Finding Irrational Roots of Polynomials

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In the honors course in Grade 11, as well as in Mathematics 1010 at Memorial, one spends considerable time teaching how to find, when they exist, rational roots of a polynomial with integer coefficients. Several worth-while topics are usually taught or recalled at this time, namely the Fundamental Theorem of Arithmetic to prove the main theorem about possible rational roots, the relationship between roots and linear factors, and synthetic division as a fast way of checking the possible rational roots. (The next natural followup to this topic at the university is the more general algorithm known as Kronecker's Theorem for factoring completely into irreducible (prime) factors any polynomial with integer (or rational) coefficients.

In addition to spending time finding rational roots of a few very carefully selected polynomials, one should spend more time finding *all* the real roots of polynomials with real coefficients. This is possible now that all students have easy access to the calculator.

As many of you know, it is impossible, in general, to write a formula for the roots of a polynomial in terms of arithmetic operations of the coefficients and extraction of roots for a polynomial of degree greater than or equal to five. For lower degrees, there are formulas for the roots. In fact, in 1830 a brilliant young mathematician, Evariste Galois, two years before being killed in a gun duel over a girl before his 21st birthday, gave a very beautiful, necessary and sufficient condition that a given polynomial equation be solvable in radicals (in terms of arithmetic operations and extraction of roots) and hence showed that, in general, fifth and higher degree polynomial equations cannot be solved in radicals. (See the historical note on pages 392 and 393 of Johnson *et al*., 1975). In particular, he showed that, $f(x) = a_p x^p + \ldots + a_1 x + a_0 = 0$, where a_i are rational numbers, cannot be solved in radicals provided p is a prime greater than or equal to five, f is irreducible (prime) over Q, and f has two complex (conjugate) roots only. Two such examples are $f(x) = x^5 - 6x + 3$ and $f(x) = 3x^7 - 7x^6 - 7x^3 + 21x^2 - 7$. By elementary calculus, one can show that the first polynomial has three, and the second, five real roots. A very easy sufficient condition for a polynomial with integer coefficients to be irreducible, known as Einstein's criterion, states that if we can find a common prime divisor of each of the coefficients of f except the leading one and it does not divide the constant term any more than once then the polynomial is irreducible. For the first polynomial, we see that the prime 3 satisfies the conditions of irreducibility; for the second, the prime 7, satisfies the conditions for irreducibility. Hence, we are guaranteed for the above two polynomials there are no explicit formulas for the roots.

The method for finding irrational zeros described by Johnson *et al* on pages 379 to 381 is a good one and is a place where linear interpolation can be reviewed. Let us consider the example, $f(x) = x^5 - 6x + 3$, the problem being to find the three real zeros to within some prescribed accuracy. Let us assume for now that we know the most efficient way to evaluate f(x) on our calculator. We find that $f(-1) = 8$ and $f(-2)$

= -17; hence, one root r lies between -2 and -1. We may wish to find the next digit by trial and error since, if the initial guesses are far apart or near a relative, extrema for the function using linear interpolation may not save us any time. So guessing intelligently, we see that f(-1.7) = -1 and f(-1.6) = 2.11 so our root r lies between -1.7 and -1.6. By comparing slopes where r = -1.6 - a, we have a/.1 = 2.11/(2.11 + 1) and hence a = .07, so we guess that r is either between -1.66 and -1.67 or between -1.67 or -1.68. We find that f(-1.68) = -0.3 and f(-1.67) = 0.03; hence r is between -1.67 and -1.68, and so we continue. Depending on the accuracy of our calculator, we could ask for r accurate up to eight or nine places after the decimal. In fact, the real roots correct to eight places are -1.67093526, 1.40164188, and 0.50550123.

For the polynomial $f(x) = 3x^7 - 7x^6 - 7x^3 + 21x^2 - 7$, one of the irrational zeros correct to eight places is -1.23707354, since one can check that f(-1.237073543) = 7.6×10^{-8} and f(-1.237073544) = -6.5×10^{-8}. Noting that f(-2) = -699, f(-1) = 11, f(0) = -7, f(1) = 3, f(2) = -43 and f(3) = 1451, we know approximately where all the zeros are. In fact, the points (-1, 11), (0, -7), (1, 3) and (2, -43) are the relative extrema of the function.

For the information of the secondary school mathematics teachers, we record here some detailed solutions to some of the problems in section 8-10 of the Johnson text. The three real zeros of $x^3 - 5x + 1$ are -2.330058739, 2.128419066, and 0.201639674. For exercises 6, 7, and 8 on page 382, the unique real zeroes of each of $x^3 + x - 3$, $x^3 - 3x^2 - x - 6$ and $x^3 + 3x^2 - 3x + 2$ are 1.213411662, 3.706527954, and -3.900571874 respectively.

The next problem is that of evaluating f(x) on our calculators in as efficient a way possible. For an arbitrary polynomial:

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ of degree n, the first method of evaluating is

the straightforward one of going from left to right evaluating term by term. For the purposes of comparison, let us assume that entering a number, storing a number, or recalling a number from memory requires *one* key operation each. Evaluating f(x) in this way on a calculator with true algebraic logic requires 6n + 1 key steps. Even for the Hewlett-Packard (HP) calculator with reverse polish arithmetic, this method requires 6n + 2 key steps. For the average student, this may well be a suitable method for finding f(x). However, there is one major problem if students have Texas Instrument (TI) calculators. The TI calculators (as well as others such as Canon), as contrasted for example with the Casio Calculators, cannot raise a negative number to an integer power, hence, cannot evaluate f(x) by the above method for negative x and degrees greater than two. (Try evaluating $(-2)^3$ on the TI calculators.)

A second, faster method of evaluating f(x) is rewrite f(x) in the nested form:

$$f(x) = (\ldots(((a_n \cdot x + a_{n-1}) \cdot x + a_{n-2}) \cdot x + a_{n-3})\ldots + a_1) \cdot x + a_0$$

and start evaluating from the inside out. The calculators with true algebraic logic will hinder us here. Instead of using the bracket feature on your calculator, use the equal button (if your calculator has true algebraic logic). If f(x) is evaluated by the key steps: enter a_n, times, enter x, store x, plus, enter a_{n-1}, equals, times, recall x, plus, enter a_{n-2}, equals, etc., then 5n+2 by steps are required. However, on the HP calculators, only 3n+5 steps are required. If we had used the brackets, then 6n+1 key steps are required, so we might as well have evaluated f(x) in the more straightforward way described earlier.

In any case, you should get to know your particular calculator, because you may be able to discover shorter methods of evaluating f(x). For example, using a TI-35, one can evaluate fourth- and lower-degree polynomials by writing f(x) in the form: $x \cdot (x \cdot (x \cdot (x \cdot a_4 + a_3) + a_2) + a_1) + a_0$, then the argument x need not be stored, since it remains in the register as "times" and "left bracket" are pressed. No higher degrees can be evaluated in this way, since there can be, at most, four pending operations on the TI-35.

On some calculators, e.g., Casio College fx-100, "times" can be omitted before "left bracket."

Using synthetic division to evaluate f(x) will not result in fewer key steps, since this method is exactly the same as the nested form above, requiring 5n+2 key steps.

Students should be given the opportunity to approximate irrational roots of polynomials. Finding rational roots requires learning some very nice algebra, but it is very limited in the polynomials selected. It would also help us in Calculus I at university if students could solve the equation $f^1(x) = 0$, where $f^1(x)$ is a polynomial of degree three or more, and where the roots need not be rational.

References

Johnson, R.E., LL. Lendsey, W.E. Slesnick, G.E. Bates, *Algebra and Trigonometry*. Addison-Wesley, Don Mills, Ontario, 1975.

Probability and Statistics Corner

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This first article for the year of 1982 coincides with another first, the first issue of the *Canadian Mathematics Teacher*. I hope that PS Corner will become a regular feature of the journal, as it has in *Vector*, the journal of the B.C. Association of Mathematics Teachers.

At the recent Leadership Conference on Statistics in the Classroom, organized by the ASA/NCTM Joint Committee on the Curriculum in Statistics and Probability, I had the pleasure of meeting teachers from British Columbia, Alberta, Ontario, and Nova Scotia who were involved at some level of teaching statistics. All of those provinces are moving strongly toward a greater emphasis on statistics and probability in all levels of the curriculum. The majority of those present at the leadership conference were working at, or had a major interest in, Grades K-8. Now that it is generally accepted that future revisions of the mathematics curriculum must "rec-