thomas	Ernest	David
John	F	T	T	F	F
Henry	T	F	T	F	T
Thomas	F	F	F	T	F
David	T	T	F	F	T
Ernest	F	F	T	T	F

Guess and Check:

Guess and check is a problem-solving strategy in which the problem solver actually guesses a solution and then checks to see if the solution satisfies the conditions of the problem. The experienced problem solver uses all of the information at hand to arrive at a reasonable solution.

THE ORDERED DIGITS Problem:

0	1	2	3	4	5	6	7	8	9

In the ten cells above inscribe a ten digit number such that the digit in the first cell indicates the total number of zeros in the entire number; the digit in the cell marked "1" indicates the total number of 1's in the number, and so on to the last cell, whose digit indicates the total number of 9's in the number. Zero is a digit, of course. The answer is unique.

Solution:

In the list below, an initial guess is made. Then a series of gradual changes are made until the correct solution is obtained.

	0	1	2	3	4	5	6	7	8	9
	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	1
	8	0	0	0	0	0	0	0	1	0
	8	1	0	0	0	0	0	0	1	0
	8	2	1	0	0	0	0	0	1	0
Answer	6	2	1	0	0	0	1	0	0	0

Pattern:

Searching for patterns can be an extremely helpful problem-solving strategy. Amazingly, problems which appear to be relatively difficult can in fact be quite simple when patterns are recognized.

THE CHINESE DINNER Problem:

Every 3 guests used a dish of rice between them, every 4 a dish of broth, and every 2 a dish of meat. There were 65 dishes in all. Can you figure how many guests there were?

Solution:

In order to "meat" the conditions of the problem the number of guests must be divisible by 12. We create the following table in which R, B, M represent the total number of required dishes of rice, broth, and meat, respectively.

# of Guests	R	B	M	Total # of Dishes	
12	4	3	6	13	increase of 13
24	8	6	12	26	

Since each increase of 12 guests will increase the total number of dishes served by 13 and there were 65 dishes served, there must have been 60 guests.

Motivations for Problem Solving

Often students need to be motivated to extend themselves in doing problems which are not going to be covered on their next test even though the techniques learned may be helpful in solving those test questions. The following suggestions have proved helpful in getting students involved in learning and using these problem-solving strategies.

The Weekly Challenge Problem. A problem may be presented on Monday with the teacher accepting solutions until Friday. The teacher keeps a weekly record of each student's correct solutions on a large chart on the bulletin board. Each week that a correct solution is turned in, the student gets a star behind his name. This chart is always there for all students to see and this chart soon motivates competition among the students in the class, each trying to outdo the other.

The Weekly Extra-Credit Problem. Each week a problem may be presented to the students for extra credit (something many students ask for). Then the teacher can use the extra-credit points at the end of each grading period to adjust the student's grade in whatever way he deems appropriate.

As a result of these efforts, the parents even get involved. Students will say, "Can I have an extra copy for my father or mother?" Or, the parent will see you and say, "Where do you get those problems?" This then gives you a great opportunity to explain to parents about the new emphasis in our profession on teaching problem solving.

Problems to Motivate and Challenge

The following problems are ones that we have used with our students. These are problems that we feel offer students good problem-solving situations in which they can exercise creative thinking and on which they can use one or more of the strategies mentioned above. The original sources for these problems are unknown to us but we thank their authors, whoever they are, for offering such fine problems for us to share with others.

Problem-Solving Exercises:

1. Miss Young has her 18 students seated in a circle. They are evenly spaced and numbered in order. Which student is directly opposite... a. student number 1? b. student number 5? c. student number 18?

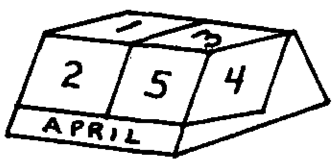
2. Mr. Evans seated his students in the same way as Miss Young's. Student number 5 is directly opposite number 26. How many students are in Mr. Evan's class?

3. Mrs. White teaches Phys. Ed. She had her students space themselves evenly around a circle and then count off. Student number 16 is directly opposite number 47. How many students are in Mrs. White's class?

4. EXTENSION: A huge number of boys are standing in a circle and are evenly spaced. The 7th boy is directly opposite the 791st. How many boys are there altogether?

5. Steve, Jim, and Calvin are married to Beth, Donna, and Jane, not necessarily in that order. Four of them are playing bridge. Steve's wife and Donna's husband are partners. Jane's husband and Beth are partners also. No married couples are partners. Jim does not play bridge. Who is married to whom?

6.

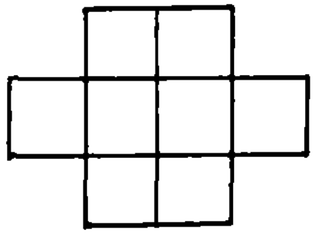


On the desk calendar above, the day can be indicated by arranging the two cubes so that their front faces give the date. The face of each cube has a single digit, 0 through 9. If the cubes can be arranged so that their front faces indicate a date 01, 02, 03, ... 31, find the four digits that cannot be seen on the left cube and the three on the right cube.

7. Find the smallest number which divided by each of the integers 2, 3, 4, 5, 6, 7, 8, 9, and 10, will give, in each case, a remainder which is 1 less than the divisor.

8. Divide a circle into four equal areas using three fences of equal length. Do not use your fences around the perimeter or on top of each other.

9. Fill in the following figure with the digits 1-8 in such a way that no two consecutive numbers are in boxes which touch at a point or side.



10. An exam has five true-false questions.
    a. There are more true than false answers.
    b. No three consecutive questions have the same answer.
    c. The students know the correct answer to problem number 2.
    d. Questions number 1 and number 5 have opposite answers. From the information above, the student was able to determine all the correct answers. What are they?

11. A man went into a hardware store and asked the clerk how much 1 cost. The clerk said 25 cents. He asked how much 10 would cost, and the clerk said 50 cents. "Good," he replied, "I'll take 1025." He then paid the clerk $1.00. What did he buy?

12. If it cost a nickel each time you cut and weld a link, what is the minimum cost to make a chain out of 5 links?

13. A goat is tied at the corner of a 20m x 40m barn with a 50m rope. If it can graze at any spot outside of the barn to which its rope can reach, what is the size of its grazing area?

14. Jim has a collection of records. When he puts them in piles of two, he has one left over. He also has 1 left over when he puts them in piles of 3 or piles of 4. He has none left over when he puts them in piles of 7. What is the least number of records he may have?

15. Golden Chain Problem: A Chinese prince who was forced to flee his kingdom by his traitorous brother sought refuge in the hut of a poor man. The prince had no money, but he did have a very valuable golden chain with seven links. The poor man agreed to hide the prince, but because he was poor and because he risked considerable danger should the prince be found, he asked that the prince pay him one link of the gold chain for each day of hiding. Since the prince might have to flee at any time, he did not want to give the poor man the entire chain, and since it was so valuable, he did not want to open more links than absolutely necessary. What is the smallest number of links that the prince must open in order to be certain that the poor man has one link on the first day, two links on the second day, etc.?

16. A man's age at death was 1/29th of the year of his birth. He was alive in 1900. How old was he in 1900.

17. Slow Horse Race: Two knights seek the hand of Princess Priscilla in marriage. Each boasts that he owns the fastest horse in all the land. So the king arranges a horse race. The king, however, is not eager to have his little girl marry, and he is especially unimpressed with her two suitors, so he decrees that the winner of the race, who will receive the princess's hand, will be the knight whose horse crosses the finish line last. It would seem that the race would never get under way; neither horseman would want to ride out ahead of the other. But Princess Priscilla, eager for marriage, thinks of a way to outwit her over-protective father. She whispers instructions to the two knights that ensure that the race will be run and that it will be fair. What did she tell the two knights?

18. Nines: Using only six nines, write a number that equals 100.

19. It is traditional in many families at Christmas time for each family member to give a gift to each of the other members. How many gifts would be given if there were 10 family members? How about for your family which has ___ members?

20. Sally has some change in her purse. She has no silver dollars. She cannot make change for a nickel, a dime, a quarter, a half dollar, or a dollar. What is the greatest amount of money she can have?

21. If a clock strikes six times in five seconds, how many times will it strike in ten seconds.

22. How much will it cost to cut a log into eight equal segments, if cutting it into four equal segments costs 60 cents?

23. Mervin was Calvin's best friend and the executor of Calvin's will when he passed away. Mervin rode his horse over to Calvin's ranch to settle the estate. Calvin had 17 horses that were to be divided among his family in the following way. Calvin's wife was to receive 1/2 the estate, his son 1/3, and his daughter 1/9. This posed a problem for Mervin. He did not want to kill any of the horses and yet he must divide the estate according to the will. How did he accomplish this task?

24. Herb the Hobo was attempting to cross a railroad bridge. When he was 3/7 of the way across he heard a train coming behind him. He ran to the far end and hopped off just as the train got to him. Later he calculated that he could have run to the other end of the bridge and still have survived. If the train was going 35 kph, how fast did Herb run?

25. In a survey of 25 college students at the University of Calgary, it was found that of the 3 newspapers, Calgary Herald, Calgary Sun, and the Globe and Mail, 12 read the Herald, 11 read the Sun, 10 read the Globe and Mail, 4 read the Herald and the Sun, 3 read the Herald and the Globe and Mail, 3 read the Sun and the Globe and Mail, and 1 person reads all 3.

   a. How many read none of the newspapers?
   b. How many read the Calgary Herald alone?

c. How many read the Calgary Sun alone?
d. How many read the Globe and Mail alone?
e. How many read neither the Calgary Herald nor the Calgary Sun?
f. How many read the Calgary Herald or the Calgary Sun or both?

26. A well is 10 feet deep. A frog climbs up 5 feet during the day but falls back 4 feet during the night. Assuming that the frog starts at the bottom of the well, on which day does he get to the top?

27. Gina and Tom raise cats and birds. They counted all the heads and got 10. They counted all the feet and got 34. How many birds and cats do they have?

28. The security guard at a bank caught a bank robber. The robber, the teller, and a witness were arguing when the police arrived. This was what the police learned in the confusion.
a. The names of the 3 men were Brown, Jones, and Smith.
b. Brown was the oldest of the three.
c. The teller and Jones had been friends for many years.
d. Brown was the brother-in-law of the witness.
e. Smith graduated from high school 5 years earlier than the robber.
Who was the robber? Who was the teller? Who was the witness?

29. How can 12 matches be arranged to make 6 regions of equal area?

30. The Editor of the Harvey School annual, The Harvey Hijinx, knows that 2985 digits were used to print the page numbers of the annual. How many pages were in the book?

31. In the Calgary Herald, the sports writing staff picked the winners for the first weekend of play in the Canadian Football League's 1982 football season. The picks are as follows:

Sportswriter "A"	Sportswriter "B"	Sportswriter "C"
Edmonton	Ottawa	Calgary
Montreal	B.C.	Edmonton
Calgary	Edmonton	Winnipeg
Saskatchewan	Montreal	Ottawa

No one picked Toronto to win. Who plays on the first scheduled weekend?

32. A certain highway was being repaired, so it was necessary for the traffic to use a detour. At a certain time, a car and a truck met in this detour which was so narrow that neither the truck nor the car was able to pass. Now, the car had gone three times as far into the detour route as the truck had gone, but the truck would take three times as long to reach the point where the car was. If both the car and the truck can move backward at one third of their forward speed, which of these two vehicles should back up in order to permit both to travel through the detour in the minimum amount of time?

33. The shuttle service has a train going from Washington to New York City and from New York City to Washington every hour on the hour. The trip from one city to the other takes 4 and 1/2 hours and all trains travel

at the same speed. How many trains will pass you in going from Washington to New York City?

34. What do the following words have in common: Deft, First, Calmness, Canopy, Laughing, Stupid, Crabcake, Hijack.

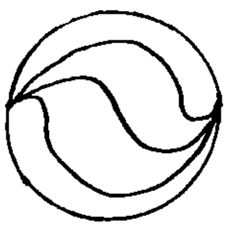
35. Supply a digit for each letter so that the equation is correct. A given letter always represents the same digit.

$$\begin{array}{r} A\ B\ C\ D\ E \\ \underline{\times\qquad\quad 4} \\ E\ D\ C\ B\ A \end{array}$$

36. A man travelled 5000 kilometres in a car with one spare tire. He rotated tires at intervals so that when the trip ended each tire had been used for the same number of kilometres. How many kilometres was each tire used?

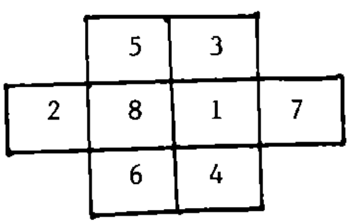
A N S W E R S

1. a. 10 b. 14 c. 9
   (Note: the difference between the numbers of directly opposite persons is always the same.)
2. 16-5=11. Thus 11x2=22, the number of pupils in Mr. Evan's class.
3. 62 pupils.
4. 1568 boys.
5. Steve and Jane; Jim and Beth; Calvin and Donna.
6. Left cube: 0, 7, 8, and 6 or 9. Right cube: 0, 1, 2.
7. 2519
8.



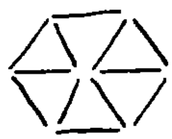
Or answers may vary.

9.



10. T, F, T, T, F (Kay is information in part C).
11. House Numbers.
12. 10 cents.
13. 2115π Sq. m.
14. 49
15. 1 Link (The 3rd link forms either end).
16. 15 or 44
17. Priscilla said, "switch horses."

18. $99 + \frac{99}{99}$
19. 90, second answer varies.
20. $1.19 (4 pennies, 4 dimes, 1 quarter and 1 half-dollar).
21. 11 times.
22. $1.40.
23. Mervin donates his horse to the estate. Then the wife gets 9 horses, the son 6 horses and the daughter 2 horses (9 + 6 + 2 = 17). So Mervin then takes his horse back and all are happy.
24. 5 kph.
25. a. 1 b. 6 c. 5 d. 5 e. 6 f. 19.
26. On day 6.
27. 7 cats, 3 birds.
28. Robber is Jones; Teller is Brown; Witness is Smith.
29.



or

30. 1023 pages.
31. Montreal - Winnipeg, Saskatchewan - Ottawa, Toronto - Edmonton, B.C. - Calgary.
32. The car.
33. 9.
34. Three consecutive letters of the alphabet.
35. A=2; B=1; C=90; D=7; E=8.
36. 4000 kilometres.

Part II

Devising a Plan – Alternatives for Related Problems

Visualization: A Problem-Solving Approach

by

Barbara Moses
Bowling Green State University

The teaching of mathematical problem solving is as complex as the problem-solving process itself. A myriad of questions faces the mathematics educator attempting to improve the problem-solving performance of students of all ages: How do I motivate the topic? How do I organize the material? Are there certain strategies which are of primary importance? Should problem solving be a group activity? What is unique about problem solving that makes it more difficult to teach and learn?

Much research has been completed concerning the analysis of the nature of problem solving. Although researchers adopt their own terminology, there appears to be some consensus concerning the stages in the problem-solving process. Krutetskii (1976) offered the following synopsis of this complex process:

> Apparently, three links or stages can always be traced in the solution of any problem (from the elementary to the very complicated). The solution of any problem seems to begin with the acquisition of initial facts, initial information about the problem, with thorough reflection, attempts to understand, and mastery. Then comes the solution proper, as a stage of processing or transforming the facts acquired for the purpose of obtaining the desired result. And finally, both the process and the result of the solution always leave some trace in the memory, somehow enriching a person's experience. (p. 183)

In other words, the three stages of the problem-solving process involve understanding the problem, planning an attack and carrying out that plan, and finally looking back and assimilating the knowledge gained. Spelling out this process points out the difficulties which arise when trying to teach mathematical problem solving. How do you teach a student how to understand a problem? How do you help a student plan an efficient attack? How do you encourage a student to learn from his mistakes, to generalize the solution, and to commit the solution to memory? Obviously, the task is not easy.

Helping a student understand a problem involves more than helping the student understand the individual words. In one study of fifth grade students' understanding of mathematical problems, the Cloze procedure was used to judge readability of a collection of ten problems. The problems were judged to be quite readable (at the fifth-grade level). Individual interviews were conducted with fifth-grade students, asking them to explain certain words. There were no glaring errors in the comprehension of the words. Yet students still had difficulty understanding the problems. If a student understands each word of a problem, what can the teacher do to help the student understand the problem?

Helping a student plan an attack is no less complicated. One of the unique characteristics of problem solving is that there is no algorithmic, step-by-step procedure for finding the most efficient solution. One fifth-grade student was feverishly working on a problem that had been assigned to the whole class. A hint for attacking the problem was given to the class, and part of the class benefited from the hint. However, this particular student seemed even more confused after having received the hint. "I don't see how that will help me get the answer," he fumed. He ignored his classmates and continued on with his own ideas. Suddenly, five minutes later, he raced up to the front of the room and asked for a point of clarification about the problem. "I got it," he whispered. The initial hint had not helped him because it was leading to a plan of attack that was not consistent with his view of the problem. How does a teacher help 30 students with different perspectives plan efficient strategies?

The final stage, generalizing the solution and assimilating the knowledge, appears to be a very individualized process, yet we expect the teacher to encourage this behavior. What types of actions occur in this third stage of the problem-solving process? How can a teacher help 30 students to assimilate this new knowledge when the previous knowledge structure for each individual student is so remarkably different?

This brief look at the three stages of the problem-solving process points out the fact that problem-solving behavior is very individualized. Each student approaches a given problem situation with his unique background, knowledge structure, inclinations, and cognitive styles. The student then reads the problem and attacks it based on his perspective of the problem. Multi-digit addition exercises, division of fraction exercises, subtraction with renaming exercises, and so on, can all be solved by step-by-step mechanical procedures; problem-solving situations cannot. The uniqueness of problem solving as a mathematical activity is that it is so very dependent on the problem solver. The major similarity between problem solving and other mathematical activities is that continual practice improves performance. Polya (1957) recognized the importance of practice and suggested that the way to improve problem-solving performance was by doing problems. He concluded:

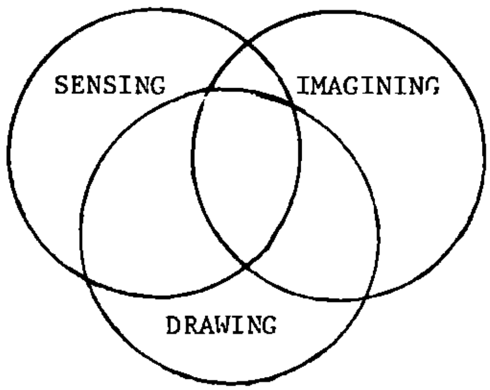
> Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation and practice... Trying to solve problems, you have to observe and to imitate what other people do when solving problems and, finally, *you learn to do problems by doing them*. (p. 4-5)

Working through the three stages of the problem-solving process in problem situation after problem situation does indeed improve problem-solving performance. The more one practises, the more adept one becomes at seeing hidden clues, gaining new perspectives and recognizing useless attempts to solve the problem. But is there some general technique, a way of thinking, that will induce good problem-solving behavior and encourage students not to feel frustrated and give up?

Visualization: A way of thinking

Visualization is a way of thinking and not merely a problem-solving strategy. It can be made to play an important role in each of the three stages of the problem-solving process. It is NOT a panacea for the classroom teacher, but it IS a useful mechanism for students to better cope with difficult problem situations.

Visual thinking is involved in numerous activities, such as when the gardener tries to imagine the garden before it blooms, when the newly-married couple rearranges furniture to make the little apartment appear spacious, when the outfielder in a baseball game knows exactly where to stand to catch the ball, or when the chemist has some insight into the molecular structure of some newly-discovered item. It involves sensing, imagining and drawing:



It involves dreaming, sketching diagrams, sculpting, manipulating concrete objects, and closing one's eyes and mentally manipulating objects. Everyone does visual thinking to some degree; creative problem-solving performance could be improved by encouraging more visual thinking in classroom activities.

In order to achieve the creativity and the flexibility that are required of problem solving, it will be a strong advantage to feel comfortable in a visual, imaginative mode of thinking.

Visualizing in the classroom

Before proceeding to suggestions for visualizing in mathematical problem-solving situations, it should be noted that visualization is a way of thinking and thus should be encouraged in all subject areas. When reading a passage from some source of literature, students should be encouraged to conjure up images of the scene. They should be asked to describe it with words or with pictures or with three-dimensional models. They should be asked to act it out. If the main character were to be placed in a given, new situation, how would she react?

In studying some historical period, students should again be encouraged to let visual imagery dominate their thoughts. How did the people of the period dress? How did they feel about the events of the period? How would these people react if they were living in today's society?

Teachers should create situations and foster imaginal thinking. For example, stimulate thinking by the following situation: Suppose you were walking all over a cube. Describe your feelings in words or in a two-dimensional picture. How would this experience be different from walking on a sphere?

With practice in all phases of the curriculum, visualization can become an integral part of all thinking. Asking "what if..." questions encourages students to become more creative, more flexible, and more aware of different perspectives in a given situation -- all essential characteristics of the problem-solving process.

Visualization in the problem-solving process

The encouragement of visualization skills can aid in all three stages of the problem-solving process. First of all, students can better understand a problem once they get a mental image of the problem situation. It may help if they restate the problem with words of their own choosing. It may help to act out the problem or to draw the situation or to construct some conrete model. In each case, the problem-solvers have translated the problem via some visual vehicle to suit their own perspective.

To organize a plan of attack and carry out that plan one may need to focus in on pictures and diagrams. Seeing is believing, and oftentimes seeing a pictorial representation of the information in a problem helps one to plan an attack and points out previously held misconceptions. Jim, the student mentioned previously who seemed bothered by the hint given in class, described what had happened when he finally came upon a solution. "The numbers just didn't add up," he said. "But then I drew a picture and saw that if zero was a number, I knew how to get the answer. And zero is a number, right?" The picture had helped Jim to see the problem, to understand it, to recognize his misconceptions of it, and to hit upon a solution. Due to his own, incorrect perception of the problem, the given hint did not aid him. He needed to re-structure his thinking, and this was accomplished most efficiently by letting Jim create a visual image of the problem situation. It should be noted that it was Jim's creation and not a representation given by the teacher. Thus, it fit into Jim's cognitive structure quite easily.

Finally, it is probably apparent that visual imagery is very important in the generalization of the problem and the assimilation of the knowledge gained. Having imagined the given situation in one's mind, variations of the situation seem to flow easily. For example, the following problem was posed to a group of in-service teachers:

What is the maximum number of pieces into which a pizza can be cut with five straight cuts?

An obvious extension is to look at the problem for n straight cuts, where n is some integral value. But a more interesting extension was posed by one teacher who suggested looking at a three-dimensional cake, where cuts were permitted in more than one plane. This problem is much more difficult, but there was a strong correlation between individuals who had used visual imagery in the solution of the original problem and those who were successful at the problem extension.

Thinking in this visual mode appears to aid the typical problem solver at all age levels. It is not a strategy as such, but a multi-sensory approach to grasping the problem. By considering a couple of specific examples, a better understanding of this approach may be gained.

<u>Some specific examples</u>

Consider the following problem situation:

A fireman stood on the middle step of a ladder, directing water into a burning building. As the smoke got less, he climbed up three steps and continued his work. The fire got worse so he had to go down five steps. Later, he climbed up the last six steps and was at the top of the ladder. How many steps were there?

Without any clues and without any training in visualization, the typical fifth-grade student will read this problem, understand each word, see three numbers, and proceed to the incorrect solution of adding the three numbers.

Training in the use of visual imagery encourages the student not to jump to the second stage of the problem-solving process prematurely. Understanding the problem is a prerequisite for organizing the plan of attack. In the visual mode, the student will first feel the fire, experience its heat, draw it on paper, act it out. The student will differentiate between climbing up the steps and going down. An entire visual image will be constructed. Perhaps a drawing of the ladder will be exhibited by the students or perhaps the mental image will be sufficient. In either case, the student will realize that climbing up the ladder is a motion in one direction and going down is a motion in the opposite direction. The numbers should not all be positive.

Extensions are up to the individual. What if the ladder were longer? What if there were an even number of steps?

Now consider a second problem:

Sally had a new bike which she takes to school every day. On some days she rides the bike to school and walks home; on the other days, she walks to school and rides the bike home. The round trip takes one hour. If she were to ride the bike both ways, it would only take 1/2 hour. How long would it take if she walked both ways?

The typical solution involved subtracting the two given lengths of time: 1 hour - 1/2 hour = 1/2 hour. The solution is not reasonable, but this would not be noticed unless some visual imagery was evoked.

The student should again be encouraged to think and visualize the problem situation. Imagine riding a bike to school; compare it to walking to school. Which is more enjoyable? Which gets you there quicker? Draw a picture or act out the scene. If it takes 1/2 hour to ride the bike both ways, how long does it take to ride the bike one way? Why?

What other modes of transportation are there? How might you change this problem to include other possibilities?

Conclusion

The teaching of problem solving is a very complex operation. To teach a student three or four strategies, such as finding a pattern, constructing a table, or establishing a subgoal, may prove beneficial once the student has reached the second stage of the problem-solving process.

A more general pedagogical idea is to consider emphasizing a different mode of thinking -- visualization. The effects of this approach are simple, but profound. As Simon (1976) points out:

> An important component of problem-solving skills lies in being able to recognize salient problem features rapidly, and to associate with those features promising solution steps. Much current instruction probably gives inadequate attention to explicit training of these perceptual skills, and the kind of understanding that is associated with them. (p. 21)

Visualization can be promoted in the classroom. Students can enhance their perceptual skills. Problem-solving performance can be improved.

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Graphic Representations of Word Problems

by

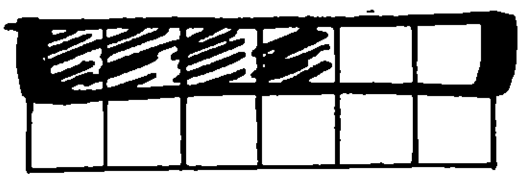
Verena Sharkey
New Castle School District

Problem solving is the ultimate goal for learning to manipulate numbers or deal with the so-called basic facts. Unfortunately, most people are not born with the ability to pull solutions to problems from either their minds or their backgrounds. This being the fact, problem-solving skills must be acquired and it becomes the responsibility of the teacher to help students learn some methods which will be reliable for finding solutions to most types of problems.

Making pictures or graphic representations of problems is one method of instructing students which can be started in the primary grades and carried through secondary school and college. Graph paper is an excellent medium for instructing students in graphic representations of problems. For primary students, large block graph paper should be used. However, for the sake of space in this paper, most examples will use centimetre paper. Rather than use pages of explanation of the method, examples will be shown using a variety of problems in the first through sixth grade level.

Problem 1. Six birds sat on a fence. Four birds flew away. How many were left?

Method. Enclose six blocks on a sheet of graph paper. Darken four blocks and count the remaining blocks.



Write the number sentence $6 - 4 = \square$
$6 - 4 = 2.$

Problem 2. On a trip through the city, Beth counted 25 red cars and Dave counted 18 blue cars. How many more red cars than blue cars were counted?

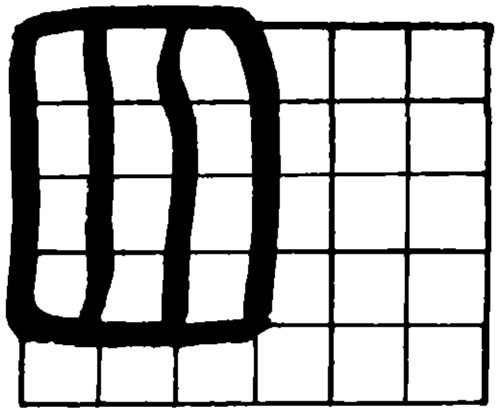
Method. Have students enclose or cut out 25 blocks to represent red cars and 18 to represent blue cars. In this case, cutting would be preferable because the 18 could be placed over the 25 and the remainder counted. Then write the number sentence

$$25 - 18 = \square$$

$$25 - 18 = 7.$$

Problem 3. Jean has 12 apples. She puts 4 apples in each bag. How many bags does she need?

Method. Enclose 12 blocks and then circle sets of 4. Count the sets of four to determine the number of bags needed.



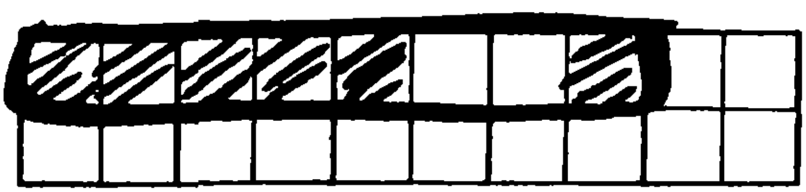
Then write the number sentence:

$$12 \div 4 = \square$$

$$12 \div 4 = 3.$$

Problem 4. James made a pan of fudge. First he put 5/8 cup of sugar in the pan and then added 1/8 cup more. How much sugar did he use in the fudge?

Method. Enclose a strip of 8 blocks. Darken 5 of them to show 5/8, then darken 1 more block to show 1/8 and add the number of darkened blocks.



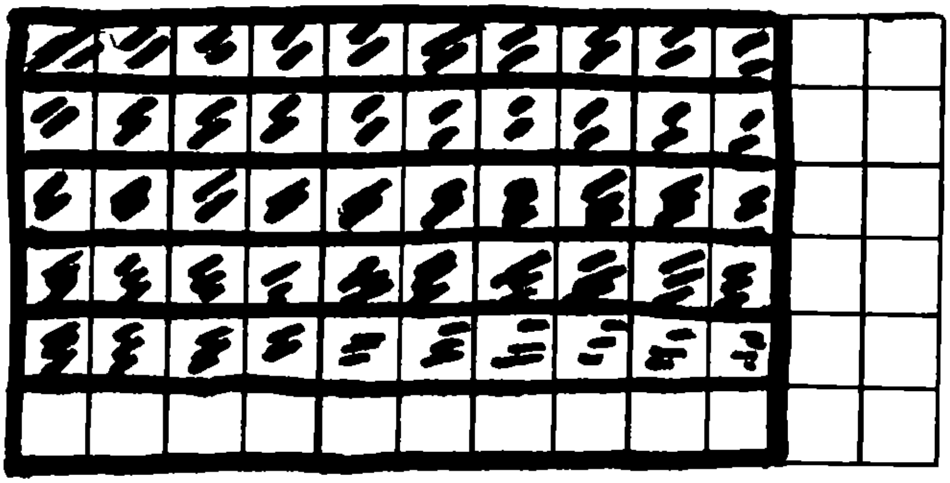
Then write the number sentence:

$$\frac{5}{8} + \frac{1}{8} = \square$$

$$\frac{5}{8} + \frac{1}{8} = \frac{6}{8}$$

Problem 5. There are 60 minutes in an hour. How many minutes are there in 5/6 of an hour?

Method. On graph paper, enclose 60 blocks, then mark off six even sets. Darken five of the six sets and count the blocks.

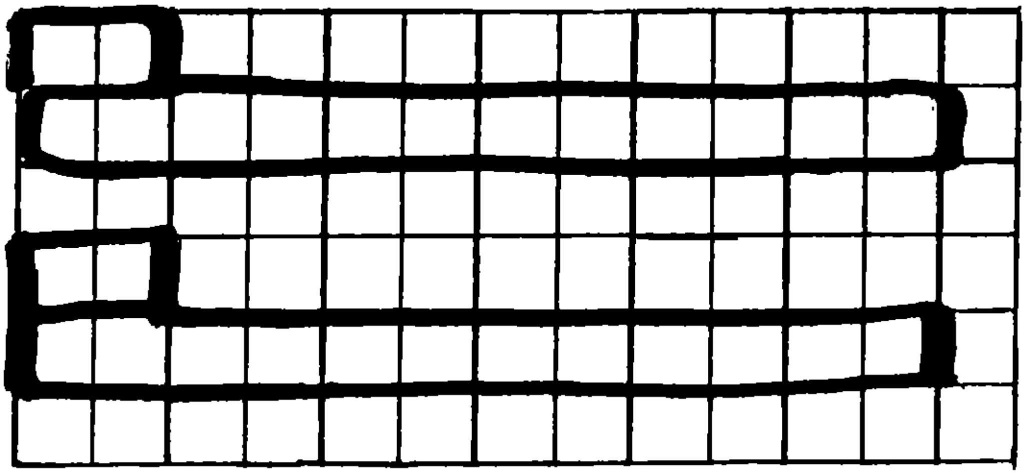


Write the number sentence: $5/6 \times 60 = \square$

$5/6 \times 60/1 = \boxed{50}$.

Problem 6. Ann rode her bike 2 kilometres in 12 minutes. At the same rate, how long will it take her to go 8 kilometres?

Method Shown.



Write number sentence: $\frac{2}{12} = \frac{8}{\square}$

$$\frac{2}{12} = \frac{8}{48}$$

Graphing, of course, is only one skill which is helpful in solving word problems. In developing the skill of graphing, the teacher plays an important role in teaching students how to relate the problem to the graph. The graphing method presents students with a successful experience in problem solving that can be applied throughout the grades.

Teaching Model Problems and the Colour Coding of Problems

by

Bruce Hedderick
John Ware Junior High School

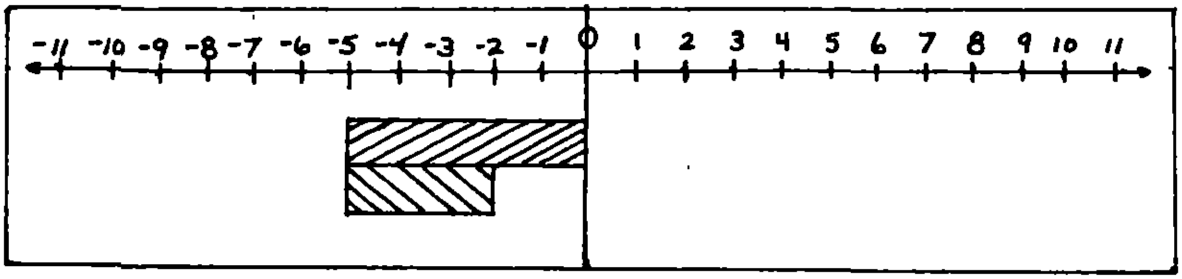
Model teaching of problems can be thought of as the excellent teaching of problems. It is hoped that my operational definition of "model problems" will encompass this idea. A model problem is a broad outline of a concept, using a particular problem, so that students may use the idea of this concept for similar questions and problems. Yaroshchuk (1969) suggests that,

> For a pupil to acquire a precise concept of a particular type of arithmetic problem, this type of problem should have a definite name. It is necessary that pupils learn to clearly isolate the mathematical structure of a model problem. In teaching model problems it is necessary to propose both numerical and subject problems, and compare them with each other. Only after they have acquired the ability to see this structure in the condition of both numerical and subject problems is it desirable to communicate the name of the given problem type to the pupils.

In other words, this material suggests that after a concept is presented using a couple of different problems, an outline for this concept would give students a general method of attack for other similar questions and problems. The outline can then be given a name to help the students remember how to attack these types of problems.

The teaching of model problems, as a concept, does not seem to be done in North American textbooks as it should. For example, the addition of integers, rational numbers, real numbers and algebra are taught, but never tied together in one large example. This example could be presented as a good review, once the pupils have been exposed to all of the ideas of integers, rational numbers, real numbers and algebra. The teaching of a model problem would incorporate these ideas: (1) a concrete example, (2) an integer example, (3) a rule, (4) a rational number example, (5) a real number example, (6) an algebraic example and (7) a problem example. The students would be asked to name this example and then some questions and problems could be given for practice.

Let us take an integer example of adding -5 + (+3) = which could be made concrete by the use of a graphic diagram. If a child can get a picture in his mind of what is happening, the problems will seem much easier to him. The students draw a line down the middle of a sheet of graph paper and then using red and green pens designed for reading material, map the addition on the graph paper:



The students could then be asked to map -5 + (-3) =, +5 + (-3) =, and +5 + (+3) =, on the graph paper. Coloured dice could be used to generate more problems. The students could be asked to explain the rule and then solve these problems.

Integers	Rational Numbers	Real Numbers	Algebra
-50 + (+30) =	$\frac{-5}{7} + \frac{-3}{7} =$	$+5\sqrt{7} + (-3\sqrt{7}) =$	+5X + (+3X) =

The students having been given this type of understanding, can now be given both a number and a word problem. An example of a number problem is, "If Johnny adds positive $5\sqrt{11}$ to negative $3\sqrt{11}$, what is the answer?" An example of a word problem is, "A submarine is on the surface of the ocean at sea level. It dives 50 metres and then rises 30 metres. How far below sea level is the submarine now?"

Kalmukova (1975) reported on the teaching methods of a Russian elementary school teacher. The pupils of V.D. Petrova attracted attention because when difficulties in problem solving arose, the pupils returned to the text of a problem, reread it, and looked through the solution they had done. They also corrected most of the errors they made by themselves. They were, in case of failure, able to change the method of solution or find a new one. They could also outline a different path of solution for a single problem.

V.D. Petrova's classes were observed systematically while she was teaching problem solving. She emphasized reading the problem carefully with intonational expression, and emphasized that each word is important regardless of how small it is. Intonation had to be varied by the students when they saw punctuation marks. The students then separated the text of the problem into individual data and the unknown. The students completed the questions and then checked for mistakes. Homework was not considered done unless the scratch sheet was turned in as well. A pupil's mistake was analyzed by the class for errors in reading or thinking. The pupils were gradually trained in controlling the operations they used, and in correcting their mistakes. A person learned to think with words. In addition to developing the pupil's speech, Petrova also developed their logical thought and increased the level of their analytic-synthetic activity.

One method of improving reading ability for problems, as well as improving the method of reading with intonation, is by colour coding the problems. This can be done by again using coloured reading pens. The red pen (represented by ◯) can be used to highlight the numerical data. The green pen (represented by ▭) can be used to mark every word considered to be important in the question, and a yellow pen (represented by ☁) can be used to point out the unknown.

Problem: A submarine is on the surface of the ocean, at sea level. It dives 50 metres and then rises 30 metres. How far below sea level is the submarine now?

Solution: X = How far below the surface now?

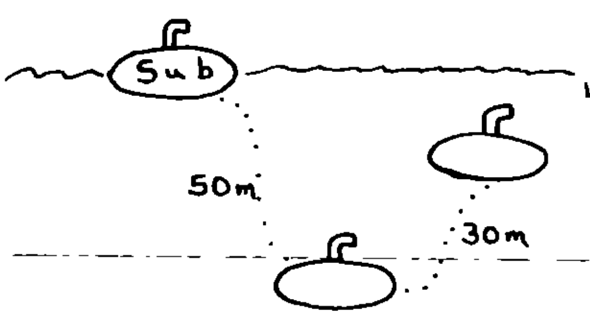
$$\boxed{-}50\ \boxed{+}\ (\boxed{+}30) = X$$

$$-20 = X$$

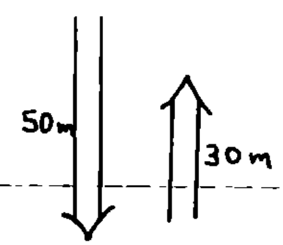
The submarine is 20 metres below the surface.

Try to solve the following number problem using colour coding. "If Johnny adds positive $5\sqrt{11}$ to negative $3\sqrt{11}$, what is the answer?"

M.E. Botsmanova (1972) suggests to us that the use of a graphic diagram could help students solve problems. A graphic diagram provides an abstracted and generalized expression of mathematical relationships. It starts with a subject analytic picture for the specific problem and leads to a graphic diagram for other cases. The following is a subject analytic picture and a graphic diagram for the submarine problem given above.



SUBJECT ANALYTIC PICTURE



GRAPHIC DIAGRAM

Many texts give us pictures that don't help us to solve problems. These pictures should be changed to subject analytic pictures and graphic diagrams to help students find methods for attacking problems.

The combination of reading with intonation, colour coding, using a graphic diagram, and checking for errors, along with a model problem reference should help students form a wider application of their concept of problem solving. Finally, the students should name the model problem with their own words. Teachers might want to call the concept "solving model problems," while students might want to call it "solving submarine problems." Whatever the name, the idea is to make the model problems have meaning so the students remember the rule and its applications.

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Teaching the Solution of Arithmetic Story Problems as a True Problem-Solving Task

by

C. Mauritz Lindvall
University of Pittsburgh

When elementary school children solve arithmetic story problems, they should be engaged in a true problem-solving activity. By "true problem-solving" I mean that the pupils should be analyzing each story in terms of what is described and what they are asked to find and then using the results of this analysis to determine what operation(s) should be applied. The desired type of activity is not taking place when children are solving a series of stories by mechanically applying the one operation they are studying at that given time (for example, solving every story by merely multiplying the two numbers that are given because this is the operation that is the focus of study for this week). Nor are they having problem-solving experience if they are using some "key word" to help them guess what operation to apply. These latter approaches to the solution of story problems are not based on a real understanding of the problem, and permitting pupils to use such procedures will not result in their acquiring a real capability for problem solving.

Work on story problems can provide the occasion for students to have the type of problem-solving experiences which teach them how to apply their mathematical knowledge and skill to practical everyday problems. But this analytical type of problem solving is not an ability that children acquire as a natural by-product of instruction and drill on basic arithmetic operations. As is suggested in the recent set of recommendations prepared by the National Council of Teachers of Mathematics (1980), the solving of arithmetic story problems is an important and separate capability and one which must be as carefully taught as any other mathematical skill. This paper describes one approach to this difficult teaching task.

Steps to be Followed in Analyzing and Solving Arithmetic Story Problems

In attempting to make arithmetic story problem solving a systematic and thoughtful process, many teachers provide a list of steps that pupils are to follow in analyzing a story and deciding what arithmetic operation to employ. Table 1 provides an example of such a list. Is such a list of any value to the pupil? Does it describe a set of teachable skills that can provide guidance for instruction?

Table 1. One Example of a Series of Steps Pupils Could Be Taught to Follow in Solving Arithmetic Story Problems

1. Find out what is given.

2. Find the question to be answered.

3. Think about the operation that should be used.

4. Select an operation.

5. Carry out the operation and find the answer.

6. Re-read the story to see if your answer appears correct.

An examination of the sequence of steps shown in Table 1 would probably convince us that steps 1 and 2 should be relatively simple to teach. Problem-solving performance would be improved if we could get our pupils to attack stories by directing their thinking to the significant information provided in the story. Since pupils will, sooner or later, learn to carry out basic operations correctly, most teachers might also feel that steps 5 and 6 would cause no great problem with the majority of students. But what about steps 3 and 4? How can one teach children to "think about the operation that should be used"? Do we really know what to teach here? The purpose of this article is to examine steps 3 and 4 and to attempt to provide some guidance for teachers faced with the task of helping students learn to do the type of analytical thinking required in true problem solving.

Can We Teach Children to "Think About the Operation That Should be Used"?

Does the injunction to "think about the operation that should be used" represent a meaningful step for pupils to follow? At first glance this statement might appear to involve a side-stepping of responsibilities on the part of the teacher. It almost suggests that the teacher is saying, "I really don't know what to tell you to do exactly, but if you think about the story, you may be able to figure out what to do yourself." Certainly, "think about" is not a very specific bit of guidance. But, in a very real sense, isn't this exactly what the teacher must say? Isn't it really a recognition of the nature of "problem solving"? If there was a specific sequence of rather mechanical steps that students could follow to always identify the correct operation to apply to a story, this activity would not be "problem solving." It would merely be the application of a set of rules, or an algorithm. Problem solving must involve the step of "thinking about" what to do. Our step 3, then, is not a "cop-out" on the part of the teacher. It is a recognition of the fact that story problems should represent a true problem-solving activity for the pupil. The task of the teacher, then, is to help pupils learn to do some effective thinking in identifying the correct arithmetic operation to apply in the case of a given story. This teaching goal can be achieved if we give students tools for analyzing the story and representing its essential information in a form that suggests the needed operation.

How Should Children "Think About" Story Problems?

Obviously, the thinking that we wish to have children do in solving any story problem is thinking which reflects a clear understanding of the problem and that results in a correct answer. We would like each child to be able to respond to the request to "explain this story to me" by giving a clear description of what states or actions the story involves and how an answer is derived. We would not be satisfied with an explanation such as "whenever I see the word 'less', I always subtract." The type of description that we want is the type given quite often by kindergarten and first-grade children when they are presented with a simple story and a set of blocks and asked to "use these blocks to show me what this story means." Here, for example, if the story involves finding the number of marbles two boys had altogether when each had a given number, we might expect a typical student to count out sets of blocks to represent the number of marbles each boy had and then put these together in one set and count the total to find the answer. Students display their understanding of the story and its solution by translating it into a form which provides a clear presentation of the information needed to solve it (i.e., the number in the given sets and the set joining operation needed to get the answer). Note that these children must "think about" the story. They don't have command of any arithmetic operations that they can apply in some arbitrary manner.

Many kindergarten and first-grade children show this kind of understanding of simple stories before they have any formal instruction on arithmetic operations. It appears that it is only after children have learned something about formal arithmetic operations and are exposed to slightly more difficult stories, that they attempt to solve stories without really understanding them, without "thinking about" them. To obtain some useful insight into why this situation develops so frequently, it may be useful to review how story problem-solving ability first develops in a typical child.

How Do Children First Learn to Solve Story Problems?

Pre-school children, after they have learned to count, will frequently take part in a type of play with a parent or some older person in which they use this newly acquired ability to answer simple quantitative questions. The adult might say to the child, "Take 3 blocks for yourself. Now give me 2 blocks. How many blocks do you and I have altogether?" Another set of instructions might take the form of "Give yourself 7 cards. Now give me 3 of your cards. How many cards do you have left?"

Activities such as the above can be thought of as the child's first exposure to arithmetic story problems. After some minimum guidance and instruction, most children appear to have little difficulty in arriving at solutions to these simple quantitative problems. It should be noted that what children do here involves an "acting out" of the story. The characters in the story, "you" and "I", are actually present as are the objects described, blocks or cards. All that is required for solution is a correct counting of the set that represents the answer. Because of the specific and concrete nature of what is described and asked, most children quickly master such "story problems."

Shortly after showing some ability with stories of the above type, children are likely to be exposed to "pretend" stories. For example:

> Pretend that these blocks are pieces of candy. Suppose that you have 5 pieces of candy. Then, I give you 3 more pieces of candy. How many pieces of candy would you have then?

Or the story might be slightly less concrete and involve even more pretending.

> Pretend that you have 4 pieces of candy. Now pretend that your friend Sue is here and that she has 2 pieces of candy. How many pieces of candy do you and Sue have together?

In solving such a story children cannot fully act out the story in the same sense that they can act out stories involving "you" and "I". When they have mastered these "pretend" stories, they have acquired a slightly more sophisticated problem-solving capability, the ability to make use of simple abstractions. They realize that to solve problems about candy they do not have to count and sort pieces of candy. Since the key to solving problems about "how many" is focusing on number, or numerosity, the children can use any easily countable elements to represent the number of pieces of candy. They have abstracted the quality of numerosity as a key component that is needed for the solution of the story. Also, they recognize that it is not necessary to have "Sue" present in any concrete form. They can represent Sue's candy by building a set of two in some convenient location and identifying this set as "Sue's candy." In doing this they have abstracted the quality of "set identity" as another key component in story representation.

The foregoing analysis has suggested that very young children may solve arithmetic story problems at one of two levels. At the simplest, or beginning, level they really "act out" the story. At a slightly more advanced level, they develop a physical model or representation of the story and manipulate this model to solve the story. This can be represented as shown in Table 2.

Table 2. The Three Levels of Representation of a Story Problem (Arrows Show Desired Steps Followed by Student Having Mastery of This Capability)

	Representation of Problem	Method of Solution
Level 1	Problem in story form ↓	Act out the story
Level 2	Essential components represented in physical model ↓	Model is manipulated to determine answer
Level 3	Essential information and operation shown in math model (e.g., number sentence) →	Solution of math model

As has been described previously, the student's development of the physical model (e.g., using two sets of blocks to model a story that says, "Ann had 3 pieces of candy and Billy had 5 pieces of candy. How many did they have together?") involves abstracting from the story as originally given, that information which is essential for solving the problem. We shall see that this is a key phase in the child's "thinking about" the story.

Learning to Write Number Sentences for Stories

Skemp (1971) described the process of solving arithmetic story problems as one of making the necessary abstractions from the actual story in order to identify the exact information needed to carry out the steps needed for solution. The process of developing a physical model of a story situation, as presented in Table 2, has already been described as involving abstracting from an actual story, that information needed for developing the model and manipulating it to arrive at a solution. This means going from the actual characters, objects, and relationships or actions described in the story to a representation of these by means of, for example, sets of blocks appropriately arranged on a table. Going from such a physical model to the writing of an appropriate number sentence, or some other representation of an arithmetic operation, involves a further task of abstraction. This third level of abstraction involves representing the numerosity of a set of blocks through the use of the appropriate numeral and representing the correct set operation with the symbol for the corresponding arithmetic operation. That is, this stage involves the development of the number sentence, or "mathematical model," that can be used to solve the story.

In Table 2 the ultimate capability which we wish to have the pupil develop is indicated by the path represented by the dotted-line arrows; seeing the problem in story form, generating some type of physical representation, writing the number sentence for the story, and solving the number sentence to determine the answer to the story problem.

It is to be noted here that if this analysis is correct, children initially learn to write number sentences for story problems by first developing a general physical model of the story and then writing the number sentence for this model. That is, they make use of a skill previously acquired (developing and using a physical model to solve the story problem) as an intermediate step in developing the proper number sentence for the story.

When children demonstrate the ability to use a physical model to solve a story, there is little doubt that they "understand" the problem. They can tell you what different elements in the model represent as far as components of the story are concerned. They can also relate operations on the model to operations described in the story. The story and model have a one-to-one relationship with no mystery associated with it.

Writing the proper number sentence, then, is also a meaningful representation of the story because it is derived directly from a set operation. That is, children understand the story and the arithmetic operation that can be used to solve it because they understand the abstracted physical representation of the story that serves to identify the correct arithmetic operation. Although the example used in this paper is that of a very simple addition story, this translation of the verbal story into some type of intermediate representation (or series of representations) that can provide a meaningful link between the story itself and the appropriate arithmetic operation is what the problem solver must do in "thinking about" how to solve any story problem no matter how simple or complex.

It should be noted that the "physical model" referred to in Table 2 is some type of simplified representation that contains the information from the story which is essential for solving the problem. With the simple story that has been used as an example in the discussion to this point, this intermediate representation (intermediate between the actual story and the number sentence that can be used for solution) could well take the form of sets consisting of physical objects such as blocks, or sticks, or the child's fingers. However, with other stories the necessary model may take the form of a diagram on paper, of some version of a number line, of a data table, or any of a number of possible simplified representations of the essential information from the story.

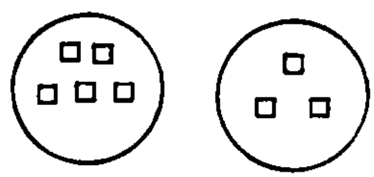
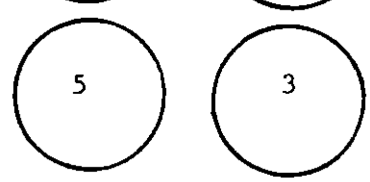
Suggested Procedures for "Thinking About" Story Problems

Table 3 gives an example of how a story problem might be represented and solved, showing representations that vary from an acting out of the actual story through increasing degrees of abstraction to the most abstract representation, the number sentence.

It is contended here that children can write a number sentence for a story, such as the one shown, with understanding, only if they first translate the story into some type of intermediate representation that makes totally clear what operation must be applied. This representation (which may be an almost instantaneous mental representation for the person who is highly proficient with the given type of story) in the case of the story shown in Table 3, must clearly show that solution of the problem requires the joining of two sets. Only when this is made clear to the problem solver can he or she proceed to write a number sentence with a full and correct understanding of why this particular mathematical operation is appropriate. That is, addition can be used to solve this story, not because it contains the word "altogether," but because the story, when correctly translated, describes the joining of two non-intersecting sets.

Of course, it can be pointed out that many pupils (and certainly most of us adults) do not have to go through any of these intermediate representations in order to solve this problem with complete understanding. Still, the evidence of our understanding must be shown by our ability, if called upon, to explain this particular story as one involving the joining of sets (or "putting groups of things together" or any comparable expression). That is, our understanding of this type of story and of the arithmetic operation needed to solve it is such that we mentally develop such intermediate representations and do this so quickly that we are not really aware of having taken this step. To achieve this type of understanding it is probably essential that all students actually be taught to use intermediate representations of stories and that they be given practice in using them at all the various points and

Table 3. Some Possible Stages that Children May Master as they Develop an Increasing Ability to Use Abstract Representations of a Story Problem in Arriving at a Solution

Representation of Problem	Method of Solution
Story	
Sally had 3. Jack had 4. How many altogether?	Student "acts out" the exact story as given
Some Intermediate Representations	
Act out, using dolls and actual objects	Story is acted out using this representation
Act out, using pictures of characters	Story is acted out using this representation
Model, using blocks (fingers, or any countable objects) but no pictures.	Model is manipulated to determine answer
Student draws sketches showing numerosity of sets and their separate identities	
Student draws diagrams to indicate identity of sets but uses numerals to indicate numerosity	
Mathematical Model	
Student writes number sentence	5 + 3 =

levels in the curriculum where new types of stories are presented, until it is obvious that they fully comprehend this process. Furthermore, instructional procedures that have the effect of causing the pupil to skip all intermediate representations of the problem will, more than likely, result in the pupil not fully comprehending the problem or its solution.

The various stages in the abstract representation of a story problem that are shown in Table 3 are presented only as examples of story representations, varying from the actual acting out of the story through intermediate representations that are increasingly abstract and provide an increasingly more direct basis for translation into a number sentence. Obviously, certain other intermediate representations might be substituted for those given here or might be inserted as additional steps in the sequence. Also, in working with any given student, only one or two of the intermediate stages might be needed to enable the student to grasp the basic structure of the problem and clearly understand what arithmetic operation to employ.

Diagnosing Pupil Difficulties in Solving Arithmetic Story Problems

The experience of most teachers provides evidence for the fact that there are large individual differences among students in their ability to solve story problems. Many students display an ability to grasp the meaning of a story quite quickly and have little difficulty in arriving at a solution. However, among the group of children who cannot solve such stories there appear to be great differences in the type of understanding (and lack of understanding) that they possess. With such students a diagnosis of specific difficulties would appear to be useful and probably essential.

It is suggested here that a sequence of stages in story representation as shown in Table 3 can provide the basis for a meaningful and useful diagnosis of pupil difficulties. It provides a means for determining the types of abstraction that a pupil can use in representing the essential meaning of a story. As an example, let's assume that Billy, one of our students, could not write the number sentence appropriate for the story shown in Table 3. We might start our diagnosis of his difficulties by seeing if he could model the story and its solution by using blocks. If he could do this, we would know that he had a good understanding of the story but needed further work on writing a number sentence for such a set operation. If he could not model the story with blocks, then we would proceed in the opposite direction and determine his ability to provide less abstract representations of the story. Could he act out the story if we provided dolls or pictures to represent the characters? If not, can he act out a similar story if we phrase it in terms of "you" and "I" and ask him to carry out the transactions described in the story? When we find the level at which Billy is able to operate, then we can build on that ability and proceed up through stages involving a greater degree of abstraction, making certain that he has ample time to master each stage in turn.

Teaching Problem Solving

The approach to the solution of arithmetic story problems that is outlined in this paper emphasizes the development of a complete understanding of each problem on the part of the pupil. It assumes that the goal of work on story problems at all grade levels is to teach general problem-solving abilities. For example, when we teach first-grade pupils to solve simple addition and subtraction stories, we are not merely teaching them "when to add" and "when to subtract." We are teaching them how to study and analyze problem situations so that they fully understand them and then, on the basis of this understanding, are able to determine what arithmetic operation(s) to apply. If this goal is to be achieved, teaching activities must be planned and carried out in a manner which emphasizes the importance of analysis and understanding. The following are some suggestions for conducting this type of teaching:

1. Classroom instruction and pupil assignments on story problems should give at least as much attention to the clear representation of the problem as to the calculation of the answer. For example, pupil written work on story problems should probably require an answer consisting of two components: (a) some type of diagram, drawing, or table that provides an abstracted representation of the essential problem components and relationships and (b) the mathematical computation used to obtain the answer. Both components should be graded.

2. Children should be taught, quite deliberately and specifically, how to develop paper-and-pencil representations of various types of stories. For example, this could include sketches of problems involving the combining of groups of things, the partitioning of groups, the combining of lengths and distances, the study of differences and relationships, and the comparison of sets of things. Continuing and frequent instruction on how to represent stories in this way should be a major feature of the teaching of problem solving.

3. The diagramatic representation of story problems must always include some type of representation of the quantities involved. This may take the form of countable elements in pictures of sets, of units on some scale of length, or of numerals.

4. It should be remembered that the arithmetic operations that will ultimately be used to solve most problems are basically efficient methods for determining quantities that would otherwise have to be found by counting. Having a clear perception, as obtained from the diagrammatic representation, of what would have to be counted to obtain the answer to a story problem is, then, essential for a correct identification of the proper operation to use.

5. Speed in arriving at a correct answer should not be emphasized in work on the solving of story problems. If importance is attached to speed of solution, this may only encourage students to make a guess concerning the operation to use and then proceed with computation. The emphasis should be placed on the careful analysis and clear representation of the problem.

Summary

Research on how adults proceed in their solution of complex problems that require a quantitative answer indicates that the most effective problem solvers do not go directly from the given problem situation to some type of equation or an arithmetic operation that immediately provides the solution. Rather they go through an intermediate stage in which they use sketches, diagrams, or other simplified representations of the problem to clarify the situation and to help them identify the needed operation(s). Research on how elementary school children solve arithmetic story problems also suggests that the successful problem solvers can make use of some type of intermediate representation to clarify the meaning of such stories. The present paper has attempted to outline how the development of such intermediate representations of story problems can be used in teaching pupils how to "think about" such stories and arrive at solutions.

Specifically, the approach advocated here suggests that understanding of a problem is gained by abstracting the essential information from the story and representing this in the form of a "model" or some type of "intermediate representation" which simplifies the problem situation. In the simple example of a problem used in this paper, the intermediate representation was presented in the form of sets of blocks or of pencil sketches of the sets involved. Such representations can be useful with problems involving groups of things that are to be joined or separated or operated upon in some way. Of course, with other types of problems other representations will be useful. For example, sketches of number lines or other indications of distances and a variety of types of charts and diagrams suggest themselves. As seen in Table 3, such diagrams may incorporate the use of numerals to indicate set size or the length of certain distances. This will be particularly necessary when amounts involved become at all large.

The paper also suggests that assessing the ability of individual students in terms of how proficient they are in developing such representations of stories can be a useful step in the diagnosis of difficulties and in identifying needed steps in instruction. It would appear that if elementary school children can acquire the ability to analyze arithmetic story problems through the type of intermediate representation proposed here, they can both become more effective in their present problem-solving tasks and acquire a basic skill in carrying out a general problem-solving procedure that will prepare them to be able to solve much more complex problems encountered in later years.

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Using Machine Technology to Enhance Problem Solving in the Middle School Mathematics Classroom

by

Karen L. Jones, Charles E. Lamb, Fredrick L. Silverman

Six of the ten basic skill areas listed by the National Council of Supervisors of Mathematics (1977) related to problem-solving skills and the use of machine technology. The listed skills are: (1) problem solving, (2) applying mathematics to everyday situations, (3) alertness to the reasonableness of results, (4) estimation and approximation, (5) reading, interpreting, and constructing tables, charts, and graphs, and (6) using mathematics to predict. Such an emphasis on problem solving and related skills along with the applicability of calculators and computer technology requires strategies for use by the teachers of middle school mathematics. It is the purpose of this article to discuss such strategies.

Bell (1978) lists five reasons that calculators and computers can enhance motivation to learn mathematics in the schools: (1) promoting internal satisfactions; (2) providing external reward; (3) enlivening the learning experience; (4) opening avenues of creativity; and (5) responding to a need for control in one's environment. All of these factors are generally evident as students plan and execute their own computer programs or address interesting situations -- often problems -- using hand-held calculators.

Some additional aspects of computers and calculators make them attractive:

(1) They provide a means for doing tedious calculations quickly.
(2) They provide immediate feedback.
(3) They may be facilitators in problem solving as they help to give partial solutions to more difficult problems.
(4) They seem to have applications with both weak and strong students.
(5) Computers and calculators are a spreading phenomenon in today's society.

Decreasing prices defy inflation, and affordability puts hand-held calculators into American pockets and micro-computers within reach of many families and businesses. Here are some examples of calculator and computer usage which may help these devices realize their potential.

Calculators

(1) Relegating tedious calculations to secondary status to enable students to make a judgement.

Consider two cars. One went 317.9 kilometres on 36.48 litres of gas; the other went 512.4 kilometres on 58.68 litres of gas. How do the cars compare in kilometres per litre?

(2) Using the calculator to find errors in computations.

Below is part of the record from a checking account. There is a $1.90 discrepancy with the bank statement. Find any errors, and correct them. How did they occur? What should the balance be?

Check #	Date	Amount	Previous Balance:	$1245.18
#1431	29 May	12.50	Bret's Hardward	1232.68
#1432	2 Jun	25.43	J.C. Peanuts, Inc.	1208.25
#1433	2 Jun	128.94	Post-Pine Furniture	1079.31
#1434	5 Jun	38.11	Mina Bird's Pets	1041.10
#1435	6 Jun	2.56	U.S.P.S.	1038.64
#1436	10 Jun	19.25	Henry's Hickory Hut	1019.39
#1437	11 Jun	29.87	Wonder Grocery	990.42

(3) Providing selected instantaneous information.

a. How useful is the calculator in finding these products?
250 x 10 =
267.5 x 10 =
2750 x 10 =
27.89 x 10 =

b. Find a decimal representation for 8/15.

(4) Regulating tedious calculations to secondary status to enable students to investigate patterns. The calculator is useful as a tool for generating, gathering, and organizing data.

a. Use the calculator to find the pattern for finding such products as those that follow.
15 x 15 = 225
25 x 25 = 625
35 x 35 = 1225
45 x 45 = 2025
55 x 55 = 3025
What is 95 x 95? (9025)
Do you see a means to find those products quickly?

Find the answer and patterns:

$15 \times 25 =$

$25 \times 35 =$

$35 \times 45 =$

$45 \times 55 =$

What is 75×85?
Do you see a means to find these products quickly? How does it compare to the one you found in the first part of this question?

b. Use a calculator to find a pattern for the units digits in the sequence $7^0, 7^1, 7^2, 7^3, 7^4 \ldots$? Try this activity with other base numbers.

c. Two sequences of numbers appear below. Investigate what happens when you add the same number of consecutive members of each sequence, starting at the beginning.
Sequence A: 1, 1/2, 1/4, 1/8, 1/16, ...
Sequence B: 1, 1/2, 1/3, 1/4, 1/5, 1/6, ...

d. A square has dimensions 16 cm on a side. If each side is halved, what effect is there on the area? Continue the process. What results emerge? Suppose you start with a square of 24 cm on a side. Apply the above procedure, and make similar observations. By what percentage does the area change?

e. Generate the Fibbonaci sequence (1, 1, 2, 3, 5, 8, 13, ...). Did you use the exchange key? If not, try to figure out a way to do that. It will be a procedure somewhat like doing "step programming" manually (Maor, 1980).

Computers

(1) Routine programs - for example, the student might receive drill on a previously learned skill.

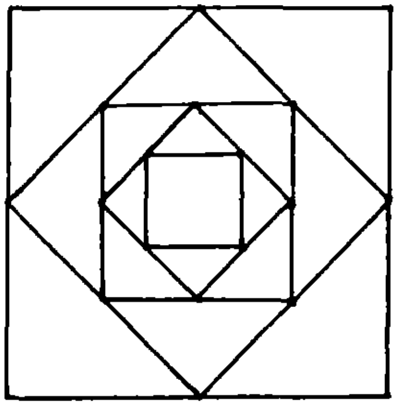
Use a programmable calculator to compute the mean, median, mode, variance, and standard deviation of a set of test scores.

(2) Debugging programs - making a program "work" may be problem solving at its best.

Debug the attached program for finding a Pythagorean Triple where all three digits are larger than 100. Do not use multiples of triples with smaller numbers.

(3) Writing programs - writing programs gives children a chance to exercise their creative abilities.

Create a program which simulates continuously inscribing squares for a sequence of iterations.



Some items appear so simple that the calculator may not simplify them (e.g., 2526 x 100). Nevertheless, children at the level of learning to "annex the zeroes" or "move the decimal point" can encounter numerous multiplication instances from which they can often discover the procedure by observing, writing, and studying the results their calculators show them. In each of these examples the calculator or the computer considerably simplifies each situation. People are using calculators and computers daily to resolve such practical concerns.

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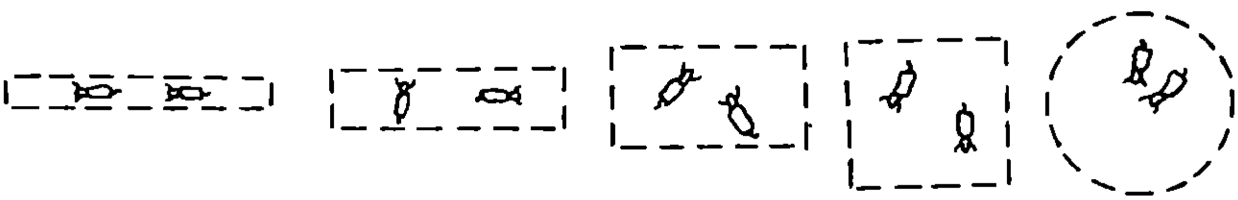
To You With Problem Solving

by

Al Anderson
Medicine Hat School System

"I want you to build a fence so that my cows will have the most possible grass to eat. That's the problem, Jim."

Jim has been given 16 five-cm rods and two plastic cows with the hidden agenda of finding a relationship between area and perimeter.



Believe it or not the above situation fits most of today's written criteria for a good problem. In short, a problem is any situation an individual faces for which no immediate solution is apparent but which holds the possibility for solution. Most descriptions would add the necessity for the person to accept the situation as a problem; otherwise, for him it is not a problem.

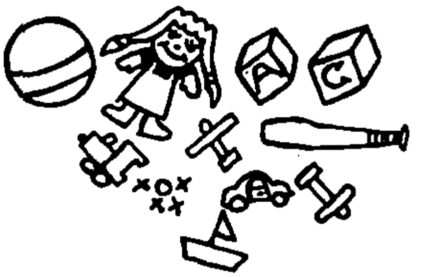
The number of students in schools who are able to experience problems like the "cow in the pasture" story is small. How many textbook problems, for example, include the following characteristics for good problems identified by Nelson and Kirkpatrick (1975)?

1. It is significant mathematically.
2. It involves a real object.
3. The child is interested in the problem.
4. The child must make modifications in the situation.
5. Several levels of solution are available.
6. The readiness with which the child attempts a solution indicates that he is convinced he can solve the problem.

Indeed we hear much about how poor students are at problem solving. What is it that they have difficulty with? Picking the equation? Getting the numerals for the sentence in the correct order? Such questions as these are indicative of a lack of experience with exciting problem-solving activities. Perhaps our students are poor problem solvers, but, given the right environment and a new mind-set toward the processes involved, both we and the students can learn. In other words, a good problem-solving experience has something for both the students and teachers. Let me illustrate through examples suitable for a range of grade levels.

Grades One-Two

Good problem-solving experiences are not limited to the upper elementary grades. The earliest pre-number development can and should be problem based. Classification tasks, for example, provide simple problem-solving experiences.



Are there more than three ways to put these toys in groups?

Even at this level pupils are required to go through a series of process actions, each one dependent on the preceding.

Eventual solution depends upon identifying minigoals and their place in the total problem.

In solving the classification task, pupils must realize that the toys have unique characteristics such as shape, color, texture, and function that relate to the "ways" to "group" the toys. They must also understand the concepts of "three" and "more than" before a realistic solution scheme can be devised. Next, there must be some type of planned actions or procedures.

The minigoals must be acted upon according to some plan toward problem solving.

Here some of the side benefits of teaching via problem solving begin to emerge. The nature of the problem will almost guarantee a pupil-teacher interaction. The setting promotes discussion. How are the toys different? Can you put some of the toys together in another way?

When the minigoals have been reached, the students must see that a solution is possible and then act on that conclusion.

The problem is to be solved.

The pupils must be able to demonstrate whether they were able to group the toys in more than three ways. A review of each setting is an important aspect of problem solving. Student and teacher should come to early and mutual agreement on the success or weakness of the strategies used.

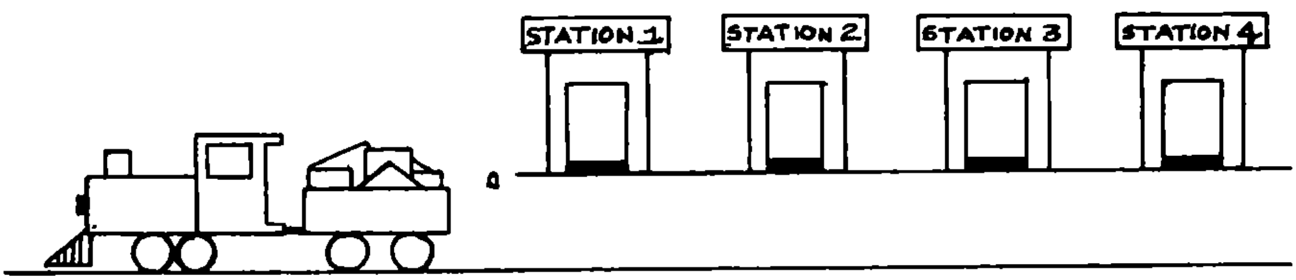
A review of the problem and solution strategy is necessary feedback.

Even though these same four steps are important at each grade level, there are obvious differences based on the prerequisite skills available to students.

Level	Prerequisite Skills	Problem-Solving Skills	Types of Problem
1-2	Sorting and classifying according to various attributes 1-1 Correspondence Equivalence Number as a class Cardinal number grouping Place value	Uses concrete materials to solve Collecting data from real experience Sequencing events in pictures and stories Discusses problems to find main parts Retells the problem Explains information in problem	Picture problem sorting, matching classification Non-number Patterns Sequences Operations concepts Numeration problems with object Calculator problems

Grades Three-Four

"You are a delivery man about to make drops at the stations but listen carefully and follow these rules:



Drop one block at Station 1 and two more blocks at Station 2 than you did at Station 1. Drop two more blocks at Station 3 than you did at Station 2, and so on. The problem is "At which station will you not have enough blocks to make a delivery (Nelson and Kirkpatrick, 1975)?"

Problems such as this can be used effectively for assessing the steps pupils take in solving problems. Do students analyze problems (identify minigoals) or do they barge ahead based on superficial hunches? What proportion of students operate as follows:

1. They do not determine the total number of blocks to be delivered (e.g., 12).
2. They tune in on problem segments and neglect the total perspective (e.g., drop one block at Station 1, two at Station 2, etc.).
3. They guess without evidence, (e.g., "Station 5").
4. They are satisfied with inappropriate solutions (no verification).

Although it is important that many problems be presented orally and that students be encouraged to talk through the problem parts and eventual solution, this oral thinking is more characteristic of young children than of children in the middle grades. Perhaps we are causing this with our abundance of written work, total class instruction and hush-hush enforcement.

Again, we can outline a distinct program of prerequisites and problem-solving skills as well as some problem types.

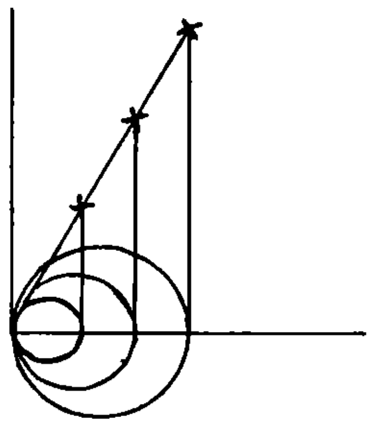
Level 1	Prerequisite Skills	Problem Solving Skills	Types of Problem
3-4	Can identify and use place value to (4-5 digits) Identifies and symbolizes operational situations Mastery of basic facts to limit of grade level Uses the algorithm to grade level limits Uses standard measuring instruments--metric linear, capacity, mass time and temperature Handles the money objectives to grade level Classifies and constructs 2-or 3-dimensional figures and objects	Uses estimation and approximation Collects data and constructs graph Interprets graphs and data charts Constructs models explores patterns Reconstructs problem Identifies relationships Makes tables for recording and interpreting data Makes projections, determines reasonableness of results Relates number sentence forms to operation situations	Real-life Can you make change for 50 cents using 6 coins, 7 coins, 8 coins? Tan-Gram puzzles Geoboard problem Number challenges If 7 cycle riders and 19 cycle wheels went by you, how many bicycles and how many tricycles passed? Calculator problems

As in our daily living, many problem breakthroughs are a result of group interaction. The following problem may best be handled in this way.

Grades Five-Six

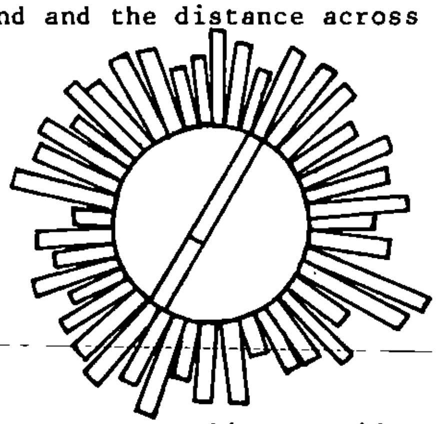
Collect sets of circular objects such as cans, cups, chips (about five different sizes). Give each grouping of two or three students one set of these circular shapes, one scissor, one metre of string, a 30 cm ruler and a large piece of paper. Students are given only one statement. "Graph the relationship of the distance around to the distance across your shapes."

Most of your groups will struggle for some time with this problem. There is a powerful temptation to rush. Don't be afraid of taking more than one class period. Some of your so-called slower pupils come through in such problems. Manipulative activities such as these often act as equalizers for these students.



Eventually, you want your pupils to cut pieces of string to fit the distance around the shapes (vertical). The horizontal distance is found by marking off the actual diameter. The result should be a straight line representation.

Another group of students can approach the same problem in a different activity. Give each group of two to three students a package of Cuisenaire rods. Ask them to illustrate or show the relationship between the distance around and the distance across a circular shape.



This method could also serve to establish the 3-1 relationship between circumference and diameter.

The next outline provides an extension of the types of problems, prerequisite skills, and problem-solving skills appropriate for grades 5 and 6.

Level	Prerequisite Skills	Problem-Solving Skills	Types of Problem
5-6	Rounding numbers Adds and subtracts whole numbers to grade limits Multiplies and divides to grade limits Ordered pairs Reads and writes coordinates Constructs and interprets graphs Uses appropriate standardized measuring units Reads distances to scale Draws diagrams to scale Knows interrelationship among units of length, capacity, and mass Uses decimals to thousand	Gains total perspective on problem Explains focus of problem Identifies required information Uses data collection and recording skills Uses diagrams and role play to solve problem Finds alternate solutions Applies equations where appropriate Checks solutions	<u>Multi-step</u> A Girl Guide troop sold 2000 boxes of cookies last year. This year they want to make \$800. If they sell the same amount of cookies and cookies cost \$1 per box, how much must they charge per box? <u>No Solution Problem</u> If a ship sinks one metre further in the water for every 200 people on board, how much of a ship will be under if 2000 people were aboard? <u>Calculator Problem</u> An average heart pumps 80 ml of blood each second. How many litres of blood has your heart pumped since birth? <u>Fun Problem</u> Two coins total 55 cents. One is not a nickel. What are the two coins?

It is good for students and teachers to realize that there are benefits to experiencing many ways of solving like problems. Such problems also provide opportunities for conducting informal assessment of pupils in terms of concepts, problem-solving process skills, or affective behavior. Anecdotal notes on students can be made while they are engaged in the problems. For example, how many give up before they try? Is there a reluctance to use pencil and paper for recording, drawing or graphing? Do students use the four problem-solving steps discussed earlier? Is there any evidence of their using previous knowledge (e.g., graphing, measurement, etc.)? Does their lack of response indicate a negative self-concept?

In conclusion, if success in problem solving is the primary goal of mathematics teaching and learning, why is it not more evident in our current mathematics programs? What can be done about improving the situation?

It is fairly clear that much confusion continues in regard to what problem solving is. Not only do the definitions vary, but we find different interpretations for its use in school programs. For example, some view problem solving only as an avenue for applying and practising newly acquired skills in a real-type setting. The basic purpose here is answer getting and refinement of skills. Others view problem solving as an opportunity to delve into the unknown. The essential goal here is for students to rediscover knowledge and to develop an awareness of skills needed. A growing view is that problem-solving process skills can be taught and thus applied to future problem situations be they science (discover-inquiry), social studies (value process) or language arts (comprehension).

Obviously, curriculum developers have failed in their attempts to build problem-solving skills, for whatever interpretation, into the current scope and sequence statements. Textbook publishers have typically had a narrow view of problem solving, mainly using word problems. We are also at fault for not going beyond what is handed to us.

Two final related cautions are in order as the various curriculum and instructional bodies attempt to rectify our failings on problem solving. It is important for all concerned not to impose rigidity on the teachers of problem solving. Second, let us not overemphasize the teaching of problem-solving processes to the extent that we are forced to swing back and forth to reach the necessary balance.

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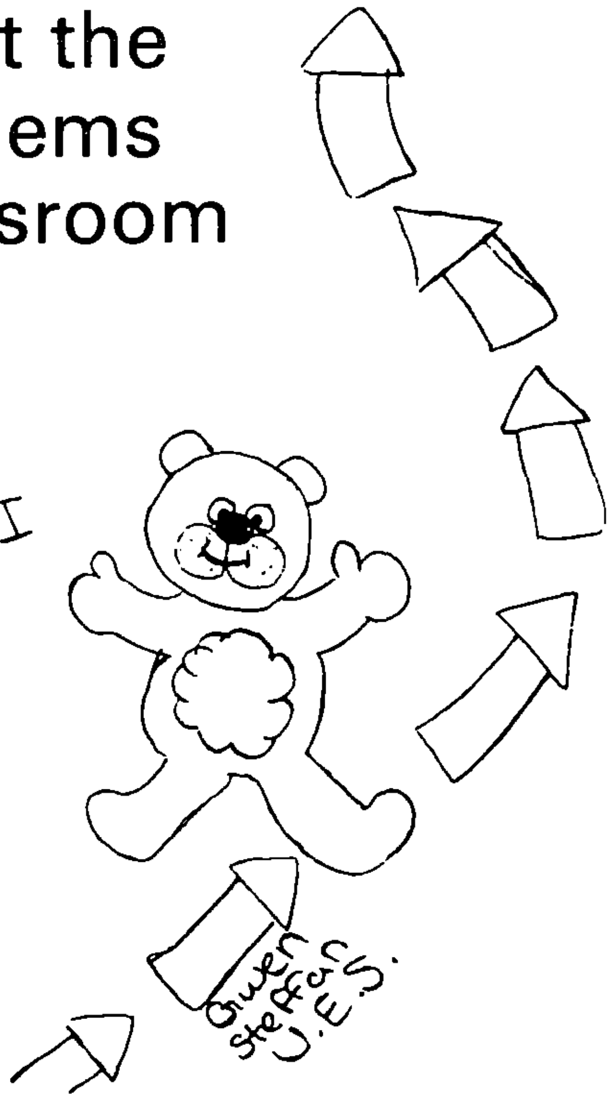
Part III

Carrying out the Plan – Problems for the Classroom

I planned well, but where do I start?



We Have This Problem with the Hall Lockers

by

James E. Riley
Western Michigan University

One of the problems of problem solving is finding good problems to solve. A good problem is one that does not yield an obvious solution. A good problem can be modeled or solved by analogy. A good problem can be studied empirically. The solution of a good problem may be arrived at from several directions. A good problem will result in the solver gaining new mathematical insights. A good problem should be an enriching experience for students with wide ranges of mathematical maturity. A good problem is hard to find.

We all have our favorite problems. I don't recall where I first came across one of my favorite problems, but I've seen it in many forms. The form I like best is found in the Indiana materials (LeBlanc, Kerr, and Thompson, 1976). It concerns a fixture found in many schools in North America. You see, we have this problem with the hall lockers.

Imagine a school with 1000 hall lockers along one side of a hallway. All the locker doors are open. Imagine 1000 children coming in from recess approaching the open lockers. The first child in line, a devilish tyke, can not resist slamming the locker doors shut.

The second child in line wishes to be involved so he starts opening the locker doors. But he cannot open them as fast as they were closed. He is only able to open every other locker starting with the second locker.

The third child in line wants to get into the act. She does so by changing the state of every third locker starting with locker number three. That is, if a locker is open she closes it, and if a locker is closed she opens it.

The rest of the children pick up the pattern. The nth student will change the state of every nth locker. When the thousandth child has passed the thousandth locker, which ones will be open and which ones will be closed?

Far be it from me to deprive the reader of the joy of solving a problem or making a discovery. Therefore, this article will occasionally be interrupted by the symbol (*) to let the reader know that this is a good place to put down the monograph and pick up a pencil and try to solve a proposed problem.

* * * * * * * * * * *

The locker problem has been presented to classes of students ranging from fourth graders to college undergraduates. Those who were able to solve the problem did so by first modeling the problem and then looking for patterns in the modeled solution. A fourth-grade class in Sparta, Michigan lined 36 English textbooks along the chalk tray. The class then lined up, like the class coming in from recess, and walked past the books turning them to represent open or closed locker doors. A book with its cover facing front represented an open locker and a book with its back cover facing front represented a closed locker door. The following pattern emerged.

1	2	3	4	5	6	7	8	9	10	11	12
B	F	F	B	F	F	F	F	B	F	F	F
13	14	15	16	17	18	19	20	21	22	23	24
F	F	F	B	F	F	F	F	F	F	F	F
25	26	27	28	29	30	31	32	33	34	35	36
B	F	F	F	F	F	F	F	F	F	F	B

When the above sequence is studied, one of two (or maybe both) patterns become obvious to the solver. What are they?

* * * * * * * * * * *

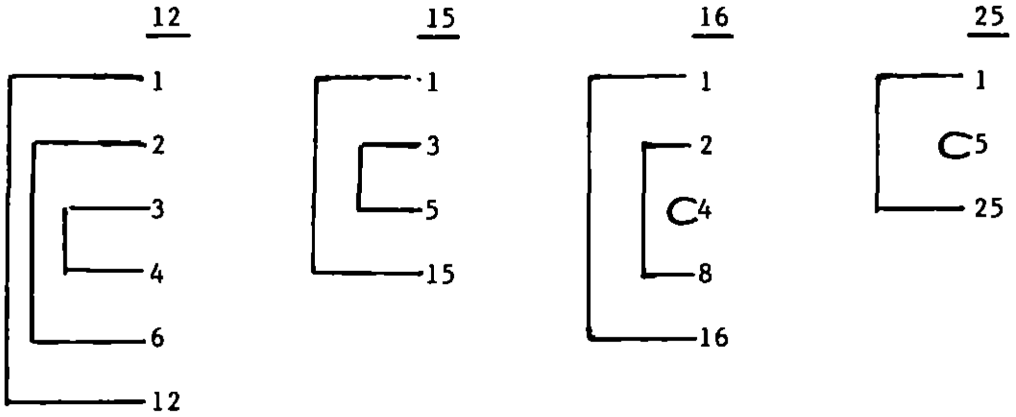
The more mathematically sophisticated solver recognizes that the closed numbered lockers are perfect squares; $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, etc. Younger children, because they are less at home with their multiplication facts, notice the following sequential pattern.

1 locker open	2 lockers closed
1 locker open	4 lockers closed
1 locker open	6 lockers closed
1 locker open	8 lockers closed
. . .	. . .

In either case, a solution to the locker problem has been found. But the solution is not mathematically satisfying. Why are the closed numbered lockers all perfect squares?

* * * * * * * * * * *

The numbered children who stop at any given numbered locker will be divisors of the locker number. The 1st, 2nd, 3rd, 4th, 6th, and 12th child will stop at locker 12. Notice that all numbers, except perfect squares, have even numbers of divisors. The divisors occur in pairs.



Any locker that has an even number of visitors will be left in the initial state because what one visitor does, the next will undo. Only those lockers with an odd number of visitors will be left in a changed state.

Generally fourth graders will stop at this point. However, the problem can be pursued a little further with fifth and sixth graders. Look again at the pattern created by the book model. Notice that the closed lockers (perfect squares) can be determined by the following sequence.

$$1 = 1$$

$$1 + 3 = 4$$

$$4 + 5 = 9$$

$$9 + 7 = 16$$

$$16 + 9 = 25$$

$$\cdots$$

Use a set of children's blocks (or a pencil and paper if your children's blocks are not at hand) to give a geometric interpretation to the above observation.

* * * * * * * * * * *

Later elementary school children have little difficulty showing that the addition of consecutive odd numbers of blocks will form a sequence of squares, with the length of a side one more than the preceding square.

Add 1	Add 3	Add 5	Add 7
*	* *	* * *	* * * *
1^2	* *	* * *	* * * *
	2^2	* * *	* * * *
		3^2	* * * *
			4^2

We leave it to the ninth-grade algebra student to show that the succeeding terms of the above sequence can be algebraically expressed as

nth square + next odd number = (n + 1)st square

or

$$n^2 + (2n + 1) = (n + 1)^2$$

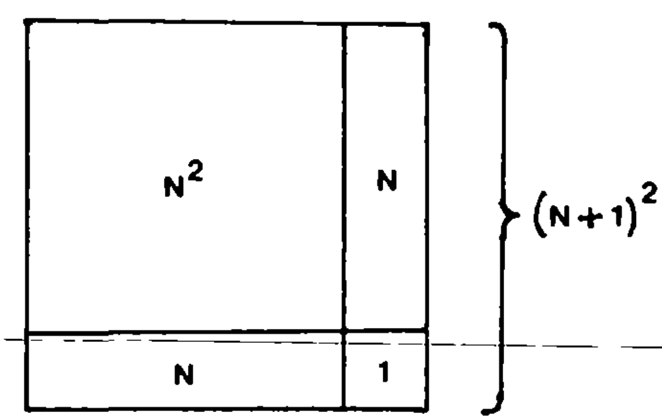
* * * * * * * * * * * *

A mathematical investigation that has fascinated students over the centuries is the finding of pythagorean triples. Pythagorean triples are positive integers (a, b, c) such that $a^2 + b^2 = c^2$. For example, 3, 4, and 5 make up a Pythagorean triple. The multiples of the triple (3,4,5) are also Pythagorian triples: (6,8,10), (9,12,15), etc. Pythagorean triples are said to be primative if a and b are relatively prime; i.e., if the greatest common divisor of a and b is 1.

Study the geometric succeeding-square model above and devise a scheme for finding infinitely many primative triples.

* * * * * * * * * * * *

A general form of the geometric model for the sequence of squares is the following.



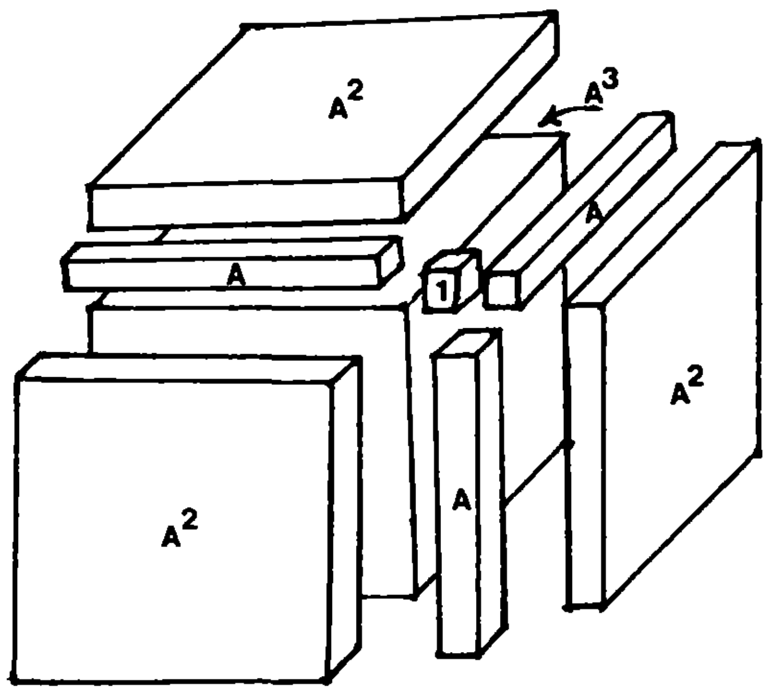
If in the formula $a^2 + b^2 = c^2$, we let $a = n$ and $c = (n + 1)$, then whenever $b = \sqrt{2n + 1}$ is a positive integer the triple (a, b, c) will be Pythagorean. Since $2n + 1$ will yield all odd numbers, it will also yield all odd perfect squares of which there are infinitely many. It can then be shown that n^2 and $2n + 1$ are relatively prime.

* * * * * * * * * * * *

Can the idea of Pythagorean triples be extended? For example, can we find triples (a, b, c) such that $a^3 + b^3 = c^3$? The Fermat conjecture states that such triples do not exist for $a^n + b^n = c^n$ where $n \geq 3$. The conjecture has been verified for all values of $n \leq 2500$ plus many more. The futility of the search can be demonstrated when one tries to extend the sequence model to the cube. Try it.

* * * * * * * * * * * *

To extend a^3 to $(a + 1)^3$, $3a^2 + 3a + 1$ must be added to a^3. This is easy to verify algebraically. The following figure shows the geometric interpretation of the extension.



If the extended Pythagorean triple is to hold for n = 3, then $3a^2 + 3a + 1$ must be a perfect cube. The following table shows the first ten perfect cubes and the values of $3n^2 + 3n + 1$ closest to the listed cube. The investigator will not be encouraged by what is shown.

n	$3n^2 + 3n + 1$	closest perfect cube
		1
1	7	8
2	19	27
3	37	
4	61	64
5	91	
6	127	125
7	169	
8	217	216
9	271	
10	331	343
11	397	
12	469	
13	547	512
14	631	
15	721	729
16	817	
17	919	
18	1027	1000

Thus we come to the end of a problem trail that started with some mischievous children and school hall lockers to an unsolved problem on the frontier of mathematics. Granted there were a number of side trails that could also be investigated such as the investigation of n-gon arrays and geometric numbers. Nonetheless, the trail we followed carried us through a number of problem-solving skills including modeling, emperical data collection, generalization, and logical thought.

References

Le Blanc, J. F., Kerr, D. R., and Thompson, M. Number theory. Reading, Massachusetts: Addison-Wesley, 1976.

Nuts!

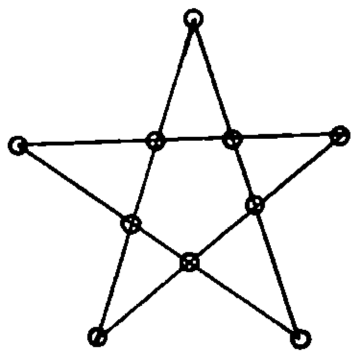
or

How Children Solve Problems

by
John Firkins
Gonzaga University

Attending NCTM meetings is always stimulating and the Grand Forks, North Dakota meeting was a special treat. As I boarded the plane to return to Spokane I ran across an ad for F. M. C. Corporation in the March issue of Republic Airline's magazine with the heading, "Drive Youself Nuts." Under this was THE PROBLEM: Place 10 nuts in five rows of four nuts each. The problem is not new. What is new is that a corporation used it in its advertising.

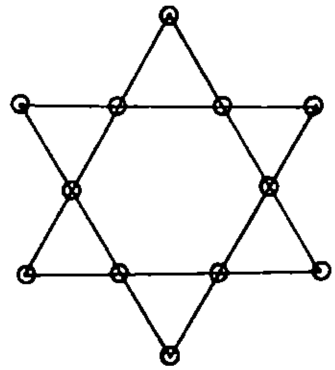
I took the ad with me and showed it to my daughters when I arrived home. Deborah, 15, scribbled on a paper for awhile and then announced that the problem was easy. Her solution looked like this:



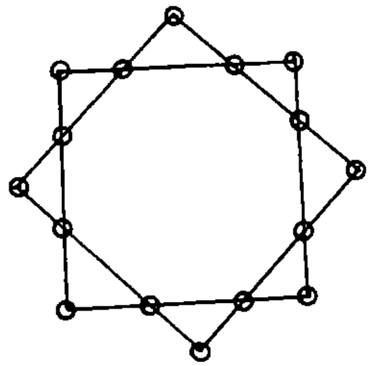
Meanwhile, Jessica, 12, had selected 10 nuts from a bowl on the kitchen counter and dashed to the basement to solve the problem. She was afraid Deborah would solve the problem and tell the answer before she had a chance to solve it.

The problem was out of a magazine not a textbook; neither was it the clever utterance of a teacher trying to stimulate a class! The effect was amazing.

Within minutes Deborah declared she could place 12 nuts in six rows of four nuts each! Her solution was to make a six pointed star and place the nuts as illustrated:



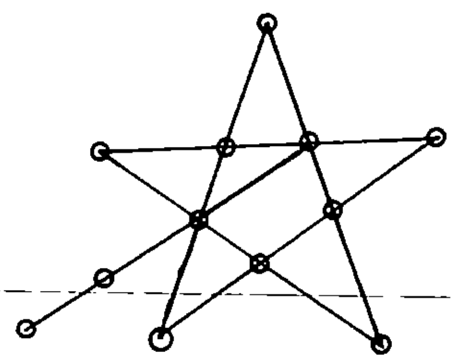
Shortly after making this discovery she declared that she could place 16 nuts in 8 rows of four nuts each! Her solution:



She was excited and rushed downstairs to tell Jessica.

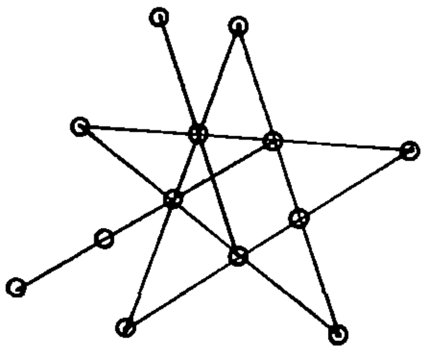
By this time Jessica had solved the original problem by placing the nuts on a table in approximately the shape of a five pointed star. Deborah didn't like Jessica's solution since some of the rows were not straight. She drew a picture of her solution for Jessica and challenged her to place 12 nuts in six rows of four nuts each.

Jessica's solution:



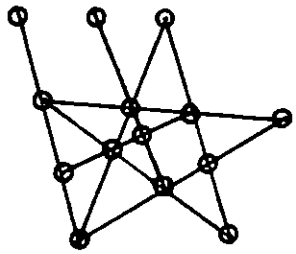
Jessica then had some surprises of her own for her older sister. "Place 14 nuts in seven rows of four nuts each."

Solution:



"Place 15 nuts in eight rows of four nuts each!"

Solution:



In this world the ability to solve problems is of paramount importance. The level at which children approach problems, the investigations they carry out and the solutions they devise depend on many variables. Fortunately, once a problem has been solved it can be explained to others who will then know as much as the original problem solver. The insight achieved can then be used by those who did not solve the original problem, or who did it a different way, to solve new similar problems. In any case, not everyone gains insight into a problem in the same way or at the same stage.

Each of the girls solved the problem. Each produced generalizations to intrigue the other.

What did Deborah learn from Jessica? Lines need not be straight. What was her last challenge?

"Place 10 nuts in 45 rows of two nuts each!"

She drew a circle, placed points on it, drew all possible chords and counted them.

It is exciting to be at the beginning!

The Great Rope Robbery

by

Elliott Bird
Long Island University

Problem: Two ropes hang 30 centimetres apart in a tall room, 10 metres from floor to ceiling. A rope thief with a sharp knife wants to take as much rope as possible, but while the thief can climb as high as necessary, a jump of more than 330 centimetres results in death. How much rope can the thief steal?

You will find a solution near the end of this article. Before you look at it, I want you to know that I regret posing my favorite problem in the mode of an article at all, preferring to have some control over my audience in its presentation and resolution. But the teaching of problem solving is much more important to me than any single problem, even my favorite one. So I relinquish the opportunity for direct contact with you in order to offer my ideas to a potentially wider audience. Because I believe strongly that to teach problem solving we must be problem solvers ourselves, I hope that before reading any further, you will spend some time working on it. By the way, this problem, like many, is best worked on for short periods of time, allowing the brain to rest over longer in-between periods.

I have posed this problem to young children, to young adults, and to teachers and other adults. With children and adults alike, my objective is the same: to provide an experience that is at once enriching, satisfying, stimulating, and pleasurable. With teachers I have an additional objective: to provide a model for the teaching of problem solving. I am using Bob Wirtz's concept that a problem poses a question which the solver understands, but knows neither an answer nor an algorithm for finding an answer. However, the solver does have enough information to find an answer with a small amount of effort.

Having posed the problem to a class, I permit a short time for discussion. In this way I can ascertain that it is understood and is being taken seriously. Often, on first hearing the problem, many people react by looking for some kind of gimmick or trick in the solution. (Indeed, when I first heard the problem from a friend, I experienced such a reaction. My friend told me the problem in a way that indicated that the thief dies if a jump is required of more than one-third the length of the rope. He did not mention the possible gap from the end of the rope to the floor. So I interpreted this to mean the thief could survive a jump of one-third the length of the rope plus whatever distance remained to the floor. Thus by cutting off both ropes at the ceiling, my thief could make a full legitimate jump to the floor. But my friend assured me I had misinterpreted him. It was then I set about to solve the problem as it was intended to be solved.) Once I take care of these initial responses and establish interest, I like to leave the problem and go on to another activity. I want my audience to go on to the activity as well, so I make sure it is exciting enough to take their minds away from the rope thief.

This is an important step in the problem-solving process: that is, leaving the problem, permitting the analytic side of the brain to rest, while the synthesizing side can operate on a subconscious level trying to obtain a total picture. A back-and-forth process of focusing on the problem and leaving it, returning and leaving again, should be repeated over a period of time. With fifth graders I would not consider the problem more than once or twice in a week. With adults, two or three times in a day is appropriate. As in real life, problems are not solved in a moment. Different minds work in different ways. Teachers who do not recognize the value of the subconscious in problem solving may overkill a problem and deny many students an opportunity to improve their sense of their own skills.

A word of caution, however: we have to be especially careful when we talk about a new problem at the very end of a class period. A young friend of mine was terribly frustrated when confronted with a homework problem whose solution required problem-solving skills and was due the next day. The child spent many frustrating hours with no success. When giving the assignment, my friend's teacher should have warned and urged the class not to spend more than five or ten minutes on it. In class the next day, an equal amount of time could have been spent discussing the difficulties and pitfalls encountered. Then the teacher should have requested an additional five or ten minutes consideration of the problem at home that evening. In this way the children learn more and more about the problem and about problem solving. They learn to savor both the difficulties of the problem and the nuances of the problem-solving process. And they gain an appreciation for the mind's intricate modes of operation and for their own ability to create and comprehend.

Now let's get back to our rope thief. It is the third day the problem is being discussed. (This may be the third week, but I do not recommend more than one week between discussions; as little as a day may be appropriate.) On day one, the problem was introduced and discussed only to the point of ascertaining that everyone understood it. On day two, solutions were presented and found wanting. These are the solutions that involve gimmicks like ladders, ceiling doors, windows. On day three, almost anything can happen.

With my own group of 5th-6th graders, on day three we spent only about 5 or 10 minutes of a one-hundred-minute math period with the problem. By this time everyone understood how the thief could easily obtain 1330 centimetres. (Climb one rope to the top, cut off the other rope at the top, climb down to 330 centimetres, cut, and jump to the floor.) And I said, "That's very good. Can you get any more than 1330 centimetres? No? Are you sure?"

If I had thought they needed more encouragement, I would have told them that I know a way for the thief to obtain more than 1330 centimetres, but I would not have told them how much more, nor would I have indicated any method. I prefer not to say any of this hoping that my stance of uncertainty will by itself accomplish the same thing.

Not much more happened on day three. We talked again about the difficulties of the problem; how to get 1330 centimetres, and how impossible it appeared to be to get more than 1330 centimetres unless the thief were to become a martyr for the rope (i.e. climb, cut, jump from ceiling, and die).

But the next week, a breakthrough occurred. When I relate this incident in my workshops, where I am in control of the problem-solving atmosphere, I tell my workshop audience that when I came into the classroom, Liz, one of my students, said, "If only there was a ______, I know a way for the thief to get more rope."

The word will be filled in momentarily--see HINT below. But, again, I want to give the reader a chance to stop and play with the problem some more. Already the quote above, even with a word omitted, is an additional clue. I would even like to urge you to consider posing the question to your class without knowing the solution. If you don't mind acknowledging your own uncertainties to your class, a satisfying and interesting discussion might follow.

My response to Liz was, "How much more rope could you get?"
"All of it."
"Oh? How?"

Now a discussion ensued involving a good part of the class. The student who first made the remark gave an explanation, but initial explanations are often unclear, and other students entered the discussion as their own understanding grew and in response to my remarks like, "You seem to have an idea, but I think you could express it better. Does anyone understand what Liz is trying to say?"

The discussion on this day lasted much longer than the preceding ones. Before coming out with a punch line I had in mind, I wanted to make certain that just about everyone in the class understood how Liz's thief could get the whole rope. By asking for repetition for the sake of clarifying, by asking who understood how this proposed device enabled the rope thief to obtain the whole rope, and by asking for omitted details to be filled in, I was able to determine the extent to which the class understood the proposed solution. And I was able to keep their interest as well. Even then, the entire discussion did not last more than fifteen minutes.

When it seemed to me that everyone in the class did understand how the rope thief could steal all two hundred feet of rope if there were a ______, I was ready for my punch line.

"That's a neat solution. If there were a ______, I can see how the thief could get all the rope. Too bad there isn't. We don't have any more time today to discuss this problem." (Many groans.)

By the next week, several students had solved the problem.

It is not totally clear to me what happened in the minds of those students during that last week. But certainly the discussion we had had was an important step in the problem-solving process. The effect of Liz's question was to make the problem easier. She changed the problem to a simpler, related problem. Once this easier problem has been solved, the original problem, too, is changed to a new one. Now the problem becomes: Is there some way to obtain or produce the desired device under the constraints of the original problem?

HINTS AND SOLUTION

First Hint

The question Liz asked that fourth week was, "How are the ropes attached to the ceiling?"

"With very strong nails. Why do you ask?"
"Well, if only there were a hook, I can figure out a way for the thief to get all the rope."
"Tell me."

Second Hint

"The thief climbs up one rope, grabs hold of the hook with one hand and cuts both ropes loose with the other, but does not let them drop. While holding onto the hook the thief ties the two ropes together to form a 20 metre length, and then slips the two ropes over the hook so that the knot is on one side of the hook.

Now the thief can climb down to the floor while holding onto both ropes. When the thief reaches the floor, the rope hanging on the side with the knot is pulled. The other rope is pulled up and over the hook."

"That's a very nice solution. It's too bad there is no hook."

Solution

The Fifth Week. "I know how the rope thief can get almost all of the rope. All the thief has to do is use a small part of the rope to make a hook. For example, the thief could climb up one rope to the ceiling, cut the other rope leaving ten centimetres. Use that ten centimetres of hanging rope to tie a loop. The loop serves the same purpose as a hook. Now hanging onto the looped ten centimetres of rope, the thief cuts off the first rope and ties together the two loose pieces of rope to form a single piece 1990 centimetres long. The thief slips one end through the loop until the knot reaches the loop. Now the two ropes are hanging down from the loop as from the hook and the solution proceeds as before."

In this way 1990 centimetres of rope can be obtained. Of course, the total amount that the thief can steal is $200 - x$, where x is the amount of rope it takes to form a loop.

Comments

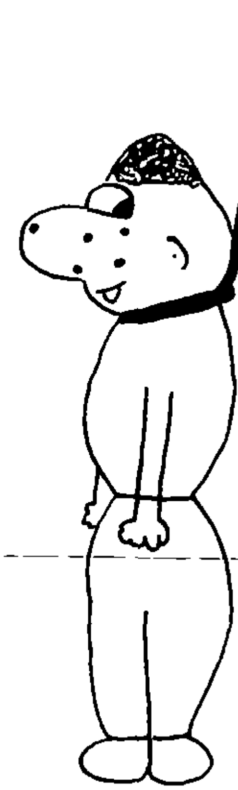
The solution presented is not unique. It is not even the one I came up with myself, but it is the one I hear most frequently. In a large group, a few individuals usually think of making a loop right away. However, it is important to keep them from saying anything aloud, thus destroying the opportunity for the others to create for themselves. At the same time these people should be credited with their ingenuity. Both objectives can be *accomplished by asking the group to whisper solutions to you or hold them until* the end of the meeting.

If you do give others the opportunity to create their own solutions, you will be surprised by the many different ideas you will hear. This may help you to become more free in your own problem-solving situations and, as a result, be a better teacher of problem solving.

FROM THE EDITOR

"The Great Rope Robbery " is reminiscent of the following problem adapted from the writings of Norman R. F. Maier:

In a large room, two ropes hang from the ceiling at a considerable distance from one another. One has a small ring on its free end. The other has a small hook on its free end. In the room are a ladder, a chair, a table, a hammer, and a book. If the ropes are too far apart to simply walk from one to the other while holding the former, how might you connect the two ropes without using any unmentioned aids? Will your method always work? How is your solution affected by shortening the lengths of the ropes?



Is this the only way?

Shannon
Anderson
U.E.S.

Problem Solving with Nim Games

by

Raymond E. Spaulding and David L. Albig
Radford University

The development of problem-solving skills should be a primary goal of any mathematics program. Nim games are an excellent vehicle for the development of problem-solving skills at all grade levels. Naturally the sophistication of the games presented to students will be a function of both grade level and previous experience in problem-solving situations.

We will examine selected one-pile and two-pile nim games. One-pile games are usually more elementary than two-pile games. A simple one-pile game has the following rules:

(1) Form a pile with ten markers.
(2) Players alternate turns, each removing one or two markers from the pile.
(3) The player who takes the last marker wins the game.

The teacher should explain the rules of the game and then allow students, working in pairs, ample time to familiarize themselves with the game in numerous contests. We believe adequate time spent allowing students to investigate the game is <u>essential</u> to the development of sound problem-solving skills. While this exploration time may appear to some as wasted and may even try the patience of teachers, it will greatly enhance the chances that:

(1) students will truly understand the rules of the game;
(2) students will appreciate the need for a winning strategy;
(3) students will develop skills for solving problems on their own and thus develop confidence in themselves.

Students who seem lost during this exploration time can be retrieved with challenges from expert players, be they other students or the teacher.

This exploration phase should evolve naturally into what we call the communication phase. Students should be encouraged to share their ideas and conjectures about winning strategies. These communication activities are valuable because they insure that students will:

(1) consider several hypotheses and therefore develop skills in the method of hypothesis construction and evaluation;
(2) practise verbal skills of self expression;
(3) develop social skills in the process of arguing for and against various conjectures;
(4) gain valuable feedback about their own conjectures by having them subjected to scrutiny.

Teachers can structure these important communication activities by asking some leading questions such as:

(1) In analyzing the game should you first consider what happens at the beginning of a game or at the end?
(2) Would it help to keep a record of exactly what happens in a few games? If so, what notation should be used to keep such a record?
(3) If you were to find a winning strategy, what would it look like?
(4) Are there any patterns which seem to develop in the games which can be used to predict a winner?
(5) Can you make a conjecture concerning a winning strategy which applies to at least part of a game?
(6) Having made a conjecture, can you test it by working out several examples and eventually find a logical basis for the conjecture?
(7) How can you expand your conjecture to the whole game? Can you test the resultant conjecture?

By spending adequate exploration time and adequate communication time, including attempts to answer some of the above questions, students will discover that leaving your opponent a pile of three markers will make it possible for you to win. Thus a pile of three markers is considered a "safe" position. Marker piles with one or two markers are considered "unsafe" because the opponent can win if you leave him these positions. A winning strategy consists of the ability to determine whether any position is "safe" or "unsafe." Any move the opponent makes on a "safe" position will leave an "unsafe" position, and from any "unsafe" position there is always a way to leave the opponent a "safe" position. If one plays so that a "safe" position is left after one's turn, a win is guaranteed.

The insights necessary to discover a winning strategy can be found by using the important problem-solving technique of organizing the data in tabular form.

TABLE 1

b	0	1	2	3	4	5	6	7	8	9	10
	s	u	u	s	?	?	?	?	?	?	?

b = the number of markers in the pile
s indicates a "safe" position
u indicates an "unsafe" position

It is apparent that zero markers is "safe"(s) to leave the opponent since this is the goal of the game. On the other hand, one or two are "unsafe" since the opponent will likely remove all the markers and win the game. Three markers in a pile is "safe" since any move the opponent makes on this position leads to an "unsafe" position. Asking the students to complete the table forces them to focus on the next step in finding a winning strategy. Since three markers in a pile is "safe," four or five markers in a pile must be "unsafe." Continuing in this way, students can complete the table and deduce a winning strategy. Students can then test their results against each other or the teacher.

The reader will note immediate possibilities for problems which require students to generalize or change their strategies. Variables in the game are:

(1) number of markers in the pile initially;
(2) maximum number of markers which may be removed in a turn;
(3) which player will take the first turn;
(4) whether the player taking the last marker wins or loses.

As just one example, consider the ten marker one-pile game with the following rule modification. Instead of removing one or two markers in one turn, allow players to remove one, two, or three markers. For this game the player who is forced to take the last marker loses. The table of "safe" and "unsafe" position then becomes:

TABLE 2

b	1	2	3	4	5	6	7	8	9	10
	s	u	u	u	s	u	u	u	s	u

b = the number of markers in the pile
s represents a "safe" position to leave an opponent
u represents an "unsafe" position to leave an opponent

In both of the nim games we have discussed the player who takes the first turn can guarantee a victory. Thus there is no hope for a player who does not get the first turn against a player who possesses and plays the winning strategy.

Two-pile nim provides an excellent opportunity for students to improve problem-solving skills acquired while studying one-pile nim. Again the rules can be modified to provide many problem-solving experiences. A good introductory game has the following rules:

(1) Form two piles of markers with five markers in one pile and ten markers in the other.
(2) Players alternate turns, and during each turn a player may remove one marker from either pile or one marker from both piles.
(3) The player who takes the last marker loses the game.

The learning steps required for the one-pile nim game remain valid for the two-pile game. We believe these steps are vital to the development of sound problem-solving strategies. Therefore we recommend:

(1) ample exploration time to investigate the game;
(2) ample time for students to share their ideas: productivity of this communication phase may be enhanced by forming teams, who work together to decide on the next move;
(3) ample time for organization of data, which, as before, can be prompted by leading questions from the teacher;
(4) ample time for constructing and testing hypotheses.

As we analyze this two-pile game, the concepts of "safe" and "unsafe" positions to leave an opponent are useful.

TABLE 3

b \ a	0	1	2	3	4	5	6	7	8	9	10
0		s	u	s	u	s	u	s	u	s	u
1	s										
2	u										
3	s										
4	u										
5	s										

a = the number of markers in one pile
b = the number of markers in the other pile.

Previous experience with one-pile games should ensure that students can fill out the first row and first column of the table. In order to find more "safe" positions, students must first discover that (1,1), (1,2), and (2,1) are all "unsafe" because they can lead to the "safe" positions of (1,0) or (0,1) in a single move. Thus (2,2) must be "safe" because any move made on (2,2) leaves an "unsafe" postion. The entire table can be completed in similar fashion, since every position which is "safe" leads to three "unsafe" positions. These "unsafe" positions can then be used to find other "safe" positions.

Once the table has been completed, a winning strategy has been found. A problem which requires generalization of this winning strategy is created by changing the number of markers placed in each pile initially. What if one pile has 5,000 markers and the other had 495? Completing a table of this size is not practical. Can students, given ample time and working together, find a simple rule which will indicate whether any given position is "safe" or "unsafe"? Problem-solving skills required for this task include looking for patterns within organized data, making conjectures concerning these patterns, and testing these conjectures. Application of these techniques will lead students to discover the "safe" positions. When only one pile has markers the "safe" positions are those with an odd number of markers in that pile and when there are markers in both piles the "safe" positions are those in which both piles have an even number of markers. After students have had time to investigate the problem, organize data, make conjectures concerning possible solutions, and establish for themselves by examples the validity of those conjectures, rigorous definitions and proofs are appropriate for mathematically mature students.

The goal of the two-pile game may be changed from (1,0), (0,1) to (0,0). That is, the player who takes the last marker wins. For many of the games the winning strategy changes drastically when the goal is changed.

In summary, nim games provide excellent opportunities for teaching problem-solving skills because:

(1) the fact that they involve actually moving physical objects implies that they are easily learned;
(2) the fact that they involve a competitive situation helps to focus the students' attention on the problem;
(3) simple rule changes create a variety of similar problems which allow students to reinforce with practice newly acquired problem-solving skills;
(4) they can be modified in order to find the appropriate game for any age group or maturity level;
(5) they provide problem-solving activities different from those usually found in complex word problems. Problem solving and word problems are too often considered the same thing. While word problems are important, they are by no means the only vehicle for teaching problem-solving skills;
(6) nim games are rewarding for students in the sense that the discovery of winning strategies:
    (a) offers students great satisfaction in the knowledge they have solved a problem through their own efforts,
    (b) allows them to win all games or fully understand why they cannot win, and
    (c) most importantly, develops problem-solving strategies which will be valuable throughout a lifetime!

The Olympics – A Problem-Solving Plot

by

Shirely S. Heck and C. Ray Williams
The Ohio State University

Problem solving involves critical thinking. Although new programs for the gifted and talented focus on critical thinking skills, there is little attention given to these skills in today's schools. Simply telling learners that something is important is not very effective. Rather, we must help them discover for themselves that critical thinking is important for societies in general, and for each of them in particular. Students should be encouraged to examine their own experiences, both direct and vicarious, to find out how critical thinking, or a lack of it, has affected their lives and the lives of others.

An interdisciplinary or thematic approach to teaching provides many opportunities for structuring and analyzing problems which extend far beyond a computational exercise. In reality, it places learning in its natural setting. In its Recommendations for School Mathematics of the 80's, the National Council of Teachers of Mathematics (1980) strongly supported this position:

> As new technology makes it possible, problems should be presented in more natural settings or in simulations of realistic conditions . . . Mathematics teachers should create classroom environments in which problem solving can flourish. Students should be encouraged to question, experiment, estimate, explore, and suggest explanations. Problem solving, which is essentially a creative activity, cannot be built exclusively on routines, recipes, and formulas. (p. 3)

The daily newspaper can serve as an excellent means for applying mathematics to local, state, national and international problems such as environmental, cultural, social and technological issues. For example, environmental and energy-related issues could be studied in terms of their mathematical problem-solving implications related to production, distribution and use of energy. The problem-solving skills of describing, comparing, contrasting, analyzing and evaluating could be reinforced by debating controversial dilemmas, such as nuclear energy versus nuclear waste disposal; strip mining of coal versus preservation of our natural landscape; school levy taxes versus quality education; local taxes versus state taxation for schools and so forth.

Numerous problem-solving skills related to mathematics could be utilized in publishing a students' newspaper or magazine. Examples of areas that might be included are: mathematics crossword puzzles; a "Dear Math Challenger" column in which students respond to problem questions submitted by other students; the new invention of the month; a scientific breakthrough; classified advertisements and problems related to these; editorials on the value of math; a science experiment corner. In addition to seeing the relevance of math to everyday problems and situations, publishing a newspaper would itself be a problem-solving situation requiring problem-solving skills. Such factors as costs, timing, communications, production, distribution, and labor are all integral to a publication and create natural problem-solving experiences.

The focus of problem solving should be on a specific problem felt to be relevant by the problem solver; it is, in fact, his or her involvement in the problem that makes it a problem. While the teacher's role as questioner is integral to the inquiry process, so too, the role of the children as questioners and problem "creators" is important. This requires a classroom environment in which children feel free to question, to take risks, to hypothesize, and to make mistakes.

Recording data, keeping track of resources, establishing assumptions and considering viable alternatives, describing a given situation, and using insight on the basis of observed patterns are desirable skills for people in all walks of life. Numerous everyday activities can be used to reinforce the development of these skills. Shields (1980) reports on how one fourth-grade class tried to solve a continuing logistics problem in the school cafeteria. The students worked on the various aspects of the problem for six weeks. The culmination of these experiences was a set of proposals which could improve service in the cafeteria. As a result of their work, several changes were implemented and the students became anxious to attack other problems they found around the school.

Another important problem-solving technique is the "open sentence." The children write open sentences about pictures or story problems before they begin to solve a given problem. Writing the open sentence beforehand requires the children to stop and think about the problem situation: What do they know? What don't they know? What is happening in the problem? How are the things they know and do not know related? When the children actually write the open sentence, they are showing how they interpret the problem. The sentence serves as a summary of the information and relationships in the situation. When children solve a problem using the open sentence as a guideline, they validate the solution and also put the solution back into the context of the problem to be sure it makes sense. This is in contrast to children who see two numbers in a story problem and immediately add or subtract without thinking about the problem.

The classroom environment can be designed to facilitate problem solving. Instead of a "problem-free," sterile environment, a more naturalistic setting with situational and simulated problems can provide opportunities which demand problem solving, decision-making, and research. A strategy used effectively with both pre-service and in-service teachers was described by Heck and Cobes (1977) in a book entitled The Creative Classroom Environment. In this strategy the classroom is perceived and compared loosely to a regular theatrical setting. For example, when the curtain opens for a play, the audience becomes aware of the time, place, and setting of the play through action, stage scenery, and property. Similarly, simple representations constructed out of cardboard can be used in the classroom at all grade levels to create an illusion of time and place, creating a more naturalistic setting for simulated problems. The ultimate development of the classroom as a stage-set design is, of course, in the classroom where bulletin boards, chalkboards, instructional centres and arrangement of furniture can reflect the time, place, or concept being studied. For example, the Olympic symbol of unity, posters of Olympic events, international flags and the symbolic torch are sufficient to set the stage for an Olympic theme. Like the scenographic elements which support the actor in his or her efforts on the stage, the stage-set design provides additional stimulus to the children in researching and role playing many of their learning experiences -- thus applying problem solving to many unstructured and unexpected situations.

Teachers need not be artistic to design the creative classroom environment. They can manipulate the classroom environment with free or inexpensive materials to make a specific unit of study come alive. A first step might be to locate illustrations that depict the era or setting of the unit being studied. Good resource materials for this activity could include encyclopedias, basic histories, brochures from travel agencies, picture files in the public library, and books or slides. A second step in drafting the design might be to visualize pieces or cutouts from the original illustrations so that when seen in isolation and out of context they will cause the learner to think about the original illustration. For example, Big Ben evokes thoughts of London, or a skyline evokes the image of a large city. Three simple ways of creating such illustrations include: (1) using an overhead projector to make a line drawing; (2) using an opaque projector to make a silhouette drawing on any type of materials including cardboard, paper or wood; and (3) using a slide projector to project pictures which depict the desired image. Using any of these three techniques, one could eliminate the precise scaling process used in theatrical stagecraft. For younger children, the teacher might provide much of this environment; older children could be given the problem of developing their own setting unique to a specific area of study. The problem solving and mathematical skills that could be developed through such an assignment are significant.

One example of a stage-set design developed with children in grades 4 to 6 was a unit pertaining to the 1980 Winter Olympics. First, the classroom teachers identified the areas of children's needs (i.e., geometry, measurement, decimals, and monetary skills). Since the winter Olympics were in process and were of great interest to the students, this theme was selected and incorporated into the instructional areas of mathematics identified by the teachers. The selection of a relevant theme and one which was of interest to the students was critical to having a positive effect on children's thought processes and enhancing confidence in their problem-solving ability. Integrating the various subjects into a high-interest theme provided a meaningful opportunity for these children to realize the necessity of considering, testing, and re-evaluating problems - a process which is an integral part of their lives.

A creative classroom environment does not necessarily teach a child what to think, but rather assists him or her in how to think. A stimulating environment provides the motivation for a child to become a miniature researcher through the processes of reading, living, and recalling. Through discoveries and explorations children can develop a more complex frame of reference from which generalizations can be derived and applied to the present and future.

An interdisciplinary unit on the Olympics can promote critical thinking in students. The teacher needs to consider involving higher level objectives. Learner tasks such as identifying the cities in which the Olympics were held, researching the history of the Olympics and reporting this on a timeline that includes national and international events represent the lowest level of information; conversely, activities which require more problem-solving skills correspond with the higher level thought processes of application, analysis, synthesis and evaluation. Examples of activities which require these higher level problem-solving skills might include simulating a trip to the Olympics with $1,000.00 per person where numerous decisions need to be made regarding lodging, transportation, food and events; forming an Olympic Committee to decide how to raise funds through advertising; inventing a more aerodynamic model of a bobsled; producing a scale model of an Olympic Village which includes the sports and housing areas and the costs involved in constructing the village; and evaluating the media coverage of the Olympic Games.

Within the stage-set design numerous research activities that promote problem-solving skills can be formulated to include areas of study such as sociology, anthropology, geography, history, economics, and political science. For example, an activity in which the students identify the cities where Olympics were held and the reason for selecting specific cities is applicable to the study of geography. In studying sociology, anthropology, or political science, a debate on the pros and cons of boycotting the 1980 Olympic Games would be an excellent activity. Role playing also becomes very natural within the stage-set design environment. Imagine the problem-solving and decision-making skills involved in role-playing the Olympic boycott decision from the viewpoint of the Canadian and Russian athletes, the parents of the athletes, the Prime Minister of Canada, the Russian and Afghanistan people, the Russian business community, the T.V. networks, the Olympic Committee, etc.

As a culminating activity to the Olympics Unit described earlier, the children were asked to develop a set of problems related to mathematics. Developing a problem is a problem itself. It serves as an excellent strategy for developing the skills of describing, observing, classifying data, and analyzing situations. While the children were given no guidelines in terms of specific problems, it was extremely interesting to observe that the problems they created were related to the areas of need identified by the teachers prior to the implementation of the unit. In addition, the children developed various levels of questioning. The problems ranged from very simple one-step processes to more complex processes.

Examples of students' problems related to measurement included the following:

If the Olympic track was 400 metres, how many times would a runner have to run it for a 5000 metre race?

Leah Mueller skated the 5000 metre race in 7 minutes and 56 seconds. The first 1000 metres took 1 minute 28 seconds. Her last 1000 metres took 1 minute 37 seconds. How long did it take Leah to skate the middle 3000 metres?

The high jumper made 3 jumps. His first jump was 2 metres 37 centimetres. His last jump was 2 metres 43 centimetres. His 3 jumps totaled 7 metres 21 centimetres. How high was his second jump?

If an Olympic swimmer swam an average of 25 metres in 38 seconds, how long would it take her to swim 100 metres? 500 metres?

The Swedish ski jumper's first jump on the 70 metre ski jump was 87 metres. His second jump was 89 metres 58 centimetres. How much shorter was his first jump? How long were his jumps when added together?

Through activities the children learned the value of a well-balanced diet for the athlete and what exercises an athlete might do in a day's training. The students also studied the effects of drugs, alcohol and tobacco on the body. Mathematical problems related to body exercise and practice became natural to the situation. These were reflected in the problems designed by the elementary school children.

For example:

Eric Heiden, the speed skater, trained an average of 6 hours a day, 5 days a week for the last 3 years to prepare for the 1980 Olympics. How many hours has he trained for the Olympics?

Design a week's nutritious menu for the 10 American hockey players. Include a daily snack for after "work-out" time. Using Safeway's newspaper ad, figure out the cost of the week's menu for the team.

Examples of monetary problems by the students included the following:

Linda Fratianne has 3 costumes she could wear for her performances. The total cost of the costumes was $261. What was the average cost per costume?

The Gold Medals are made out of gold and silver. There is about $126.00 worth of gold, and $332.00 worth of silver in each Gold Medal. Eric Heiden won 5 Gold Medals. How much are his medals worth altogether?

One meal (supper) for a hockey player cost $6.25. How much would the meal cost for the whole Canadian team if all 20 players ate?

Mary Wilson bought dinner for her 3 friends at a good restaurant at Lake Placid. Mary's lobster dinner cost $18.00 for everything. Together, all 4 meals cost $75.87. What was the average cost for one of Mary's friend's meals?

You used 400 litres of gasoline to drive to Lake Placid and back to your home. The gasoline cost $148.00. How much did the gasoline cost per litre?

An Olympic visitor bought 8 tickets for 3 events. 2 tickets cost $8.50 each, 3 cost $15.25 each, and 3 cost $9.00 each. How much did the visitor pay for all 8 tickets?

The problems created by the students often included irrelevant data, as illustrated in the following examples:

John Smith drove to Lake Placid for the Olympics. It took him 8 hours to get there. He used 250 litres of gasoline, and traveled 548 kilometres. How much did John average in speed per hour?

The four men in the 4-man bobsled totaled 780 lbs. The weight of the men and the bobsled totaled 1985 lbs. What was the average weight of the men?

Juri Svenson of Norway is a ski jumper. He is 23 years old and has trained 36 hours a week for the last 3 years. His two jumps at the 90 metre jump totaled 245 metres. His first jump was 122 metres. How long was his second jump?

All children should have the chance to develop their problem-solving abilities, and in turn, develop their minds in order that they might enjoy a fuller life. An interdisciplinary unit of instruction with diversified problem-solving activities allows children to utilize their particular learning styles. Whether they learn best through concrete experiences or more abstract presentations, this approach provides for meaningful experiences.

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Romance in Problem Solving

by

H. Laurence Ridge
University of Toronto

So as not to disappoint those who have read the title in a vernacular sense, here is a math dilemma based on matters of the heart.

On Friday evenings, planes leave Edmonton International at hourly intervals for Vancouver and at hourly intervals for Calgary. Ms. Thema Mattix, a liberated Edmonton math teacher, has an admirer in both of these cities and decides which one she will visit each weekend by taking the first plane out after she reaches the airport. Although school circumstances and variable traffic conditions make her arrival time at the airport completely unpredictable, she finds herself in Vancouver four weekends out of five. How is this possible?

The term 'romance' actually refers to the first stage of Whitehead's three cycles of intellectual activity: "romance, precision, and generalization" (as set out in The Aims of Education). The romantic stage is characterized as follows:

> The first procedure of the mind in a new environment is a somewhat discursive activity amid a welter of ideas and experience. It is a process of becoming used to curious thoughts, of shaping questions, of seeking for answers, of devising new experiences, of noticing what happens as a result of new ventures. This process is both natural and of absorbing interest. (p. 32)

In case you read into these remarks a call for Summerhill-style (total) freedom, Whitehead hastens to add:

> This stage of development requires help and even discipline. The environment within which the mind is working must be very carefully selected... [However,] a block in the assimilation of ideas inevitably arises when a discipline of precision is imposed before a stage of romance has run its course in the growing mind. ... The cause of so much failure in the past has been due to the lack of careful study of the due place of romance. [Precision] has been the

sole stage of learning in the traditional scheme of education. [To sum up,] without the adventure of romance, at the best you get inert knowledge without initiative, and at worst you get contempt of ideas - without knowledge. (pp.32-33)

In the realm of problem solving as considered in most schools, I see the stage of romance as a period of 'mucking about' - as the British vernacular would have it - brainstorming and experimenting with various strategies virtually uninhibited by rules or algorithms. Call it "development of heuristics" if you will. But - the problems have to be appropriate. The environment must be right - as per Whitehead.

Rather than go right away to the currently fashionable non-algorithmic process problems à la Carole Greenes and others, let's consider a more romantic approach to some traditional text-book-style problems, an approach which may both assist the lesser able student and also offer further insights to the more mathematically able folks.

The treatment presented in this paper is essentially one of numerical analysis which involves the freedom of romance as a reasonable starting point. The methods suggested do lead to precision and even generalization stages. Overall, the message is that much more can be derived from text-book-style problems than arises from the traditional algebraic algorithmic approach. For some readers the approach may smack of too much precision albeit a different form than the traditional algebraic algorithmic approach. From a comparative and realistic point of view, however, the amount of investigative freedom encouraged by the approach is considerable.

Traditionally, when we encourage students to develop 'alternative solutions', our expectations are usually at the precision stage. Here is a case in point.

> Solve in as many ways as you can.
>
> The sum of three consecutive integers is five more than the sum of the least and the greatest of the consecutive numbers.
> What are the numbers?

What we as teachers might expect, or at least, what students would think we expect is a variety of algebraic solutions such as:

I	II	III
Let x represent the smallest of the three consecutive integers. Then the other two integers are x+1 and x+2.	Let x represent the greatest of the three consecutive integers. Then the other two integers are x-1 and x-2.	Let x represent the middle integer of the three consecutive integers. Then the other two integers are x-1 and x+1.

(The third approach is considered to be the most clever since a sum is involved and the constant terms will vanish.)

Now, for students who may be having difficulty with algebraic expressions and equations and in order to introduce what can be an insightful approach, suggest that they try a "guesstimate" of what the three numbers might be. The idea is not to come up with a correct answer via guessing, nor to keep guessing until a correct answer is found, nor to keep making adjustments on the basis of 'too high' or 'too low'. The purpose of this romantic playing with numbers is to investigate the structure underlying the problem. What is really happening to numbers when they are combined as the problem dictates? No matter what trio of consecutive integers is chosen, there has to be a check as to whether this trio solves the problem. For example, should 7, 8, and 9 be selected, the sum of the integers is 24 but when 5 is added to the sum of the greatest and the least, the sum is 21.

It is at this stage that some teacher guidance would be appropriate. Students should be encouraged:

a) to write down what they are doing, and
b) to write it down in unsimplified form
so that patterns can be observed more readily.

The above check of 7, 8, and 9 could easily be done mentally as could checks on several other guesstimates. This style of romantic venture, however, could deteriorate into a guess-til-you-get-there approach which would not serve the solver particularly well in the long run.

The following is a suggested checking format. The assumption here is that for lesser able students it is easier to check a proposed answer than to formalize one in the first place.

GUESSTIMATE: 7, 8, 9

CHECK: $7 + 8 + 9 = 24$ $\qquad$ $7 + 9 + \boxed{5} = 21$

The fact that 7, 8, 9 is not the sought after trio pales upon realization from the check that the $\boxed{5}$ must be the middle number of the trio in order to make the sums equal. Should such insight into the 4, 5, 6 solution not arise, the development of a routine for checking any guesstimate leads to a pattern for solution by more precise means.

The romantic aspect lies in the solver's being able to start anywhere and continue in such a fashion, not looking for the answer so much, as for a pattern which may lead to a more precise or insightful approach to the problem. Should a solution arise during such an initial procedure, that is a bonus. It must be emphasized that this romantic playing with numbers is not expected to yield an answer directly.

If a successful trio has not emerged, let's finally take the "numbers that work." For three consecutive integers, we'll need a smallest 'number', a second one or 'number + 1', and a third one or 'number + 2'. (These expressions arise from student experience with specific consecutive triples of integers.) Using the format of the 'check', we have:

$$\text{Number} + (\text{Number} + 1) + (\text{Number} + 2) = \text{Number} + (\text{Number} + 2) + 5$$

[EXPLANATION: The "=" sign is used because these are the "numbers that work." The sums have to be equal!]

or simply,

$$n + (n + 1) + (n + 2) = n + (n + 2) + 5 \text{ , etc.}$$

What we have then, is a romantic approach to a precise method.

'Age problems' can be interesting. Note that this one is set in something of a puzzle motif - again - creating an environment.

> "You've got to be kidding!!"
>
> John is 19 years old and his sister Susan is only 1 year old. In how many years will John be:
>
> a) 7 times as old as Susan?
> b) 4 times as old as Susan?
> c) only twice as old as Susan?
> d) the same age as Susan?

<u>Strategy I</u>: Use <u>guesstimation</u> to help form an equation.

Guesstimate	John's age	Susan's age
8 (years)	$19 + \underline{8} = 27$	$1 + \underline{8} = 9$

$$\text{but } 27 \neq 7 \times 9$$

(After a sufficient number of guestimates to see the pattern of checking ...)

<u>Number that works</u>: n (years) $19 + n = 7(1 + n)$, etc.

<u>Strategy II</u>: <u>Systematic numerical analysis</u> (which adds a dimension of precision to a romantic beginning)

Such an approach may well be suggested during a romantic interlude with numbers. It might be noted by chance that

$$19 + \underline{8} = 3(1 + \underline{8})$$

or that there are other integral age relationships not even mentioned in the problem. This raises the question as to what other integral multiples are possible. (Hence we have a suggestion of generalization even within a romantic context.)

Let's start right from ground zero to see just what is going on.

Number of years from now	John's age (years)	Susan's age (years)	Integral multiple ?
0	19	1	19 = 19 x 1
1	19 + 1	1 + 1	20 = 10 x 2
2	19 + 2	1 + 2	21 = 7 x 3 AHA! - part a solved!)
3	19 + 3	1 + 3	No
4	19 + 4	1 + 4	No
5	19 + 5	1 + 5	24 = 4 x 6 (Part b)
6	19 + 6	1 + 6	No
7	19 + 7	1 + 7	No
8	19 + 8	1 + 8	27 = 3 x 9
9	19 + 9	1 + 9	No
10	19 + 10	1 + 10	No
11	19 + 11	1 + 11	No, but get 30 = 2.5 x 12 (How
12	19 + 12	1 + 12	No much farther to go?
13	19 + 13	1 + 13	No
14	19 + 14	1 + 14	No
15	19 + 15	1 + 15	No
16	19 + 16	1 + 16	No

We might note at this time that only odd numbers added to 19 will give an even result (divisible by 2).

17	19 + 17	1 + 17	Finally! 36 = 2 x 18 (Part c)

It is interesting to note that no matter what the age difference, if any two people live long enough one will be twice as old as the other, and, regardless of their birthdays, will be twice as old for a period of time totaling one year.

Now, how long will it take to get to 1, that is, the same ages?

Although the question in part d is absurd on the basis of common sense (since there is always a difference of 18 years) it leads to a more precise consideration of the relations among numbers based on the above analysis.

Number of years from now	John's age (years)	Susan's age (years)	Integral multiple
n	19 + n	1 + n	k

If n and k are "numbers that work," then

$$19 + n = k(1 + n) \quad (\#)$$

Now, to be 'precise', let's see what limitations there are on k.

$$k = \frac{19 + n}{1 + n}$$

As n gets increasingly large, $\frac{19 + n}{1 + n}$ gets closer to 1. More precisely,

$$\frac{19 + n}{1 + n} = \frac{\frac{19}{n} + 1}{\frac{1}{n} + 1} \longrightarrow 1 \text{ as } n \rightarrow \infty$$
(n increases indefinitely)

$$\left(\text{since } \frac{1}{n} \text{ and } \frac{19}{n} \longrightarrow 0\right)$$

(A practical application of this phenomenon can be seen with older people. The older they get, the less difference there appears to be in their ages even though the numerical difference is constant.)

To see mathematically that k cannot be one even though the ratio $\frac{19 + n}{1 + n}$ can be made as close to one as we please by taking n large enough, write equation (#) as $19 + n = k + kn$ and solve for n to get

$$n = \frac{19 - k}{k - 1}, \text{ so long as } k \neq 1.$$

Hence for integral n, k cannot be 1. This treatment could be generalized and thereby lead to the concept of 'limit of a sequence'.

Another way of looking at the relative behaviours of n and k is to apply a number theoretic approach when

$$k = \frac{19 + n}{1 + n}$$

and write it in the form $k = 1 + \frac{18}{1 + n}$ upon division by $1 + n$. From this form it can be seen that k is integral only when $n + 1$ is a factor of 18, that is, when $n = 0, 1, 2, 5, 8,$ and 17. These values of n correspond to $k = 19, 10, 7, 4, 3,$ and 2, respectively. Similarly, the equation

$$n = \frac{19 - k}{k - 1}$$

can be written in the form

$$n = -1 + \frac{18}{k - 1}$$

From this form it can be seen that n is integral only when $k - 1$ is a factor of 18, the difference in the ages. This condition yields the same pairs of integers as above.

Here we have seen a case of an initial romantic investigation suggesting a more systematic search of a relationship within a seemingly inocuous problem which in turn has led to an intuitive treatment of the limit of a sequence. Once a romance has been started who knows where it may lead? Oh, yes. Each plane for Calgary leaves 12 minutes after a Vancouver plane.

Part IV

Looking Back – Looking Ahead

Recent Advances in Mathematics Education: Ideas and Implications

by

Alan H. Schoenfeld
Hamilton College

There have been major changes in mathematics education research over the past decade. Research in education is now highly interdisciplinary, with contributions from cognitive psychologists, workers in artificial intelligence, etc. There are new people, new perspectives, new methodologies -- and most important, new results. Taken as a whole, these results promise to re-shape our understanding of the learning and teaching processes. In this paper I will discuss one aspect of recent work, and its implications.

The three examples I'm going to discuss in this paper seem on the surface to have little to do with each other. John Seely Brown and Richard R. Burton have done a detailed analysis of the way elementary school children perform certain simple arithmetic operations. John Clement, Jack Lochhead, and Elliot Soloway have studied the way that people translate sentences like "There are six times as many students as professors at this college" into mathematical symbolism. My work consists of an attempt to model "expert" mathematical problem solving, and to teach college freshman to "solve problems like experts." Yet all three of these studies share a common premise, and their results tend to substantiate it. That premise is the following:

> There is a remarkable degree of consistency in both correct and incorrect mathematical behavior on the part of both experts and novices. This consistency is so strong that it may often be possible to model or simulate that behavior, at a very substantive level of detail.

The implications of this assumption for both the teaching and learning processes are enormous. First, consider the notion that much of our students' incorrect behavior can be simulated -- and hence predicted. This means that many of their mistakes are not random, as we often assume, but the result of a consistently applied and incorrectly understood procedure. In consequence, the student does not need to be "told the right procedure"; he needs to be "debugged." This idea lies at the heart of the Brown and Burton work. It is also central to Lochhead and Clement's work, where we will see that the simple process of translating a sentence into algebraic symbols is far more complex than it at first appears. The other side of the coin has to do with the consistency of expert behavior. That, of course, is the assumption made in artificial intelligence -- where the attempt is made to model expert behavior in enough detail so that it can be simulated on a computer. If that seems plausible, then another step should seem equally plausible: model expert behavior so that humans, rather than machines, can simulate it. That is, teach students to "solve problems like experts" by training them to follow a detailed model of expert problem solving. That is the idea behind my own work.

1. A Close Look at Arithmetic.

In this section I offer a distillation of Brown and Burton's paper "Diagnostic Models for Procedural Bugs in Basic Mathematical Skills." There is much more in that paper than I can summarize here, and it is well worth reading in its entirety.

The key word in the title of their paper is "bug." It is, of course, borrowed from programming terminology -- and is fully intended to have all of the connotations that it usually does. While a seriously flawed program may fail to run, a program with only one or two minor bugs may run all the time. It may even produce correct answers most of the time. Only under certain circumstances will it produce the wrong answer -- and then it will produce that wrong answer consistently.

Often one discovers a bug in a computer program when it produces the wrong answer on a test computation. One might hope to find the bug by reading over the listing of the program and catching a typographical error or something similar. It is usually easier, however, to trace through the program and see when it makes a computational error. At that point, one knows where the source of difficulty is and can hope to remedy it. If the basic algorithm were simple enough, it might be possible to guess the source of error by noticing a pattern in the series of mistakes it produced. Thus one might be able to find the bugs in a program -- without even having a listing of it. For example, see if you can discover the bug in the following addition program from the five sample problems.

41	328	989	66	216
+9	+917	+52	+887	+12
50	1345	1141	1053	229

Of course, if you don't have a listing of the program, you can never be certain that you have the right bug. However, you can substantiate your guess by predicting in advance the mistakes that the program would make on other problems. For example, if you have identified the bug which resulted in the answers in the previous five problems, you might want to predict the answers to the following two:

446	201
+815	+399

This particular bug is rather straightforward. We can get the same answers as the program for each of the five sample problems by "forgetting" to reset the "carry register" to zero: after doing an addition which creates a carry in a column, simply add the carry to each column to the left of it. For example, in the second problem, 8 + 7 = 15, so we carry 1 into the second column. That gives us a sum of 4. If the 1 is still carried to the third column, that gives us 1 + 3 + 9 = 13. The same difficulties arise all the way across the board. Using this bug, one would predict answers of 1361 and 700 to the two extra problems.

A student might have this "bug" in his own arithmetic procedure, just as the computer program might. In fact, a child might well use his fingers to remember the carry, and simply forget to bend the fingers back after each carry is added. This would produce exactly the bug above.

The finding of bugs is far more than an exercise in cleverness: it has tremendous implications for the way we teach. The naive view of teaching is that the teacher's obligation is to present the correct procedure coherently and well, and that if anything goes wrong, it is simply because the students have not yet succeeded in learning that procedure. The above example (and many more in the text) suggest that something very different is happening. Suppose a student is making consistent mistakes. The teacher who can diagnose such a bug in that student stands a decent chance of being able to remedy it. The teacher who looks at the student's mistakes and concludes from them simply that the student has not yet learned the correct procedure, is condemned simply to repeat the correct procedure -- with much less likelihood that the student will perceive his own mistakes and begin to appropriately use the correct procedure.

If one makes the assumption that a student's behavior is consistent when it is wrong, then the issue appears to be theoretically simple. You begin with the correct procedure, and then at each step generate what might be considered plausible bugs. Next, you create a series of test problems so that the student's answers to those problems indicate his bugs. Finally, after identifying the bugs, you intervene directly to remedy them.

While this theory may sound remarkably simple, the implementation is actually quite complex. First, it is a surprisingly complicated task to write down all the operations that one has to do to add or subtract two - three digit numbers. Primitive operations involved in subtraction, for example, include knowing the difference between any two single digits, being able to compare two digits, knowing when it is appropriate to borrow, being able to borrow, knowing to perform operations on the columns in sequence from right to left, and many, many more primitive operations. Any flaw in one of these procedures causes a bug which needs to be diagnosed; flaws in more than one procedure cause compound bugs which may be even more difficult to diagnose. Brown and Burton hypothesized the following list of nine common procedural mistakes in the simple subtraction algorithm. When one considers possible combinations of these, things start to get out of hand very rapidly.

143 -28 125	The student subtracts the smaller digit in each column from the larger digit regardless of which is on top.

143 -28 125	When the student needs to borrow, he adds 10 to the top digit of the current column without subtracting 1 from the next column to the left.
1300 -522 878	When borrowing from a column whose top digit is 0, the student writes 9 but does not continue borrowing from the column to the left of the 0.
140 -21 121	Whenever the top digit in a column is 0, the student writes the bottom digit in the answer; i.e., 0-N = N.
140 -21 120	Whenever the top digit in a column is 0, the student writes 0 in the answer; i.e., 0-N = 0.
1300 -522 788	When borrowing from a column where the top digit is 0, the student borrows from the next column to the left correctly but writes 10 instead of 9 in this column.
321 -89 221	When borrowing into a column whose top digit is 1, the student gets 10 instead of 11.
662 -357 205	Once the student needs to borrow from a column, he continues to borrow from every column whether he needs to or not.
662 -357 215	The student subtracts all borrows from the left-most digit in the top number.

Based on the premise that students do indeed follow certain consistent procedures, Brown and Burton tested this list empirically with the scores of 1325 students on a 15-item subtraction test. Their data indicates that more than 40 percent of the errors made on the test could be attributed to "buggy" behavior. In particular, more than 20 percent of the solution sheets were entirely consistent with one of their hypothesized bugs. (That is, all of the answers were exactly what that particular faulty algorithm would produce.) Another 20 percent of the solution sheets indicated behavior which was strongly consistent but not identical with such a bug.

Further, the analysis of the students' performance on this test, led to the identification of new "bugs." Of the 1325 students tested, 107 students had a bug in their "borrow from zero" procedure. In consequence, they had missed all 6 of the 15 problems on the test which called for borrowing from zero. In the original interpretation of the data, those 107 students were simply identified as students who scored 60 percent. Later they were identified as students who have not yet mastered the technique of borrowing from zero.

2. A Look at "Simple" Word Problems.

For a number of years, a group at the University of Massachusetts at Amherst has been studying a variety of students' misconceptions in college-level physics and mathematics. This discussion is based primarily on two of their working papers, "Translating Between Symbol Systems: Isolating a Common Difficulty in Solving Algebra Word Problems" by John Clement, Jack Lochhead and Elliot Soloway, and "Solving Algebra Word Problems: Analysis of a Clinical Interview" by John Clement. These papers deal with college-level students, and (at least at first) with subject matter "appropriate" for students at this level. Yet, there are two very strong similarities between this work and the work described in section 1. First, a process which is "simple" to do correctly may be a rich source of potential errors. Second, there is an almost remarkably perverse consistency in the way that students make mistakes -- to the point where remediation is rather difficult, even if one understand what the student is doing. Finally, there is an interesting contrast between the "static" nature of mathematical language and the "dynamic" nature of a programming language.

Since Clement, Lochhead, and Soloway were dealing with college-level students, the authors began with problems of some complexity. One problem, for example, asked the student to determine what price, P, to charge adults who ride a ferry boat, in order to have an income on a trip of D dollars. The students were given the following information: There were a total of L people (adults and children) on the ferry, with 1 child for each 2 adults; children's tickets were half price. The students were asked to write their equation for P in terms of the variables D and L. When fewer than 5 percent of the students given the problem solved it correctly, the authors began to use simpler and simpler problems. After a sequence of increasingly easier problems, they wound up using problems like the ones given in Table 1.

Table 1

1. Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this University." Use S for the number of students and P for the number of professors.

2. Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there are five people who ordered strudel." Let C represent the number of cheesecakes and S represent the number of strudels ordered.

3. Write a sentence in English that gives the same information as the following equation: A = 7S. A is the number of assemblers in a factory. S is the number of solderers in a factory.

4. Spies fly over the Norun Airplane Manufacturers and return with an aerial photograph of the new planes in the yard.

R R R R B B B

R R R R B B

They are fairly certain that they have photographed a fair sample of one week's production. Write an equation using the letters R and B that describes the relationship between the number of red airplanes and the number of blue planes produced. The equation should allow you to calculate the number of blue planes produced in a month if you know the number of red planes produced in a month.

The correct answers for these four problems are (1) $S = 6P$, (2) $5C = 4S$, (3) "There are 7 assemblers for every solderer," and (4) $5R = 8B$. The success rates for these four problems were 63, 27, 29, and 32 percent, respectively.

It might seem at first that the researchers had simply found a bunch of students who were extremely defective in their algebraic skills. However, the students had been given the following six questions:

1. Solve for x: $5x = 50$

2. Solve for x: $\frac{6}{4} = \frac{30}{x}$

3. Solve for x in terms of a: $9a = 10x$

4. There are 8 times as many men as women at a particular school. 50 women go to the school. How many men go to the school?

5. Jones sometimes goes to visit his friend Lubhoft driving 6 miles and using 3 gallons of gas. When he visits his friend Schwartz, he drives 90 miles and used _?_ gallons of gas. (Assume the same driving conditions in both cases.)

6. At a Red Sox game there are 3 hotdog sellers for every 2 Coke sellers. There are 40 Coke sellers in all. How many hotdog sellers are there at this game?

On average, more than 95 percent of these problems were solved correctly. Therefore, the difficulties of these college students were not in simple algebraic manipulations. The difficulties were in translating a statement from a sentence into a suitable algebraic form. Actually, the students were very competent in courses beyond algebra. Clement's paper provides a detailed analysis of the transcript of a problem-solving session with one student who was doing B+ work in a standard calculus course at the time of the interview, and had been able to differentiate the function $f(x) = \sqrt{x^2 + 1}$ rapidly, using the chain rule, without difficulty. Yet, the student was unable to solve any of the problems in Table 1.

As in the Brown and Burton work, the students' errors were remarkably consistent for all of the problems in Table 1. More than four-fifths of the incorrect solutions to the problems were of the form $6S = P$, $4C = 5C$, "Seven solderers for every assembler", and 8R = 5B, respectively. In other words, there was a consistent reversal of the symbols and their role in the equations.

Through an analysis of clinical interviews, the authors identified two major causes for the reversal. The first explanation for the reversal was that the students made a "syntactic" translation of a sentence into algebraic form; i.e., the student reads along the sentence, replacing words where appropriate by algebraic symbols. Thus, "six times as many students" becomes 6S; "as" becomes equals, and "professors" becomes P. The resulting equation is 6S = P.

The second explanation for the reversal was that although the students recognized that an equation does stand for a relationship between two quantities, the way that the students represented that relationship to themselves resulted in a reversal. Many of the students, for example, drew pictures such as:

On one side of the desk is the professor; on the other side are the 6 students. Thus the equality is 6S = P.

To the mathematician, an equation for the "students and professor's" problem is a device which allows him to calculate the number of students given the number of professors, or vice-versa. Since there are 6 times as many students as professors, one must multiply the number of professors by 6 to get the number of students (for example, 10 professors yield 60 students). Thus, S = 6P. Obviously, students do not have this perspective.

In another experiment, the authors provide some dramatic evidence of the difference between the static and dynamic interpretations of an equation. Their "subjects" were 17 professional engineers who had between 10 and 30 years of experience each. The engineers had come to take a course in the BASIC programming language. On the first day of the course, the engineers were asked to write an equation for the following statement:

> At the last football game, for every four people who bought sandwiches, there were five who bought hamburgers.

Only 9 out of 17 of the engineers solved the problem correctly. The following day, without any discussion of the previous problem and the solution to it, the engineers were asked to write a computer program for the following:

> At the last company cocktail party, for every 6 people who drank hard liquor, there were 11 people who drank beer. Write a program in BASIC which will output the number of beer drinkers when supplied with the number of hard liquor drinkers.

All 17 of the engineers solved the problem correctly. The authors further substantiated these results with a study of some college students in a programming course. The notion of programming suggests a possible means of remediation: If we train students to think of an equation as a "program" with inputs and outputs, we may increase the likelihood of their getting the correct answers.

3. A Look at Problem Solving.

Apparently random problem-solving behavior can actually be quite consistent. In the work with BUGGY and with elementary word problems, the focus was on consistent patterns of mistakes, for purposes of diagnosis and remediation. In this section we look at the flip side of the coin. Just as a look beneath the surface discloses consistency in novices' incorrect behavior, a look beneath the surface will also disclose great consistency in the problem-solving behavior of experts. To make the point that experts and novices approach problems in dramatically different ways, consider the following three problems -- all of which are ostensibly accessible to high school students.

Problem 1: Let a, b, c, and d be given numbers between 0 and 1. Prove that $(1-a)(1-b)(1-c)(1-d) > 1-a-b-c-d$.

Problem 2: Determine the sum $\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!}$.

Problem 3: Prove that if $2^n - 1$ is a prime, then n is a prime.

On problem 1 most students will laboriously multiply the four factors on the left, subtract the terms on the right, and then try to prove that $(ab+ac+ad+bc+bd+cd-abc-acd-bcd+abcd) > 0$ -- usually without success. Virtually all the mathematicians I've watched solving it, begin by proving the inequality $(1-a)(1-b) > 1-a-b$. Then they multiply this inequality in turn by $(1-c)$ and $(1-d)$ to prove the three-and four-variable versions of it.

Likewise in problem 2, most students begin by doing the addition and placing all the terms over a common denominator. A typical expert, on the other hand, begins with the observation, "That looks messy. Let me calculate a few cases." The inductive pattern is clear and easy to prove.

The expert who read problem 3 and said "That's got to be done by contradiction" was typical (given the structure of the problem, one really has no alternative). Yet this almost automatic observation by experts was alien to students. A large number of the students to whom I have given the problems either responded with comments like "I have no idea where to begin" or tried a few calculations to see whether the result is plausible and then reached a dead end.

Of course these are special problems, for which expert and novice performance are each in their own way remarkably consistent. While the experts did not consciously follow any strategies, their behavior was at least consistent with these "heuristic" suggestions:

a. For complex problems with many variables, consider solving an analogous problem with fewer variables. Then try to exploit either the method or the result of that solution.

b. Given a problem with an integer parameter n, calculate special cases for small n and look for a pattern.

c. Consider argument by contradiction, especially when extra "artillery" for solving the problem is gained by negating the desired conclusion.

Many of the novices were unaware of these strategies, and many others "knew of them" (that is, upon seeing the solution they acknowledged having seen similar solutions), but hadn't thought to use them. Expert and novice problem solving are clearly different. The critical question is: Can we train novices to solve the problems as experts do?

There are a number of obstacles. First, we have to factor out simple subject matter knowledge: There is no way that one can hope to give the students experience before they have it, or to compensate for it. Rather, we would like to provide the students with strategies for approaching problems with flexibility, resourcefulness, and efficiency.

Second, we must realize that the heuristic strategies described by Polya are far more complex than their descriptions would at first have us believe. Consider the following strategy and a few problems.

"To solve a complicated problem, it often helps to examine and solve a simpler analogous problem. Then exploit your solution."

Problem 4: Two points on the surface of the unit sphere (in 3-space) are connected by an arc A which passes through the interior of the sphere. Prove that if the length of A is less than 2, then there is a hemisphere H which does not intersect A.

Problem 5: Let a, b, and c be positive real numbers. Show that not all three of the terms $a(1-b)$, $b(1-c)$, and $c(1-a)$ can exceed $1/4$.

Problem 6: Find the volume of the unit sphere in 4-space.

Problem 7: Prove that if $a + b + c + d = ab+bc+cd+da$, then $a=b=c=d$.

These four problems, like problem 1, can be solved by the "analogous problem" strategy. Yet, it is unlikely that a student untrained in using the strategy would be able to apply it successfully to many of these. Part of the reason is that the strategy needs to be used <u>differently</u> in the solution of each problem.

In solving problem 1, we built up an inductive solution from the two-variable case, using the result of the analogous problem as a stepping stone in the solution of the original.

In contrast, analogy is used in problem 4 to furnish the idea for an argument. The problem is hard to visualize in 3-space but easy to see in the plane: We want to construct a diameter of a unit circle which does not intersect an arc of length 2 whose endpoints are on the circle. Observing that the diameter parallel to the straight line between the endpoints has this property enables us to return to 3-space and to construct the analogous plane.

Problem 5 is curious. It looks as though the two-variable analogy should be useful, but I haven't found an easy way to solve it. At first the one-variable version looks irrelevant, but it's not. If you solve it, and think to take the product of the three given terms, you can solve the given problem. So again we exploit a result, but this time a different result in a different way.

Problem 6 exploits both the methods and results of the lower-dimensional problems. We integrate cross-sections, using the same method; the measures of the cross-sections are the results we exploit.

In problem 7 it would seem apparent that the two-variable problem is the appropriate one to consider. However, "which two-variable problem" is not at all clear to students. A large number of those I have watched tried to solve

Problem 7': Prove that $a^2 + b^2 = ab$ implies that $a = b$, instead of
Problem 7": Prove that $a^2 + b^2 = ab + ba$ implies $a = b$.

The description "exploiting simpler analogous problems" is really a convenient <u>label</u> for a collection of similar, but not identical, strategies. To solve a problem using this strategy, one must (a) think to use the strategy (this is non-trivial!), (b) be able to generate analogous problems which are appropriate to look at, (c) select from among the analogies, the appropriate one, (d) solve the analogous problem, and (e) be able to exploit either the method or result of the analogous problem appropriately.

If we assume now that we can actually describe the strategies in enough detail so that people can use them, we run right into another problem. That is: a list of all the strategies in detail would be so long that the students could never use it! Knowing how to use the strategy isn't enough: The student must think to use it when it is appropriate.

Consider techniques of integration in elementary calculus. There are fewer than a dozen important techniques, all of them algorithmic and relatively easy to learn. Most students can learn integration by parts, or substitution, or partial fractions, as individual techniques and use them reasonably well, as long as they know which techniques they are supposed to use. (Imagine a test on which the appropriate technique is suggested for each problem. The students would probably do very well.) When they have to select their own techniques, however, things often go awry. For example, $\int \frac{x\,dx}{x^2-9}$, a "gift" first problem on a test, caused numerous students trouble when they tried to solve it by partial fractions or, even worse, by a trigonometric substitution!

In "Presenting a Strategy for Indefinite Integration" (The American Mathematical Monthly, 1978) I discuss an experiment in which half the students in a calculus class (not mine) were given a strategy for selecting techniques of integration, based on a model of "expert" performance. The other students were told to study as usual -- using the miscellaneous exercises in the text to develop their own approaches to problem solving. Average study time for members of the "strategy" group was 7.1 hours, while for the others it was 8.8 hours; yet the "strategy" group significantly outperformed the others on a test of integration skills -- in spite of the fact that they were not given training in integration, just in selecting the techniques of integration.

The "moral" to the experiment is that students who cannot choose the "right" approach to a problem -- even in an area where there are only a few useful straightforward techniques -- do not perform nearly as well as they "should." If we leap from techniques of integration to general mathematical problem solving, the number of potentially useful techniques increases substantially, as does the difficulty and subtlety in applying the techniques. An efficient means for selecting approaches to problems, for avoiding "blind alleys," and for allocating problem-solving resources in general thus becomes much more critical. Without it, the benefits of training in individual heuristics may be lost.

In consequence of the above, an attempt to teach general mathematical problem solving would need these two components: first, a detailed description of individual strategies, and second, a global framework for selecting these strategies and using them efficiently. One way of presenting such a framework is with a "model" of expert problem solving. That model takes a semester to unfold, so there is no sense in my attempting to summarize it here. What I have done is simply to give the outline of the model (see Figure 1), and a description of the most important heuristic strategies which fall within each of the major blocks of that strategy (see Figure 2).

Of course, documenting improved problem-solving ability is rather difficult. I am slowly amassing evidence, in a variety of different ways, that instruction in problem solving actually can have an impact on students' problem-solving performance. The material on integration provided some evidence of this. A "laboratory study" demonstrated that "problem-solving experience" in and of itself is not enough: In the experiment, two groups of students worked on the same problems for the same amount of time and saw the same solutions, but one saw in addition heuristic explanations of the solutions. The differences in their performances were dramatic. (See "Explicit Heuristic Training as a Variable in Problem-Solving Performance.") Third, there is a large amount of "before and after" data on the students in the problem-solving course. These data indicate both an improved problem-solving performance on the part of the students and an improved ability to generate plausible approaches to problems, as opposed to a control group. There is much data to be analyzed by a variety of different means -- means which were unavailable just a few years ago, and which come from a variety of disparate sources. As one such example, let me discuss briefly the notion of "hierarchical cluster analysis." Consider the following three problems.

Figure 1. SCHEMATIC OUTLINE OF THE PROBLEM-SOLVING STRATEGY

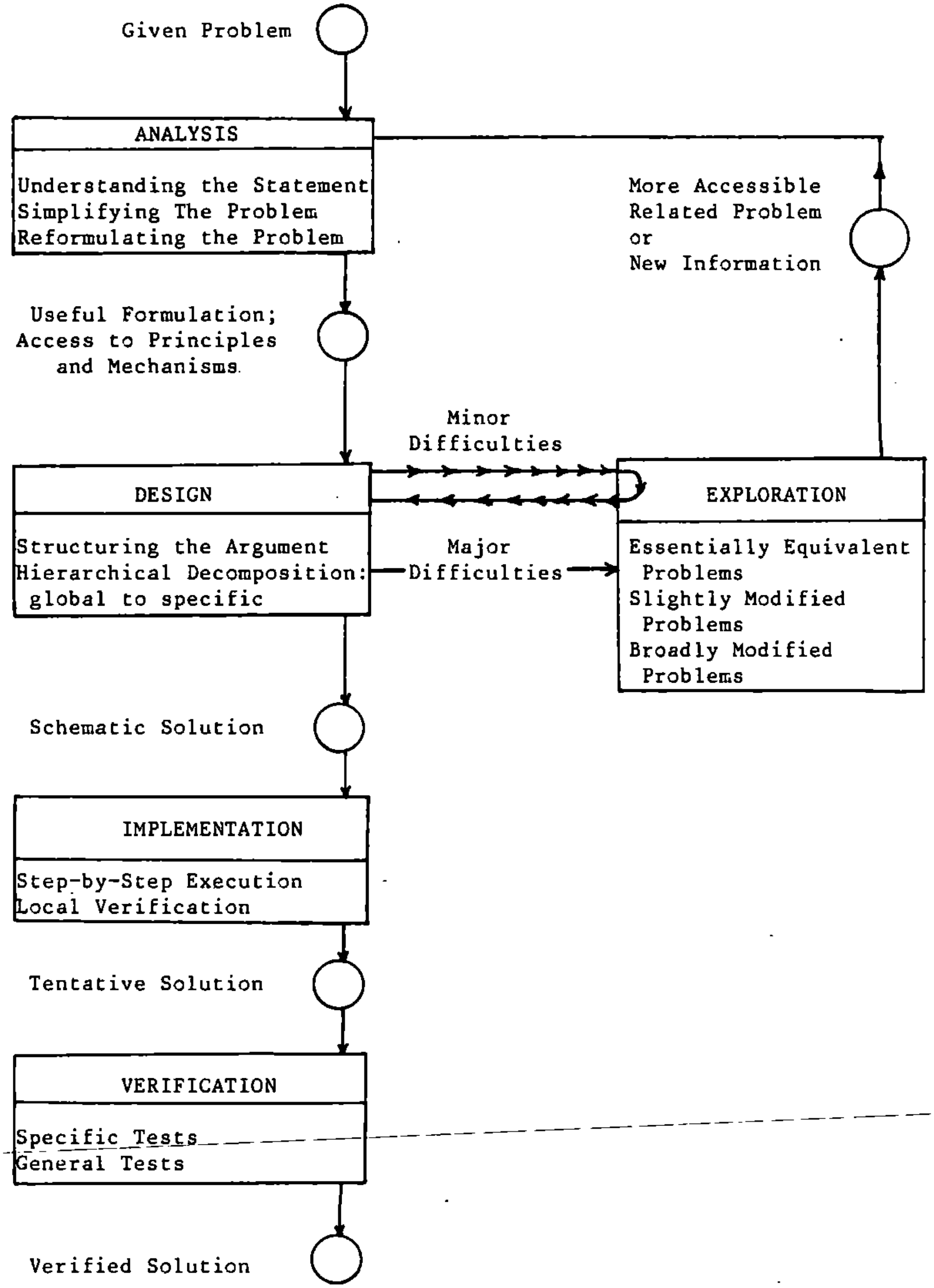

Figure 2. SOME IMPORTANT HEURISTICS IN PROBLEM SOLVING

For Analyzing and Understanding a Problem:

1. Draw a Diagram if at all possible
2. Examine Special Cases
   (a) to exemplify the problem,
   (b) to explore the range of possibilities through limiting cases,
   (c) to find inductive patterns by setting integer parameters equal to 1,2,3,... in sequence.
3. Try to simplify it, by using symmetry or "without loss of generality."

For the Design and Planning of a Solution:

1. Plan solutions hierarchically.
2. Be able to explain, at any point in a solution, what you are doing and why; what you will do with the result of this operation.

For Exploring Solutions to Difficult Problems:

1. Consider a variety of equivalent problems
   (a) replacing conditions by equivalent ones,
   (b) recombining elements of the problem in different ways,
   (c) introducing auxiliary elements,
   (d) reformulating the problem by (i) a change of perspective or notation, (ii) arguing by contradiction or contrapositive, or (iii) assuming a solution and determining properties it must have.
2. Consider slight modifications of the original problem:
   (a) choose subgoals and try to attain them.
   (b) relax a condition and try to re-impose it.
   (c) decompose the problem and work on it case by case.
3. Consider broad modifications of the original problem:
   (a) examine analogous problems with less complexity (fewer variables).
   (b) explore the role of just one variable or condition, the rest fixed.
   (c) exploit any problem with a similar form, "givens," or conclusions; try to exploit both the result and the method.

For Verifying a Solution:

1. Use these specific tests: Does it use all the data, conform to reasonable estimates, stand up to tests of symmetry, dimension analysis, scaling?
2. Use these general tests: Can it be obtained differently, substantiated by special cases, reduced to known results, generate something you know?

Problem 8: Given that lines intersect if and only if they are not parallel, and that any two points in the plane determine a unique line between them, prove that any two distinct nonparallel lines must intersect in a unique point.

Problem 9: Given 22 points on the plane, no three of which lie on the same straight line, how many straight lines can be drawn, each of which passes through two of those points?

Problem 10: If a function has an inverse, prove that it has only one inverse.

Let us take an extreme case. The student who understands virtually nothing of these problems may think that problems 8 and 9 are related because they both deal with lines in the plane. On the other hand, the mathematician sees that both problems 8 and 10 deal with the uniqueness, and are likely to be proved by contradiction. Therefore he may perceive of those problems as being similar.

Suppose 100 students were given these 3 problems, and asked to group together those problems which they thought were related. (They might decide that none of the problems was related or that two of them were, or that three of them were.) One could then create a 3 by 3 matrix, where the i,j-th entry represented the number of students who considered the i-th and j-th problems to be related. A comparison of these matrices before and after instruction, for both experimental and controlled groups, could indicate changes in the students' perceptions of the way these problems were structured mathematically.

In fact, my cluster analysis used 32 problems, with a 32 x 32 matrix for analysis. There were clear differences between experimental pre- and post-test scores, and controlled pre- and post-test scores. Further comparison with "expert" sorting of the problems is also planned. The full tally is yet to come, but the preliminary results are encouraging.

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Issues in Mathematical Problem-Solving Research*

by

Frank K. Lester, Jr.
Indiana University

During the seventies mathematics education researchers devoted more attention to problem solving than any other topic in the mathematics curriculum and there is every indication that this condition will exist for some time. While there is some evidence that problem-solving research is beginning to be investigated in a systematic way, it is difficult to synthesize the myriad of studies due to such factors as lack of agreement on what constitutes problem solving, how performance should be measured, what tasks should be used, and what the key variables influencing behavior are. Indeed, the nature of mathematical problem solving appears to a certain extent to be so complex and subtle as to defy description and analysis. However, there are some factors (variables) associated with problem solving that are inextricably linked together. These factors can be classified into four categories with each category involving many parts. It is immediately evident that these categories are not disjoint; in fact they are so closely related that it often is difficult to determine to which category a particular factor belongs. The four categories are:

I. Subject Factors - what the individual brings to a problem.
II. Task Factors - factors associated with the nature of the problem.
III. Process Factors - the overt and covert behavior of the individual during problem solving.
IV. Environment Factors - features of the task environment which are external to the problem and the problem solver; instructional factors comprise an important class of factors.

Categories I and III are so closely related that some further clarification is warranted. Variables within the Subject Factors category are associated with individual traits and background (e.g., previous mathematical background, age, sex, cognitive style, familiarity with certain problem types). Variables in this category serve to characterize the subject. Category III variables (Process Factors) relate directly to the behavior of the individual during problem solving. The manner in which the problem solver

Issue III. Characteristics of problem solvers greatly affect behavior and consequently severly limit generalizability of results. The kinds of subjects to use in problem-solving research is a topic of much discussion. For example, while knowledge about the processes good problem solvers use is clearly important, it is less clear that average ability problem solvers can be taught to use these processes. Should subjects used in mathematical problem-solving research be "mathematically talented" or of "average" ability?

Instruction-Related Issues.

There is every reason to believe a substantial portion of future problem-solving research will focus on instruction. For this reason it is appropriate to point out the key issues directly associated with instruction.

Issue IV. There is little agreement regarding how best to improve problem-solving performance beyond the obvious fact that attempting to solve problems is a necessary ingredient. Common points of view regarding problem-solving instruction include:

a. Having students solve many problems - no direct instruction;
b. Teaching unitary skills (tool skills);
c. Teaching heuristic strategies;
d. Modelling good problem-solving behavior and having students imitate this behavior;
e. Some combination of the above.

Issue V. In addition to a lack of consensus regarding the best ways to enhance problem solving, there is no accord about what should be the nature of problem-solving improvement. Some researchers interested in problem-solving instruction have focused on the improvement of students' abilities to use particular strategies or skills, while others have considered improvement only in terms of an increase in the number of correct solutions. Also, in many cases no attention has been given to whether newly acquired facility in solving a particular type of problem transfers to solving a different type of problem. Indeed, the extent to which various types of transfer of training should be expected is an open question.

Issue VI. The extent of instructional treatments in recent mathematical problem-solving research varies from about one week to several months with relatively short treatments being the most common. Treatments should be extensive enough to allow not only for full explication of ideas and procedures, but also to provide ample opportunity for students to practise the procedures being taught.

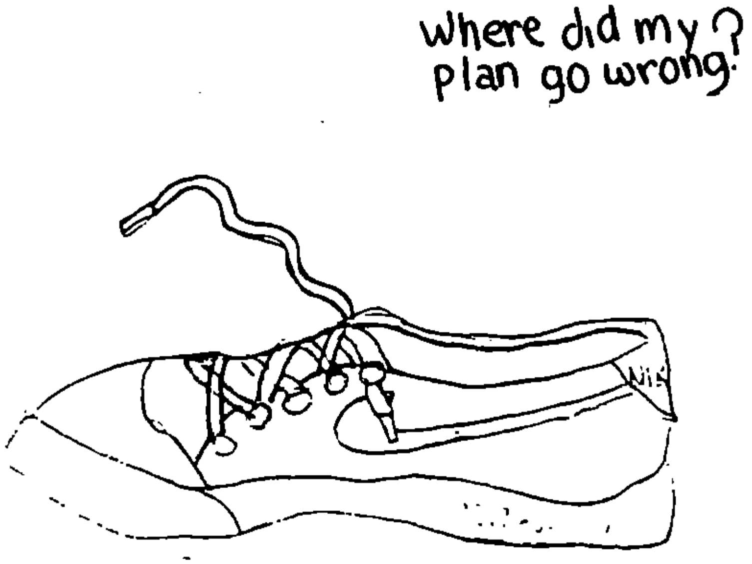
Research Methodology Issue.

There is a single issue related to research methodology. Typically, methodological issues become less important when a sound theoretical basis guides the conduct of inquiry. However, the present lack of adequate problem-solving theories makes issue VII a current, although possibly short-term, concern. This issue is neither the unique domain of problem-solving researchers, nor of the same level of importance as the first six issues but it is important enough to warrant serious attention.

Issue VII. There are no generally accepted methods or instruments for measuring performance or observing behavior during problem solving which are clearly reliable and valid. Thus, the kind of instrumentation which is appropriate for a particular purpose remains an issue. The most popular instruments are of two types: paper-and-pencil tests and protocol analysis based on "thinking aloud" or retrospection. Each of these types has serious weaknesses. Paper-and-pencil tests are notoriously unreliable measures of problem-solving processes and often use only routine problems. Protocol analysis suffers equally serious limitations. Forcing the problem solver to think aloud during problem solving may have a deleterious effect on performance and the problem solver typically is unable to articulate all, or even the most important, thought processes. Retrospective analysis is often criticized for the unreliability of accounts of behavior, including all the cognitive processes used, which are reconstructed by a problem solver after an attempt to solve a problem. Should more or less emphasis be given to the development of paper-and-pencil tests? Should more or less emphasis be placed on the development of procedures for collecting and analyzing problem-solving protocols?

The individual researcher must make personal decisions regarding some, or all, of these issues before undertaking problem-solving research. At the same time the problem-solving research community as a whole should give overt attention to discussion of the controversies involved with these issues. It is only through the open exchange of ideas and points of view that progress can be made toward building a large and stable body of knowledge about the nature of mathematical problem solving.

* The ideas expressed in this paper are abstracted from "Problem Solving Research," in (R.J. Shumway, Ed.) Research in Mathematics Education, Reston, VA: NCTM, 1980.



Ye Shall Be Known by Your Generations

by

Stephen I. Brown
State University of New York at Buffalo

I. Introduction

A quarter of a century ago with the advent of the "new math," we were persuaded that understanding the structure of mathematics through such pedagogical strategies as discovery learning would attack frontally the most pervasive issues regarding the meaning of mathematics and the roles of the teacher and student as well. That myth has passed for the most part, but we are now bombarded by a new set of slogans as we are cajoled to teach problem solving as our new salvation.

Why have we not been led through the pearly gates in the past, and why is the prognosis not much better now? There are many reasons, of course, not the least of which is that curriculum specialists frequently do not appreciate a valuable intuition that is built into the bones of the best of practitioners: that schools involve a complicated interaction among people whose interests are frequently and fundamentally in conflict, and "diddling" a little bit with a curriculum or with a teaching strategy may bypass some of the most basic components that must be confronted if change is to occur.

Sarason (1971), a clinical psychologist, carefully observed efforts to implement a new math program in a school system several years ago. He articulated a number of characteristics of the school setting that may have accounted for a great deal of the failure of the new curriculum. Among these were:

1. The relation between teacher and pupil is characteristically one in which the pupil asks very few questions.
2. The relation between teacher and pupil is characteristically one in which teachers ask questions and the pupil gives an answer.
3. It is extremely difficult for a child in school to state that he does not know something without such a statement being viewed by him and others as stupidity.
4. It is extremely difficult for a teacher to state to the principal, other teachers, or supervisors that she does not

understand something or that in certain respects her teaching is not getting over to the pupils.

5. The contact between teacher and supervisor (e.g., supervisor of math, or of social studies) is infrequent, rarely involves any sustained and direct observation of the teacher, and is usually unsatisfactory.
6. One of the most frequent complaints of teachers is that the school culture forces them to adhere to a curriculum from which they do not feel free to deviate, and, as a result, they do not feel they can, as one teacher said, "use (their) own heads."
7. One of the most frequent complaints of supervisors or principals is that too many teachers are not creative or innovative but adhere slavishly to the curriculum despite pleas emphasizing freedom. (p. 35)

His main point is that no amount of development and delivery of a new curriculum per se could succeed if efforts were not made to take into account some of the above characteristics. If these characteristics threaten the success of any new curriculum project, how much more must they tend to abort our efforts to implement a problem-solving curriculum -- a curriculum that supposedly not only honors the intelligence of the student, but that suggests a reconception of the authority of the teacher!

But our disinclination to appreciate the complexity of the social context of school is only part of what has doomed earlier curriculum movements to something less than smashing success. Even if it were legitimate to isolate the issues of curriculum from those of the social setting, we have tended to foist a unidimensional view of curriculum issues on teachers who once more frequently intuit correctly that things are more complicated than theorists would have them believe.

Our intention in this paper is to attempt to point out how it is that the slogans of the 1950's and 1960's cannot exist in isolation from those of the 1970's and 1980's, and that any serious efforts at curriculum and instruction reform must search for important linkages. Thus, while the focus of this paper is more modest than the concerns of Sarason, it would be a mistake to implement a program that neglects to incorporate the two areas of concern.

Before beginning our analysis, it is worth admitting a bias that will soon become very obvious. That is, I believe it is a serious error to conceptualize mathematics as anything other than a human enterprise which, among other things, helps to clarify who we are and what we value. That bias will "ooze out" rather than be dealt with frontally at least in the first few sections. It will assume a central position, however, by the end of the paper.

We turn first to a consideration of a concept that was at the forefront of the modern math movement in the 50's and 60's -- that of understanding.

II. On Understanding

How poorly understood both in terms of pedagogical practice and psychological research is the notion of understanding! Let us begin with some comments made by Henri Poincare (1961) in an essay of his in which his focus was on mathematical creativity as a way of seeing some of the difficulty with regard to mathematics:

> How does it happen there are people who do not understand mathematics? If mathematics involves only the rules of logic such as are accepted by all normal minds; if its evidence is based on principles common to all men, and that none could deny without being mad, how does it come about that so many persons are here refractory?
>
> That not everyone can invent is no wise mystery. That not everyone can retain a demonstration once learned may also pass. But that not everyone can understand mathematical reasoning when explained appears very surprising when we think of it. And yet those who can follow this reasoning only with difficulty are in the majority: that is undeniable, and will surely not be gainsaid by the experience of secondary school teachers. (p. 33).

Now it is one thing to attempt to answer reasonable sounding questions but it is frequently far harder to find unwarranted assumptions that relegate such questions to the class of pseudo-questions. In some cases the "excess baggage" is obvious (e.g., "When did you stop beating your husband?"). In other cases it takes the wisdom of centuries to expose pseudo-questions. Mathematics itself is a beautiful example of a discipline in which such "unpacking" required enormous labor pains over hundreds of years. Almost since the creation of Euclidean geometry, questions were asked about derivability of the parallel postulate from other postulates. It seemed that the fifth postulate (through a given outside point one line can be drawn parallel to a given line) was much less fundamental than the others (like "two points determine a line"). For a very long period of time, people tried in vain to prove the fifth postulate from the others. It was, however, not until people began to have the courage to rephrase their questions -- exposing hidden assumptions -- that progress was made. Notice the subtle difference between the following two questions?

> "How can you prove the parallel postulate from the rest of Euclidean geometry?" versus
> "What happens if we assume that the parallel postulate cannot be proven from the rest of Euclidean geometry?"

It was the courage to ask the question in the second way that gave birth not only to non-Euclidean geometry but to a totally new conception of the nature of mathematics.

Enough of a digression! What has Poincare done in inquiring why people have difficulty understanding mathematics? I believe that he has brought along excess baggage that fits somewhere between the obviousness of the husband-beating question and the extreme subtlety of the original parallel postulate question.

In contrasting mere understanding and creating, he assumes that they are different states of mind or different kinds of activities. Understanding mathematics is one thing -- creating is another! What is it that leads us to believe that "mere understanding" is so simple a construct and so divorced from an act of creation?

We have been misdirected partly by a technological input/outgo view of the world to conclude that "coming to understand" is a relatively straightforward matter. The viewpoint is connected to a commonly held myth regarding good teaching. Good teachers are supposed primarily to be able to explain things well and to be able to "get us" to understand things that we could not do well on our own! I would like to explore a more dynamic model of understanding mathematics. I will do so by reflecting on personal experiences in teaching or learning and by examining curriculum as well.

Part/Whole Thinking and Mathematics

We begin with one of the most serious problems in understanding -- that of the attempt to relate the part to a whole or to a context in coming to understand a concept.

Consider the following two problems:

(1) In the set of natural numbers N = 1, 2, 3, 4, 5, ... , we define a prime number as a number with exactly two different divisors. So, 5 is prime because 1 and 5 are its only divisors. 4 is not prime because it has 3 divisors: 1, 2, 4.

Now instead of focusing on the set of natural numbers, look at E = 2, 4, 6, 8, 10, 12 . . . . Using the same definition of prime as in N, list the primes in E.

(2) Amy Lowell (the poetess of human liberation of her day) goes out to buy herself some cigars. She has a bunch of change in her pants pocket. Reaching inside, she feels around and finds that:

-- she has nickels, dimes and quarters
-- there are 25 coins all together
-- there are three more nickels than dimes
-- the total amount of money is $7.15.

How many coins of each kind does she have?

We invite you to think about these two problems before reading on, without considering your level of mathematical sophistication as a particular hindrance or a help in working them out.

Recently, I discussed problem (1) with Zvi, a mathematics teacher (Brown, 1978). Below is a rough replay of our dialogue:

Zvi: The only prime in E is 2.
Me: Why?
Zvi: Because 2 is *the only* even prime.
Me: Why isn't 6 prime in E?
Zvi: It can't be.
Me: Why?
Zvi: Because 6 is divisible not only by 1 and 6 but by 2 and 3 as well.
Me: Is it?
Zvi: Yeah.
Me: *How do you know?*
Zvi: Just do it.
Me: Can we forget about E for a minute and look back only at N?
Zvi: Sure.
Me: I think that 5 is not prime in N.
Zvi: Why?
Me: 'Cause 5 is divisible by 2.
Zvi: No way!
Me: Why not?
Zvi: 'Cause you get 2 1/2 and you can't get "1/2's" when you divide.
Me: Why can't *you get* halves?
Zvi: You can't because when you divide the answer has to be "even" -- no fractions.
Me: What's wrong with fractions?
Zvi: They're not allowed when you try to divide in the natural numbers.
Me: Why not?
Zvi: They're just not. When you divide in the natural numbers, things have to go "evenly."
Me: Can we look again at E?
Zvi: Sure.
Me: Does 2 divide 6 in E?
Zvi: Yes, and so 6 is not a prime as I said before.
Me: Can you think of a way of conceiving of "divides" that would make the statement "2 divides 6" false in E?
Zvi: No! 2 does divide 6.
Me: But does it do it in E?
Zvi: Yes.
Me: *How do you know that 2 divides 6?*
Zvi: *Because* 3 x 2 = 6.
Me: But 3 doesn't belong to E.
Zvi: So?
Me: Why wouldn't you let me say that 2 divides 5 in N?
Zvi: 'Cause then you'd get a fraction for an answer.
Me: What's wrong with that again?
Zvi: I told you already. They're not allowed when you try to divide in the natural numbers.
Me: Can you give me a reason for why they're not allowed?

Zvi: They're just not. If you divide in N, it has to go evenly.

Me: Can you look at the situation in E again and find a way of <u>excluding</u> 3 as an answer when you try to see if 2 divides 6?

Zvi: No.

Me: Well, suppose we think about 2 1/2 as not being permissible as an "answer" when you try to divide 5 by 2 not because things don't go "evenly" but because 2 1/2 <u>isn't</u> a member of N!

Zvi: That's not really why. But so?

Me: Suppose you use that reasoning in E. Then 2 does <u>not</u> divide 6 because the only candidate 3, that could make it true does <u>not</u> belong to E! <u>Therefore 6 is prime in E.</u>

Zvi: You can't do that.

Me: Why not?

Zvi: Prime makes sense only in N, and it's only because 2 does not divide 5 "evenly" that 5 remains a prime in N.

Me: What does "evenly" mean again?

Zvi: No remainders when you try to divide!

I apologize for the long dialogue, but I hope the interchange is beginning to raise some questions about the nature of understanding. Before discussing things, let us turn to the second problem.

I have given a modified version of the Amy Lowell problem to many people over the past few years, and I have met with astounding results. Those people who have studied a great deal of mathematics almost always begin with something like:

Let d = number of dimes
d + 3 = number of nickels
25 - (d + d + 3) = number of quarters.

They then set up an equation taking into consideration the fact that the total amount of money is $7.15. In attempting to solve the equation, they frequently end up with a negative, fractional value for d. What do they do? Most sophisticated mathematicians then either look over the calculation to see where they may have made an error or they take out a new sheet of paper and do the same thing over again -- once more ending up with a fractional negative answer for d. I have seen this type of behavior continue for a half hour resulting in an even greater sense of frustration than when Pavlov presented an ellipse to a dog after training it to expect food if the event is preceded by a straight line and punishment if by a circle!

Let us explore the Amy Lowell example a little bit first. After some initial frustration, perhaps, it is possible to explore this problem <u>intelligently</u> and no amount of repeated equation solving will in itself reveal an intelligent approach to the problem. One intelligent approach would be to attempt to see the larger picture instead of immediately committing oneself to setting up an equation. In trying to relate the pieces to each other, it is possible to solve the problem by observing that if you had 25 coins and even if all of them were quarters, then it would be impossible to have $7.15. There is no way to relate 25 nickels, dimes and quarters so as to yield $7.15!

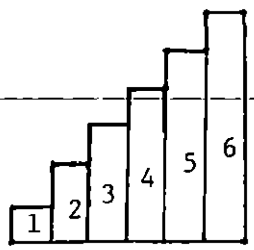
Well, what are we getting at here other than demonstrating an insightful vs. a "plodding" approach to a problem? The difference between the two approaches goes much deeper than that. In one case a primarily linear approach is taken to solve a problem -- an approach in which information is added bit by bit without regard for the large picture, and more importantly without any serious attempt to have intelligence prevail. In the other approach, an effort is made to view the pieces in relationship to the whole and to other pieces and to see how an intelligent reformulation of the problem reduces it to one of mental arithmetic rather than algebra.

Though we see this distinction (linear vs. holistic approaches) clearly in the more creative act of trying to solve the Amy Lowell problem, it is also present in more subtle form in Zvi's inability to "merely understand" what I was driving at in the dialogue. His problem was apparently not *only* that he would not allow us to extend the use of the word "prime" to an unfamiliar context, but that the *concept* of prime number could not be extended because he was unable to view failure of divisibility in N in any terms other than whether or not the "answer comes out even." He was incapable of seeing "coming out even" as only a partial view of what divisibility in N might mean and more importantly he was not capable of seeing that the concept of prime was not a concept in isolation but rather one that made sense in a context. That is, he had conceived of "prime" in such a way that it lacked "hinges" to the broad context of domain.

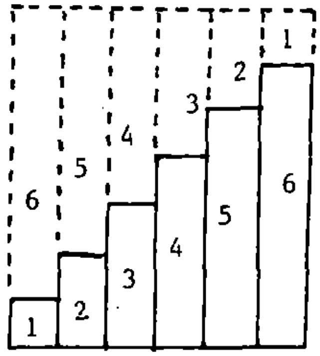
All of this from a mathematics teacher who could follow and teach any number of procedures involving primes in N -- getting prime factors of a number, adding fractions, reducing to lowest terms and so forth! He could do everything expected of him -- except perhaps understand the *concept of prime*.

The problem as we have identified it *so far* is one that Wertheimer (1945) addressed over a quarter of a century ago. Concerned with gestalt psychology and its ability to point out what distinguishes productive from non-productive thinking, he chose many mathematical examples to illustrate the point. As a matter of fact, he made the famous mathematician Gauss an almost popular hero by exploring in gestalt terms an alleged story of him as a youngster faced with the task of finding the sum of the natural numbers from 1 to 100.

It is by looking at 1 + 2 + 3 + 4, + ... + 97, + 98 + 99 + 100 in gestalt terms that we can begin to see how a shortcut might have emerged historically. Anyone who thinks of adding up the pieces in terms of a geometric staircase model (below) might readily see how the pieces could be viewed as part of a whole.



It is possible to view the above structure as only half a configuration embedded in a rectangle as in the figure below:



Compare this illuminating approach with the following found in many texts:

Start with: $1 + 2 + 3 + 4 + 5 + \ldots + 97 + 98 + 100$. Now count backwards and arrange so that we have the following:

1	+	2	+	3	+	...	+	98	+	99	+	100
100	+	99	+	98	+	...	+	3	+	2	+	1

If we now add vertical pairs, we end up with:

$$101 + 101 + 101 + \ldots + 101 + 101 + 101$$

Instead of finding the desired sum, we thus have twice the desired sum. How many times do we have 101 as a term in the sum? It is obviously 100 times. But then $100 \cdot 101$ gives us the value of <u>twice</u> the sum of the numbers from 1 to 100. Since we want the sum only once, the answer is $1/2 \cdot 100 \cdot 101 = 5050$.

It is not all clear how one might have thought up this scheme for finding the answer by examining twice the sum and writing one of them "backwards" unless one has seen a geometric type scheme as above.

Explaining how gestalt thinking works, Wertheimer comments,

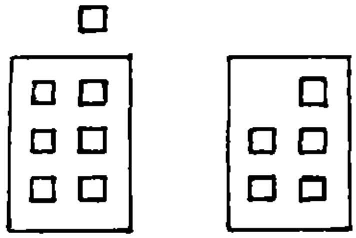
> The aim of discovering the inner relation between structure and task leads to regrouping, to structural reunderstanding. The steps and operations do not in the least appear to be a fortuitous, arbitrary sequence; rather they come into existence as parts of the whole process in one line of thinking. They are performed in view of the whole situation, of the functional need for them, not by blind accident nor as thoughtless repetition of an old rule-of-thumb connection.

A Second Look at Part/Whole Thinking

One reading of cognitive gestalt psychology is that its focus on the relationship of the part to the whole is essentially an inner state of mind. This is especially so if one reviews the experiments in perception and pays attention to references such as "flashes of insight" and the like in the literature. This is certainly suggested when Wertheimer claims:

> Often it is not even necessary to assign a task for sensible response to appear: it grows out of the inner dynamics of the situation. (p. 108)

And he illustrates his point with figures such as:



Apparently, it is a sign of gestalt thinking to place the "lonely" square from the top of the left figure to the inside of the right one.

Though the gestalt metaphor is valuable, we can find much of value in encouraging people to relate parts and wholes in ways that go beyond the purely cognitive "inner state" construct.

That is, a concerted effort on the part of educators to explore with youngsters the many different ways in which parts and wholes do or should relate to each other would seem to have enormous payoff. So much of our educational experience places us in the position of having or being parts of a whole, and yet we are given almost no encouragement to reflect upon that experience. In the previous subsection we focused on part/whole relationships from the perspective of specific problems, and we pointed out shortcomings that result from an inability to attempt to see how parts and wholes relate to each other. But this inability exists not only with regard to a problem and its components. It is also an issue with regard to a course in the context of one's mathematical experience and with regard to one's mathematical experience in relationship to other experiences.

Schools are notorious for encouraging a "piece-meal" approach to virtually everything. Youngsters are given very little opportunity to reflect upon how the pieces fit together. Frequently, there is no rationale, and if there is one, it may be frightening -- dealing more with conformity and authority than with the fostering of intelligence. That is, as we have come to understand dimensions of the "hidden curriculum," we see that much of what passes for education is not necessarily in the best interests of the children, nor of their teachers. Learning to wait and to be obedient are hardly designed to serve the intellectual interests of children, though they serve an important rite of passage in a technological society that has internalized these values.

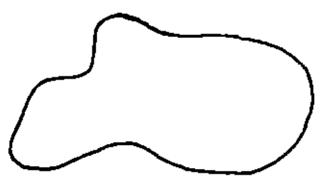
The problem is poignantly expressed by Matthew Lipman, Ann Sharp, and Frederick Oscanyan (1977) in their program of teaching philosophy for children.

> One of the major problems in the practice of education today is the lack of unification of the child's educational experience. What the child encounters is a series of disconnected, specialized presentations. If it is language arts that follows mathematics in the morning program, the child can see no connection between them, nor can he or she see a connection between language arts and the social studies that follow, or a connection between social studies and physical sciences.
>
> This splintering of the school day reflects the general fragmentation of experience, whether in school or out, which characterizes modern life...The result is that each discipline tends to become self-contained, and loses track of its connectedness with the totality of human knowledge...(p.6)

How can we as educators help students at all levels make better sense out of their fragmented lives and ours?

Consider for example the issue of relating parts to a whole not with regard to a specific mathematics problem, but with regard to an entire course. The basic assumption that students are not wise enough to see a whole picture until they have experienced completely all the pieces and thus that pieces are perceived temporally prior to wholes is at best a mischievous assumption and one that is responsible for a great deal of student malaise, animosity, and rejection.

One of my most educationally worthwhile teaching experiences occurred when I had the courage to begin a calculus course not by defining derivatives and definite integrals as I had done for a number of years, but by giving each student in the class a shape like:



I spent three weeks having them try (on their own and in collaboration with others) to find out an area for that region. A great deal of frustration ensued. Some very brilliant investigations took place. Beyond a number of individual differences, however, what emerged was an almost "instantaneous" (3 weeks compared to an academic year) appreciation for what calculus was getting at.

Halmos (1975), a first generation student of R. L. Moore, captures the essential elements of this experience in the following remarks:

For almost every course one can find a small set of questions...questions that can be stated with the minimum of technical language, that are sufficiently striking to capture interest, that do not have trivial answers, and that manage to embody in their answers, all the important ideas of the subject. The existence of such questions is what one means when one says that mathematics is really all about solving problems, and my emphasis on problem solving (as opposed to lecture attending and book reading) is motivated by them. (p. 467)

Having begun to explore the part/whole phenomenon as an essential and poorly appreciated ingredient in understanding -- even in the mild sense of following an argument -- let us now turn to another dimension that challenges a passive interpretation of what is involved in coming to understand: problem generation.

Problem Generation

We begin once more with a small anecdote. First consider the problem below:

The ten's digit of a two digit number is one half the unit's digit. Four times the sum of the digits equals the number. Find the number.

This problem was shown to me by a beginning mathematics teacher who was distressed upon discovering it in a text for one of her high school classes. She worked it out and based upon the solution decided that it was a mistake and that she would not assign it to her students. Why? If you tried to work this out algebraically, you most likely arrived at something like:

Let t = ten's digit
u = unit's digit

Then $t = 1/2u$
$4(t + u) = 10t + u$
.
.
.

you probably then ended up with something like:

$$6u = 6u$$

Therefore, any u should work and the only restriction on t would be that $t = 1/2u$. Her point is that unlike all digit problems she had done before, this one seemed highly irregular in that it implied many solutions. For example, 12, 24, 36, 48 at least would work. Thus:

$$4(3 + 6) = 36!$$

Her method of handling this irregularity was to dismiss it (though she did privately make inquiries).

How might one try to make sense out of this anomaly? In addition to asking "why?" directly, one reasonable way of proceeding would be to "probe" the phenomenon by asking any number of questions such as:

(1) Are there any other problems like this digit problem for which a similar phenomenon results? For example, when can I get the same results if the ten's digit is three times the unit's digit?
(2) To what extent is the result a consequence of the base selected? Would I get the same result in a base other than ten?
(3) What kind of problem for a <u>three</u> digit number would yield similar anomalies?

Lest we lose sight of the larger picture here, let us consider what is behind "probes" of the kind we are suggesting. At bottom is an inclination to <u>generate problems</u>. Though problem solving has become an explicit area of concern of mathematics educators at all levels, we seem to have lost sight of the fact that problem solving is rooted in a much more fundamental activity: <u>problem generation</u>.

Students who understand that it is legitimate to expect them to solve problems do not believe that it is similarly reasonable to expect them to <u>pose</u> problems. The irony of it all is that no one ever is capable of solving a <u>problem</u> (not just doing an exercise) without formulating some new problem along the way. The fact that students are disinclined to see mathematics (or perhaps any school activity) as a problem-posing enterprise first occurred to Marion Walter and me a number of years ago at which time we were team teaching a course to Harvard Master of Arts in Teaching students. Having a definite "lesson plan" in mind, we began by asking the students to give us some answers to:

$$x^2 + y^2 = z^2$$

We got dutiful responses like 3, 4, 5; 5, 12, 13, and even a few "wiseacre" ones like: 1, 1, $\sqrt{2}$; -1, 2, $\sqrt{5}$.

It occurred to us afterwards that the students were answering a non-question. No one (including us at the time) had realized that: $x^2 + y^2 = z^2$ is <u>not</u> a question, but an open form about which <u>many</u> questions could be asked or problems posed. For example, find x, y, z so that the Pythagorean equality misses by 1; or find three bona fide fractions that satisfy the equality; or give a geometric interpretation of the equality; and so forth.

If people are disinclined to generate problems even when the context is a "natural" one -- that is inspired by anomaly, surprise, doubt -- then how much more are they reluctant to do so when they are just being asked to "follow" or to "understand" someone else's presentation? Let us return again to the problem of Zvi and primes.

Zvi had learned very well what a prime number is according to the definition he was given. However, because he viewed "understanding" as a passive affair, it never occurred to him to go beyond the conception which he was "given": A number is prime if the only things that "go evenly" into the number are 1 and the number itself.

What else might he have done -- even if he were asked to accept that definition? If he had been inclined to see the world in less authoritarian and more "elastic terms," he might have asked, for example:

1. What's so special about numbers that have only two divisors? Can numbers have 3 divisors? (4 and 9 being two examples).
2. Can numbers have four divisors?
3. How many numbers are there with 5 divisors?
4. I wonder if there is some way to visualize prime numbers.
5. What is the biggest prime number?
6. Why are we focusing on divisibility? Is there something like primes with subtractions?
7. Are there any fractions that are prime?

These questions could be expanded at will, and we perhaps should be cautious in criticizing Zvi for never generating such a list. After all, we all have a finite time to invest in any activity and this was one that Zvi chose to "accept." The problem is, however, that believing that "mere understanding" is what Poincare depicted it to be -- a passive activity or achievement in which one keeps his "nose clean" -- Zvi had acquired very little understanding of any aspect of mathematics.

If you accept that in some sense one must create knowledge (as implied by the criticism that Zvi never asked any of these questions) in order to understand anything, then you might reasonably ask why a teacher (as opposed to the students) could not generate these questions to initiate understanding. The problem at least is that each of us comes to any experience with a highly idiosyncratic view of the world. The kinds of questions that make sense to me in terms of solidifying understanding are very different from those that make sense to you. Some of the questions I have asked above imply a need for visualization which others do not; some are asking for a very large context and some for a smaller one. Some are open to many alternative conceptions and others to a limited number.

It is not the disinclination to view any one phenomenon as "elastic" and "probe-able" that limits one's ability to understand so much as world view that conceives of understanding in such an inert way.

Summary

~~Using anecdotes and reflecting upon personal educational experiences, I~~ have attempted to suggest that a behavioristically rooted model of "understanding" has grave limitations. Referring both to the issue of part/whole and of problem generation, I have tried to illustrate how it is that understanding is a personal and aggressive construct in the sense that no one is capable of doing your understanding for you. There is perhaps a sense of human liberation captured by a stance which makes the "it" in the most "cool" of all subjects less mechanistic and more of a private phenomenon. In what ways, however, is this "mushier" conception of the "it" capable of shedding light on the self as part of an educational experience? We turn now to that question.

III. Towards An Integrated Notion of Self

In 1913, Dewey (1975) produced an extremely important work that has been a well-kept secret in educational circles, Interest and Effort in Education. He focused on a question of fundamental importance to practitioners and one which dividends the advocates of "free," "open," and "traditional" education. He asked whether teachers ought to take major responsibility for "interesting" children in the perhaps dull substance of their education, or should they expect youngsters to exert "effort" on their own in order to master material even (or especially) if it is "uninteresting" to them? All of us have, perhaps, heard or made arguments that support these two conflicting points of view. Opinions about the benefit or harm of "sugar coating" content frequently falls back upon disagreements with regard to these two poles.

Dewey begins his book by making a plausible case for each point of view, and then proceeds to point out what he conceives of as a basic fallacy in both of them.

> The common assumption is that of the externality of the object, idea or end to be mastered to the self. Because the object or end is assumed to be outside self it has to be made interesting; to be surrounded with artificial stimuli and with fictitious inducements to attention. (p. 7)

Having linked the need to make things interesting to the erroneous notion of separation of self and object, he finds the same fallacy in "effort" as a fundamental obligation of the student.

> Or, because the object lies outside the sphere of self, the sheer power of "will," the putting forth of effort without interest has to be appealed to. (p. 7)

He sees a resolution of the dilemma to be in the direction of unification of object and self.

> The genuine principle of interest is the principle of the recognized identity of the fact to be learned or the action proposed with the growing self; that it lies in the direction of the agent's own growth, and is, therefore, imperiously demanded, if the agent is to be himself. Let this condition of identification once be secured, and we have neither to appeal to sheer strength of will, not to occupy ourselves with making things interesting. (p. 7)

In further blurring the sharp distinction between "self" and "object," Dewey reveals himself as the unacknowledged originator of the new popular concept of "hidden curriculum" in education. He comments:

> The question of educative training has not been touched until we know what the child has been internally occupied with, what the predominating direction of his attention, his feelings, his disposition has been while he has been engaged upon this task. <u>If the task appeals to him merely as a task, it is as certain psychologically as is the law of action and reaction physically, that the child is simply engaged in acquiring the habit of divided attention; that he is getting the ability to direct eye and ear, lips and mouth to what is present before him so as to impress those things upon his memory, while at the same time he is setting his thoughts free to work upon matters of real importance to him.</u> [emphasis my own] (p. 8,9)

If there is any portion of the curriculum that has become the hallmark of separation of object and self, it is mathematics. What kind of thinking is needed in order to provide a different conception of their relationship? We shall in the following subsections provide possible directions for integrating the two, without attending to any detailed scheme of implementation.

<u>Part/Whole Thinking A Third Time</u>

We ought, perhaps, to be more cautious in making such harsh judgments of Zvi and "blind" efforts on the Amy Lowell problem. How might we expand some of our criticism in the subsection entitled "A Second Look at Part/Whole Thinking" so as to focus not primarily upon "making objective sense," but upon greater self understanding?

Consider those people who approach the Amy Lowell problem in an algorithmic way. Now, it is possible for them to justify their approach. After all, the problem did resemble ones they had done before and there is certainly considerable efficiency involved in placing similar problems in an already well-worked-through mold. Such an argument would then select the existing algebraic structure as a "whole" within which this small problem is a part. Those who decide to view the problem in such a way as to relate the parts to the whole <u>within the problem itself</u> (rather than to the whole of an algebraic structure) could justify their procedure on other grounds. They might argue, for example, that though efficiency may be a virture -- all other things being equal -- <u>this</u> case appeared to be <u>different</u> enough from others they explored to warrant a reconsideration.

Well, why did they consider it different? Why were the algorithmic thinkers willing to run the risk of missing the <u>uniqueness</u> of the Amy Lowell problem for the sake of efficiency? To what extent were they out to see each experience in mathematics as part of a more general phenomenon, and thus easily incorporated into existing structures, and to what extent were they desirous of viewing new phenomena in a unique way? To say that something is to be viewed uniquely does <u>not</u> imply that it is not to be seen as a part of something larger -- but only that the something larger need not necessarily be an already well-established structure.

Now a great deal of deliberate mathematics education does err in the direction of diminishing novelty. In fact the search for order, for isomorphic structures and substructures, for harmony where apparent disharmony exists, is frequently taken as the hallmark of mathematics. Whether or not this _ought_ to be the case is an interesting and important question, but it is perhaps desirable for us to transform this philosophy of mathematics question into an educational one.

An educational transformation would have us provide many opportunities for students to approach problems and to view solutions either as _unique_ experiences or as something fitting into existing structures. To what extent and under what circumstances do they feel comfortable with the uniqueness of a particular mathematical experience? Why? How does that reflect upon the desirability of finding uniqueness in non-mathematical circumstances as well?

It is quite conceivable that by understanding their stance towards the value of uniqueness or the unexpected in a mathematical context, students of mathematics may begin to understand how they value uniqueness and novelty in other areas as well.

An effort to relate in a personal way the role of the unique and the unexpected in attempting to assimilate and accommodate new worldly input may move us in the direction of self-understanding.

Problem Generation Revisited

Earlier we suggested how understanding mathematics _per se_ requires a form of problem generation. Here, we turn to relationships between problem generation and self-understanding.

There is an important sense in which we are known to others as well as to ourselves by the kinds of questions we ask and the problems we generate. Such activity is frequently more courageous and involves considerably more risk than appears on the surface. The asking of questions and the generation of problems when done in a spirit of inquiry not only reveals an initial state of ignorance and a desire to know, but also has embedded within a set of assumptions. Such activity tells the world something about the specifics of what we believe and in addition has the ability to inform others of the intensity of these beliefs.

Are we willing to propose "foolish sounding" questions and under what circumstances? For what purposes? Earlier we discussed how far several hundred years investigating the parallel postulate revealed basic assumptions which were in fact incorrect (i.e., that the parallel postulate can be proven from the rest of Euclidean geometry). What is less obvious, however, is the enormous courage required to even ask the question in what has become a twentieth century spirit (showing logical independence of propositions).

In 1822, Johann Bolyai wrote a letter to his father Wolfgang in which he disclosed his new and daring form of the parallel postulate question. Johann did so with intellectual curiosity, but also with great fear that he would be perceived as risking his sanity by even asking the question. How could it make sense to even conceive of a world in which there were no lines parallel to a given line through a given outside point? How indeed! It was only after the dust was cleared from an intellectual revolution from Copernicus through Darwin through Freud through Einstein that we could say that sanity had prevailed.

How much do we and our students risk when we ask questions that have embedded the potential for even minor revolutions? Especially if our question verges on foundational issues, we run a thin line between meaninglessness and revolutionary finds.

Consider the following incident that occurred in a number theory course of mine. We were trying to show that if a perfect square is even, then the square root of that number is also even. (For example, 16 and 36 are even, and so are 4 and 6.) An indirect proof led us after several stages to the following assertion:

$$2n + 1 = 2m.$$

That is, an odd number would have to equal an even number. Just as we were about to "cap" the proof by a <u>reducto ad absurdum</u> claim, someone shyly asked:

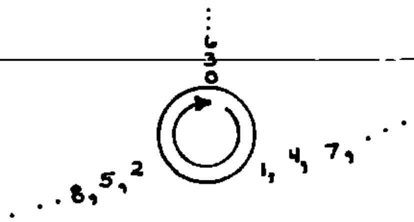
"Why can't an odd number equal an even one?"

Why indeed! Using any number of experiential arguments in the set of natural numbers, we can come to believe that it is impossible for an odd number to equal an even one.

Despite all that, we tried to push the logic further. If $2n + 1 = 2m$, then simplifying we'd get $1 = 2(m - n) = 2 \cdot x$.

So now, we are led to conclude that twice an integer must equal 1. All our experience rebels against the conclusion, but where do we go from here? A natural inclination would be to try to prove that $1 = 2x$ has no solution in the set of natural numbers or integers. We had just begun the course and no one had adequate machinery to pursue that issue at the time, so we tried another tack. Instead of trying to prove the equality false as we know it to be in the set of natural numbers, we began to explore where it might be true. The equation $2 \cdot x = 1$ obviously has a solution in the set of fractions, but that system appears to be different in so many ways from the natural numbers that the find was unrevealing.

After some highly creative exploration, we found a system that "felt" closer to the natural numbers but within which $2 \cdot x = 1$ has a solution: Clock arithmetic.



Starting with zero and moving clockwise through 1 and 2, and then circling back to the zero for 3 and 1 for 4, and so forth, we can "wrap" all the integers around the circle forming 3 separate classes. Choosing to define addition and multiplication in a "natural" way, we find that there is a number x so that $2 \cdot x = 1$; the equivalence class generated by two works.

But what does that say about odds and evens in clock arithmetic? And how does clock arithmetic compare with the natural numbers? What properties does one have that the other lacks which enables us to find numbers that are both odd and even in one system but not in the other?

What have we done here? By shifting the context slightly (from natural numbers to clock arithmetic) a foolish question emerges as the starting point for some deep exploration -- including the opportunity to re-explore the question in the original context with greater insight! On a minor scale, we too have performed a "Bolyai." We took a very foolish-sounding question seriously and found a home within which we emerged a hero!

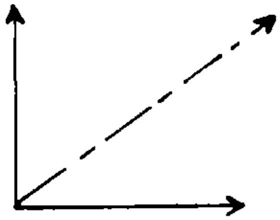
As with Bolyai, pushing the question challenged every bit of experience, and finding a non-trivial home for the question was a testimony to our ability momentarily, at least, to suspend logic in favor of a creative leap (keeping in mind the level of experience of the class at the time).

That the exploration was mathematically rewarding and successful in some sense should not blind us to the potential for interplay between logical and creative thinking in mathematics. We tend to stress the former as the hallmark of mathematical thought so much that we lose sight of the fact that problems are generated by human beings and that such generation makes use of the mind not as a logic machine alone but as an instrument for poetic thought as well.

We are capable of generating not only by modifying the attributes of a given problem (as we suggested in the Zvi example) and not only by refuting experience and logic, but also by making use of extralogical tools of thought such as imagery, metaphor, and the like.

Unfortunately, so much of mathematical training at all levels unnecessarily constricts rather than liberates us by focusing on the narrowly conceived end product of following a <u>proof</u> that we lose both the ability and the inclination to generate ideas through the use of these tools.

I recall as a junior taking my first graduate level mathematics course. It was finite dimensional vector spaces offered by a world-famous mathematician. The first day we were told that the only things that count are the axioms and definitions together with rules of logic, and that it was solely that apparatus to which we ought to appeal in the doing of mathematics. Anything else was to be interpreted as a bastardization of the discipline. He proceeded to list the axioms of a vector space, and as sometimes happens under such circumstances, he got stuck. He stood before us, mumbled a few words and then turning his back to the class, and blocking the blackboard with a stomach that was adequate for the purpose, he sketched a diagram that looked something like:



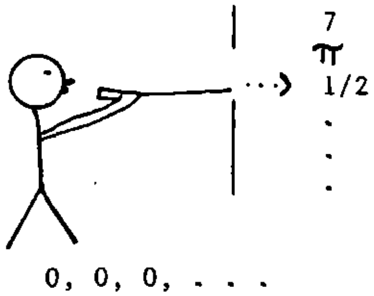
In an attempt then to be consistent with his original advice, he quickly erased his sketch and proceeded to list a few more axioms and to prove a few theorems "based solely upon definitions, axioms and logic."

If there is one thing I look back on proudly with regard to that experience, it is that I dropped the course immediately, and took it the following semester from a mathematician who, though less world-renowned, was more in tune at the very least with his own style of learning.

Now, this is an extreme case of confusing generation and verification, but if we are warned against using even isomorphic type diagrams in this extreme case, how much more of a heinous crime to use imagery of a looser nature!

All kinds of images and metaphors direct my activity not only at problem generation but at problem solving and in just plain recalling as well. This machinery is apparently the most well-guarded secret when it comes to mathematical thought.

For me, "zero" is not the midpoint of an infinite line, nor is it primarily the identity element under addition. Instead, it is the following "fellow" from multiplication.



He holds a machine gun, looks through a peep-hole and as each of the numbers marches before the wall he annihilates them and collects them as little images of himself.

It seems to me than an important part of a humanistic education and experience is disclosing sharing, and understanding the significance of the images that direct our thinking. If that can be done well within the context of mathematical thinking, where can it not be done?

In addition to imagery, use of metaphor is a powerful problem generator. Two brief personal illustrations will make my point. One day I was "doodling" with the following multiplication facts:

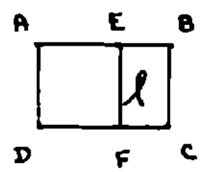
$$\begin{aligned}1x3 &= 3\\ 2x4 &= 8\\ 3x5 &= 15\\ 4x6 &= 24\\ 5x7 &= 35\end{aligned}$$

I was wondering what sense to make out of that when the metaphor of "striving" popped into my mind. I saw each number to the right of the equation in an existential sense as "striving" to become something it had not yet become. Instead of what I had there, I saw:

1 x 3 is almost 4
2 x 4 is almost 9
3 x 5 is almost 16
4 x 6 is almost 25

The right hand side formed perfect squares, and what started out as a metaphor ended up as an exploration that led to a totally new algorithm for doing multiplication (Brown, 1974).

At another time I was learning about the golden rectangle:



ABCD is a golden rectangle if I can construct a line ℓ parallel to a side so that a square (AEFD) is created together with another rectangle (EBCF) similar to the original. I saw this phenomenon not as squares and rectangles, but as organisms giving birth to another generation yielding the most ideal form possible and one offspring that is a miniature version of the original. Again this metaphor led to the generation of many different problems that had never been dreamed of before (Brown, 1976).

IV. Summary

Where are we now? What do we do with these various observations? We are suggesting that if for purposes of understanding mathematics an important part of the curriculum is part/whole thinking and problem generation then for purposes of understanding self a reflection on those same dimensions is needed. What is there that encourages or inhibits each of us from generating problems? To what extent do we make use of (and perhaps hide) images, metaphor and fantasy in generating problems? What kinds of risks do we personally take in the questions we ask and the problems we generate?

To what extent are we influenced not only by the machinery of logic, anti-logic and poetry as we attempt to generate, but by the presence of a role diet of questions whose solution is simple once the teacher reveals the "trick," serves to confirm these feelings.

To a math teacher or a strong math student, a question whose solution goes counter to intuition can be a delight. To a math-anxious student, mat -- in general -- tends to go against intuition. Some textbooks have made overt attempts to discourage students from trusting or using their intuitio My old text for introduction to geometry began with a series of optical illusions. The message was "don't trust your eyes because things are often not what they seem." Geometry was not supposed to make sense. Until you could establish something with a rigorous deductive proof, it was best not believe it or use it.

If we are trying to teach children that making reasonable guesses and then testing the conjectures is a legitimate problem-solving technique, the we must also teach that math is basically reasonable and intuition can be trusted.

Questions which require an "intuitive leap" can be most satisfying for students who persevere to the point where the breakthrough occurs. The majority of students, however, probably lack whatever it takes to persevere that point. The result will be one more wrong answer to a question which "tricked" them.

For example:

> A visitor arrives at a hotel seeking accommodation for a week. He has no cash but the hotel owner agrees to accept one tiny gold ring as payment for each night the visitor stays in the hotel. The problem is that the seven gold rings are linked together in a chain. The visitor doesn't trust the proprietor enough to make seven day's payments in advance and the proprietor doesn't trust the visitor to withhold payment until the end of the week. The visitor, therefore, must cut links of the chain so that payment of one link can be made on a daily basis. The question is, what is the least number of rings the visitor would have to cut so that he could make payments for his accommodation on a daily basis?

Very little thought will show that it can be achieved with six cuts. little more thinking and most students will arrive at three cuts. As this a considerable improvement over six, and there is no way of knowing whether this is the correct solution or not, most students will probably be satisfi with this. A few students may persevere to the realization that the proprietor can use rings he already has to make change and, in fact, the payments can all be made after cutting just one ring (third from one end). Will the majority who got an answer of three be amused or stimulated by thi clever solution or will they feel that they have been fooled again and be l likely to want to try another one? Questions like this, whose solution is maximum or minimum, don't provide a method for determining whether a soluti is correct or not. This lack of the possibility for verification can resul in a question being abandoned before a correct solution is found, or in a l of time being spent on an unproductive line of reasoning.

A problem which recently appeared on a Canadian math contest paper goes as follows:

> You are given one hundred coins in ten piles of ten coins each. Ninety of the coins are genuine and weigh exactly one gram each. All of the coins in one of the ten piles are counterfeit. These coins look and feel like the genuine coins but each one of them is .1 grams heavier than a genuine coin. The question is, if you are given a miniature bathroom type scale which can measure correct to .1 grams, what is the minimum number of weighings you could make to identify with certainty the counterfeit pile of coins?

The inclusion of this question on the contest paper was unfortunate for a couple of reasons. First, it was not an original problem and had appeared in problem books previously. Some test writers may have encountered the question before and could acquire full marks with almost no expenditure in time or effort. Those seeing it for the first time could spend a considerable amount of time creating any number of schemes for weighing the coins, without knowing when to quit, as they could not know whether their solution was a minimum. If this work was done in their heads or on rough paper (with no breakthrough to the correct solution) they could receive no marks for some high quality thinking.

(For those who have not seen this problem solved, the correct solution is a single weighing. This can be accomplished by placing one coin from the first pile, two from the second, etc., and noting the discrepancy between what the scale should read and what it does read. This multiple of .1 grams will reveal the counterfeit pile of coins.)

It would be a rather different question if it asked, "How is it possible to identify the couterfeit pile with a single weighing?" Then a person doing the problem would know when a correct solution had been found.

Problems like these need to be presented to those students who have the tenacity and abilities to arrive at correct solutions or learn from their failures. If handled appropriately they can be valuable for the majority of students, provided they are given credit for effort and headway in the process of solving the problem, not just for getting the correct solution. This might be achieved by presenting a problem like the coins question, limiting the time spent by students on the problem, and discussing the progress towards its solution, in class, for a few minutes each day. At first any procedures which revealed the counterfeit coins would be considered as good solutions. Then strategies for identifying the coins in fewer weighings could be discovered and discussed. Finally, with feedback and some hints, the "best" solution might be discovered. Such a procedure could reduce a sense of having failed in many students and provide insights into the problem-solving methods.

Some years ago I ran off class sets of mazes for students to work on when they had finished their work. The initial reaction from almost the entire class was "no thanks." This changed when one of the weaker students picked up a maze one day and managed to work through it in two or three minutes.

He was rather surprised and blurted out, "Hey! I got it - it's easy!" The mazes were popular from that day on and students often asked if they could take some to do at home.

The significant thing seems to be that they were afraid to try until it seemed clear that they would succeed. Many stronger students, for whom the mazes could not be considered as challenging, seemed pleased to attempt and complete the mazes.

Many problems will yield to a certain amount of perseverance on the part of almost any student. The following question, which I have used with a number of classes, is usually solved by anybody who keeps at it for a period of time. A certain amount of luck also helps and it is often one of the weaker students who arrives at a correct solution first.

A "ruler" is designed from a blank strip of wood exactly 13 centimetres long. You want to be able to measure integral values from 1 cm to 13 cm with this ruler without moving the ruler and with only four marks on the ruler at four points of your choosing. Where should you put the marks?

If we are to teach children to be effective problem solvers we will have to develop in them some confidence in their ability to solve problems and a predisposition to attempt to solve problems in the first place.

Confidence can be built through exposure to problems which are easy to solve or which require only perseverance and effort rather than the brilliant "aha" which usually eludes the majority of students.

Questions which by their nature have solutions which are not verifiable should be used in such a way that feedback can be injected at appropriate times. Similarly, questions with the possibility for a lot of work being spent chasing down blind alleys should be monitored to reduce frustrations. Trick questions should be used with discretion.

There are hundreds of problems available to teachers from a variety of sources. A judicious choice in the problems we present and the way in which we present them can help in attaining the goal of producing better problem solvers.

Problem Solving: Goals and Strategies

by

Jesse A. Rudnick and Stephen Krulik
Temple University

What is problem solving? Haven't we been teaching problem solving all along? What are we doing when we go over all those word problems in the textbooks; isn't that problem solving? The answers to the latter two questions are yes, no, or maybe.

To clarify this state of confusion, let's first define what we mean by problem solving. Problem solving is the means by which an individual uses previously acquired knowledge, skills, and understanding to satisfy the demands of an unfamiliar situation. The existence of a problem implies that the individual is confronted by an unfamiliar situation, one for which no apparent solution or path to a solution is readily seen. The key words here are unfamiliar, and for which no apparent solution or path to a solution is readily seen. You see, once a student has seen a problem and been shown a method of solution, additional exercises similar to the original (even though the names, the setting, and the numbers have been changed) are no longer problems. They are exercises, merely drill and practice. You might say that the students are engaged in solving problems, but they are not engaged in problem solving.

Problem solving is a process, a systematic search by the individual through the given data, and a synthesis of the findings into a neatly executed solution. Indeed, the solution is not the final answer, but rather the entire process from the original confrontation to the final conclusion. Thus the word problems that appear in most textbooks do not provide problem-solving experiences because most teachers do the initial thinking for the students and then provide them with a model, a method, or an algorithm for doing each of the several types of problems. The fact is that most of us and the children as well, try to identify the "type" of problem and then attempt to recall how we do that particular type. There is really nothing wrong with this procedure, if you are trying to solve specific problems, or problem types. Unfortunately, this does not serve the purposes of problem solving.

Most of the word problems that appear in a textbook are designed to support mathematical skills that have just been developed. Indeed, this is the crucial point of the recommendations of the National Council of Teachers of Mathematics (1980) that skills should be developed to give the students the power to resolve problems; the problems are not materials to support the skill. Skills, in the absence of the ability to utilize them in appropriate settings, are useless!

It is probably unrealistic at this time to believe that the school mathematics curriculum will be immediately rewritten with a central theme of problem solving. If it ever happens, it will not be for a considerable length of time. However, in the meantime, or until such time as substantial changes are made, classroom teachers can, on their own, implement the N.C.T.M. recommendations for the 1980's by making problem solving an ongoing activity in their classrooms. Some problem-solving experience can probably be worked into each and every classroom lesson, if the teacher feels it is important enough and prepares the lessons accordingly.

Before we consider problem-solving activities for the classroom, let's discuss problem solving itself. How do <u>we</u> solve problems that <u>we</u> have not seen before? What do <u>we</u> do when confronted by a perplexing situation that needs resolution? Perhaps if we can respond to these questions, we will gain some insights that will prove helpful in the school classroom.

Polya (1957) states that successful problem solving involves four steps:

1. Understanding the problem
2. Selecting a strategy
3. Solving the problem
4. Looking back at the problem.

When we are confronted by a problem, the first thing that we usually do is read the problem. This means (1) look for key words; (2) try to understand the situation; (3) visualize the situation in your mind; (4) look for the relationships that exist between the data; and (5) find out what is being sought. Then, according to Polya, a strategy should suggest itself which, if treated carefully, will result in a correct answer. (We consciously avoid using the word "solution" at this point, because we want to emphasize the fact that in problem solving the solution is the entire process, not just the answer.)

But, strategy selection is not just an automatic outcome of reading or understanding the problem. How does one select a strategy? Indeed, what are the strategies one can employ to find an answer?

We have identified eight of the most widely used strategies. They are not unique, nor is the list by any means exhaustive. We list them below:

1. Pattern recognition
2. Working backwards
3. Guess and test
4. Simulation or experimentation
5. Reduction or looking for a simpler problem
6. Exhaustive listing
7. Logical deduction

8. Data representation
    8.1 graph
    8.2 equation
    8.3 *algebraic expression*
    8.4 table
    8.5 chart
    8.6 diagram

Although it would be impossible to illustrate each of these strategies in this paper, let us take a look at some of them, with some illustrative problems for each. As an illustration of the Guess and Test strategy, consider the following problem:

> A textbook has been opened to pages 26 and 27. If we multiply these two numbers, their product is 702. Jane opened her math book and found that the product of the numbers on the two facing pages was 8,556. To what pages was her book opened?

As your students read this problem and consider the information it contains, they should be brought to realize that the numbers which appear on facing pages of a textbook are always consecutive. Thus, one of them is even, one of them is odd, and their product will always be even. If the students are encouraged to try a pair of successive numbers, say 34 and 35, they find that the product of these numbers is less than 8,556. If they try a larger pair, say 110 and 111, they find that this product is too large. Thus the pair of numbers we wish to find lies somewhere between these two pairs. This is a good notion for the students to learn at this time -- the idea of approaching a limit from both sides. Now, by trying various products within this range, they should find the correct pair, 92 and 93. Notice too, that this is an excellent time for the teacher to introduce the concept of a square root. If the students take the square root of 8,556 (92.4986) they need only take the whole numbers which lie on either side of the square root, namely 92 and 93. If practice in multiplication is not needed at this time, this problem is a good one to explore with a calculator. (This is basically a consecutive integer problem, which, in Algebra, would be solved with the equation $x (x + 1) = 8{,}556$.)

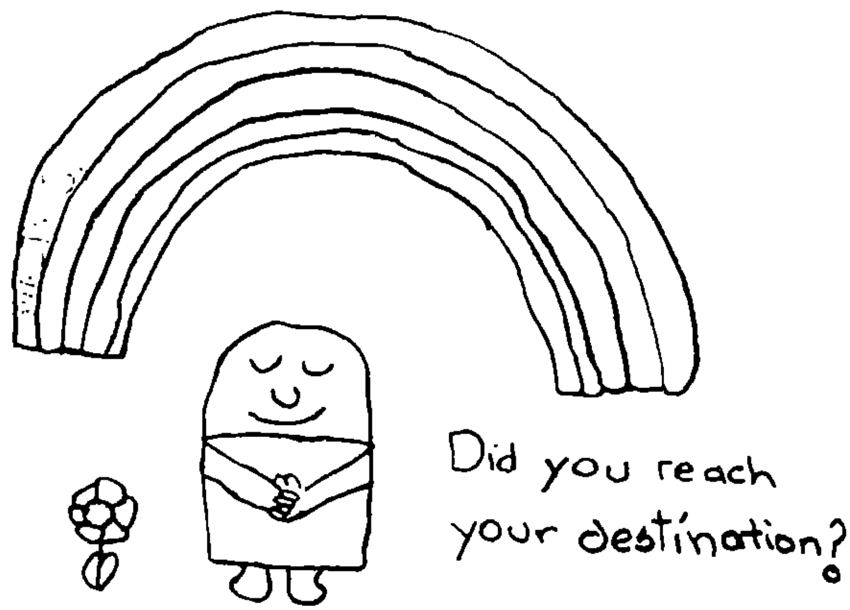
In many cases, it requires a combination of strategies to resolve a problem. For instance, an elegant solution to the following problem employs three of the strategies on our list: simulation, recording data in a table, and pattern recognition.

> Eight members of the Harlem Globetrotters are warming up for their game with the Washington Generals. The players are in a circle. Each player passes the ball to each of the other players. How many times is the basketball tossed?

This problem could be solved by actually having eight students stand in a circle and *toss a basketball or crumpled piece of paper to each of the other* students while counting. However, you may wish to help the students prepare a table similar to the following:

Carrying Out the Plan

Looking Back



The Problem-Solving Cartoons used in this monograph were created by the students in West Block and North West Block of the University Heights Elementary School, Calgary, Alberta.