So much for a start as to what should be in the curriculum. I have not addressed the issue of content for the range of courses at the senior levels, consumer math, statistics courses, computer science, etc., nor have I addressed the issue of what should be omitted. Some people might note certain omissions, such as standard deviation and the normal distribution. Such things will, I hope, be considered in the ensuing discussion.

Teaching Problem-Solving

Dr. Don Kapoor

Associate professor and editor Saskatchewan Math Teachers' Society journal, University of Regina, Regina, Saskatchewan.

The history of mathematics education shows that problem-solving has always been an issue of concern. Most recently, the National Advisory Committee on Mathematics Education (NACOME) and the National Council of Supervisors of Mathematics (NCSM)[1] have indicated that problem-solving is one of the most important basic skills that every student should master in order to survive in our society. The National Council of Teachers of Mathematics (NCTM) has also taken a position supporting the infusion of problem-solving into the school mathematics curriculum for the 1980s.[2] However, there is a great deal of misunderstanding over what problem-solving is and even what constitutes a problem. Jeremy Kilpatrick[3], in his article entitled "Stop the Bandwagon, I Want Off," decries the "use of 'problem-solving' as an empty vessel that we can fill with our own meanings." In many cases, mathematics teachers teach solving problems with the mistaken idea that they are teaching problem-solving.

This article has basically two purposes: to illuminate and reflect on problem-solving and to illustrate, by way of an example, how to teach problem-solving skills in a classroom.

What is a problem?

One common concept of a problem is that of a question proposed for an answer or solution. The teacher has this concept in mind when he/she says to a mathematics class, "Your assignment for tomorrow is to work problems one to ten on page 164." The question that may be either *explicit* or *implicit* in each problem is, "What is the answer?"

A second concept of a problem considers the existence of a question to be necessary, but, unlike the first concept, existence of the question is not regarded as sufficient. The additional conditions pertain to the individual who is considering the questions. What may be a problem for one individual may not be a problem for another. A problem for a particular individual today may not be a problem tomorrow.

The necessary conditions for the existence of a problem for a particular individual are:[4]

1. The individual has a clearly defined goal of which he/she is consciously aware and whose attainment he/she desires.
2. Blocking of the path toward the goal occurs, and the individual's fixed patterns of behavior or habitual responses are not sufficient for removing the block.
3. Deliberation takes place. The individual becomes aware of the problem, defines it more or less clearly, identifies various possible hypotheses (solutions), and tests them for feasibility.

This concept of "problem" holds that when these three necessary conditions are met, a problem exists for the particular individual. Moreover, Cronbach[5] points out, "...it is not posing the question that makes the problem, but the person's accepting it as something he must try to solve." The second concept of a problem appears to be the more useful concept in most educational contexts.

Problem-solving and problem-solving skills

Problem-solving is a process. It is the means by which an individual uses previously acquired knowledge, skills, and understanding to satisfy the demands of an unfamiliar situation. When students solve various "types" of textbook problems (age problems, coin problems, motion problems, etc.), they are simply applying a previously learned algorithm or model to a familiar situation–they are merely solving problems. Only when a student synthesizes what he/she knows and applies it to a NEW situation is the student problem-solving.

Problem-solving is more than a single skill–it is a set of skills that includes the ability to:

Read and understand.
Explore. (That is, play around with a problem, trying different approaches.)
Select an appropriate strategy.
Carry through the strategy.
Look back and extend.

Skills are the building blocks used in solving a problem. Some of the skills are unique to mathematics; others are interdisciplinary. The categories used by the Lane County Mathematics Project (LCMP) are:[6]

1. Problem discovery, formulation
2. Seeking information
3. Analyzing information
4. Solving–putting it together–synthesis
5. Looking back–consolidating the gains
6. Looking ahead–formulating new problems

These categories are further broken down into 46 different skills.

According to Polya[7], the process of problem-solving has four phases. First, we have to understand the problem; we have to see clearly what is required. Second, we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a plan. Third, we have to carry out our plan. Fourth, we have to look back at the completed solution, review and discuss it. An awareness of these skills being used to solve a problem is probably the most important step in the development of a pupil's problem-solving abilities.

George Polya contends that the technique for teaching problem-solving has two aspects: abundant experience in solving problems and serious study of the solution process. He expresses the need for the first in this way: "Solving problems is a practical art, like swimming or skiing or playing the piano; you can learn it only by imitation and practice." However, he warns that imitation and practice are not sufficient. Not only must problems be solved, but the learner's attention must be directed to the methods used. These must be general enough so that they become available for use in solving similar problems in the future. Polya uses the term *heuristics* to describe this way of teaching problem-solving in mathematics.

Implications for mathematics education

Solving problems is human nature itself. We may characterize humans as "problem-solving animals." If education fails to contribute to the development of the intelligence, it is obviously incomplete. Yet intelligence is essentially the ability to solve problems in everyday life. The student develops his/her intelligence by using it, he/she learns to do problems by doing them. Which kind of problems should a secondary school student do to develop his/her ability to solve problems? What is the teacher's role in teaching problem-solving? George Polya[8] in his article originally published in 1949 and later appearing in NCTM 1980 Yearbook sums up the thoughts as follows:

"A boy or girl of high school age and average ability can solve on a scientific level mathematical problems, but no other kind of problems. An average boy of fifteen can obviously not acquire the technique or knowledge or judgment needed in treating on a scientific level a problem of biology or history or physics. Yet, if he has a good teacher, he can, after a while, solve a problem of geometric construction or invent by himself the proof of a simple theorem on the level of Euclid, and Euclid's level is fully scientific.

"This is the great opportunity of mathematics: mathematics is the only high school subject in which the teacher can propose and the students can solve problems on a scientific level. This is so because mathematics is so much simpler than the other sciences. Because of this simplicity, the individual, just as the human race, can arrive so much earlier to a clear view in mathematics than in the other sciences.

"In my opinion, the first duty of a teacher of mathematics is to use this great opportunity: he should do everything in his power to develop his students' ability to solve problems.

"First, he should set his students the right kind of problems: not too difficult and not too easy, natural and interesting, challenging their curiosity, proportionate to their knowledge. He should also allow himself some time for presenting the problem appropriately, so that it appears in the proper light.

"Then, the teacher should help his students properly. Not too little, or else there is no progress. Not too much, or else the student has nothing to do. Not ostentatiously, or else the students get disgusted with the problem in the solution of which the teacher had the lion's share. Yet, if the teacher helps his students just enough and unobtrusively, leaving them some independence or at least some illusion of independence, they may experience the tension and enjoy the triumph of discovery. Such experiences may contribute decisively to the mental development of the students."

However, here Polya reiterates the first condition for discovery: No teacher can impart to his/her students the experience of discovery if he/she has not got it. Therefore, future teacher-education programs should capitalize on discovery approaches much more than they have in the past with emphasis on the practical ability to solve not too advanced problems and the methods of solution.[9]

How to teach problem-solving skills—an illustration

This problem is designed to help teachers teach specific problem-solving skills. Four common but powerful problem-solving skills[10] are:

1. Guess, check, and refine.
2. Look for and/or use a pattern.
3. Make a systematic list.
4. Make and use a drawing or model.

Pupils might use other skills to solve the problems (for example, working backwards, etc.). They can be praised for their insight, but it is usually a good idea to limit the emphasized list of skills directly taught during the first few lessons.

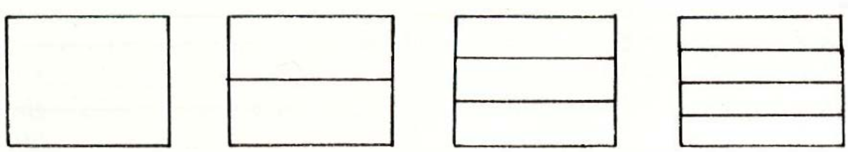
Aim. Looking for pattern.
Level. Grades 7-12.
Problem. Ten people are at a party. Each person shakes hands with each of the others. What is the total number of handshakes?
Heuristics. Start with easier cases. Complete the table. Look for patterns.

Number of people	Number of handshakes
2	1
3	3
4	6
5	
6	
7	
8	
9	
10	
•	
•	
•	
n	

Can you generalize the problem?

Variations of the problem

1.



1 3 6

Notice that:
The first drawing contains one rectangle.
The second drawing contains three rectangles.
The third drawing contains six rectangles.

How many rectangles are in the fourth drawing?

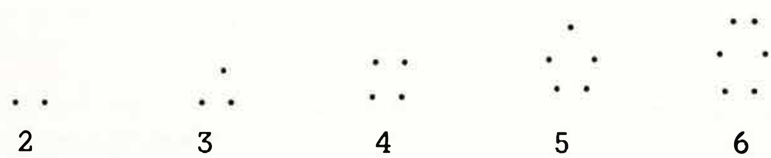
If we continued the pattern, how many rectangles would be in:
The fifth drawing?
The sixth drawing?
The tenth drawing?

Complete the following table.

Drawing	**Number of rectangles**
first	one
second	three
third	six
fourth	
fifth	
etc.	

Can you generalize the problem?

2. Two points determine at most one line segment.
   Three points determine at most three line segments.
   Four points determine at most six line segments.
   Five points determine at most ten line segments.
   Six...
   Seven...
   Eight...
   •
   •
   n



Generalize and check.

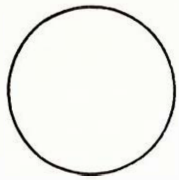
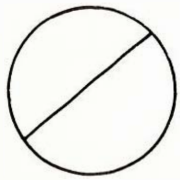
3. Detect the regularity:

1	= 1
1 + 2	= 3
1 + 2 + 3	= 6
1 + 2 + 3 + 4	= 10
1 + 2 + 3 + 4 + 5	= 15

   Generalize and check.
   1 + 2 + 3 + ... + n

Extension of the problem
Find the maximum number of parts into which the interior of a circle can be divided by a given number of lines in the plane.

Complete the following table:

Number of lines in plane (n)	Drawing	Maximum number of regions (n)
0		1
1		2
2		
3		
4		
n		

Can you generalize the problem?

Can you check to see that the formula works?

Footnotes

[1] *National Council of Supervisors of Mathematics Position Paper on Basic Mathematical Skills.* Minneapolis, Minnesota: National Council of Supervisors of Mathematics, 1977.

[2] *An Agenda for Action: Recommendations for School Mathematics of the 1980s.* Reston, Virginia: National Council of Teachers of Mathematics, 1980.

[3] Jeremy Kilpatrick, "Stop the Bandwagon, I Want Off." *The Arithmetic Teacher*, 28(8), April 1981, p. 2.

[4] NCTM: 21st Year Book, pp. 230.

[5] Lee J. Cronbach, "The Meaning of Problems," *Arithmetic, 1948.* Supplementary Educational Monographs, No. 66. Chicago: University of Chicago Press, 1948, pp. 32-43.

[6] LCMP: Introduction to the LCMP Mathematics Problem-Solving Programs, (Oregon ESEA Title IV–C Program).

[7] George Polya, *How To Solve It.* New York: Doubleday and Company, Inc., 1957.

[8] NCTM: *1980* Yearbook.

[9] *Ibid.*, p. 2.

[10] LCMP, *Op. Cit.*