THE CANADIAN MATHEMATICS TEACHER
Winter 1982

PUBLISHED JOINTLY BY
The Alberta Teachers' Association Mathematics Council
The British Columbia Association of Mathematics Teachers
The Manitoba Association of Mathematics Teachers
The Mathematics Council of the Newfoundland Teachers' Association
The Saskatchewan Mathematics Teachers' Society

INSIDE THIS ISSUE

From the Editors

Les Dukowski

This publication is a co-operative venture of the associations of mathematics teachers in Alberta, British Columbia, Manitoba, Newfoundland, and Saskatchewan. It is an experiment. The contributing editors believe that there may be a place for a national mathematics education journal in Canada, but you, the readers, are the ones who should let us know. The editors are of the opinion that there may be issues in mathematics instruction that are particular to Canada as a whole and that should be discussed in a journal of this type. We would like to hear your views. Please write me, Les Dukowski, at the address below. If the general consensus of the readership is that this publication is worth while, then the provincial associations will consider continuing to jointly publish a journal one or more times a year and have it perhaps replace one issue of the journals currently published by each association.

As this is a co-operative effort. I would like to take this opportunity to thank the contributing editors who submitted articles for publication. They are:

George Cathcart, The Alberta Teachers' Association Mathematics Council
Dale Drost, The Mathematics Council of the Newfoundland Teachers' Association
Don Kapoor, The Saskatchewan Mathematics Teachers' Society and
Alan Wells, The Manitoba Association of Mathematics Teachers.

If you have prepared an article that you would like to have considered for publication in a future issue of this journal, please submit it to the editor of your provincial association. All articles dealing with issues in mathematics education, teaching ideas, and topics of general mathematical interest are welcome.

See you in Toronto for the 1982 NCTM Annual General Meeting!

Les Dukowski
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Individualizing Instruction Through Multilevel Performance Problem-Solving Activities

Jim Vance

Jim Vance is a member of the Mathematics Education Department at the University of Victoria.

One of the greatest challenges facing teachers of mathematics in elementary and middle schools lies in providing for the needs of learners with differing abilities and interests. That individual differences among students exist at all grade levels and that the range of these differences increases from grade level to grade level is common knowledge among teachers and has been well documented by research. Jarvis (1964), for example, found that Grade 6 students may vary by as much as seven years in arithmetic achievement, and that even among students with IQs of 115 or higher, the range of achievement is about five years.

Several procedures for accommodating individual differences in mathematics have been explored over the years. One approach that has received considerable attention is self-paced instruction. Self-paced or individualized programs are designed to enable each student to progress at his/her own rate of speed through a sequence of learning units. Research comparing the effectiveness of such programs with conventional teacher-directed settings, however, indicates that in general, and particularly in Grades 5 through 8, the self-paced approach has been ineffective in mathematics (Schoen, 1976). The lack of interaction among students and the reduced contact of students with their teacher are two of the major drawbacks of such programs.

Another organizational scheme at the school level for meeting individual differences is homogeneous grouping of students. Although differences among students will still exist, that approach can reduce the range of ability of achievement within a class, permitting the teacher to adjust instruction to suit the needs of more able, average, or less able students. Grouping on the basis of ability has been found in some studies to be effective, particularly for students at upper ability levels. Where grouping is based on achievement, research findings are more variable. It appears that the teacher is the most important factor in determining the success of any system of grouping (Suydam and Weaver, 1970).

Grouping may also be used within a self-contained class. For example, three instructional groups, studying different content and working at different levels, could be created. An alternative procedure involves flexible grouping with all students studying the same topics. Under this plan each new unit is introduced to the whole class. Following a period of instruction relating to the basic objectives of the unit, a diagnostic test is administered. Three groups—reteach, practice, and enrichment—are then formed (Underhill, 1972). The enrichment group, having mastered the basic material, studies related topics or investigates the content at a deeper level. Students from the three groups come together again for the unit culmination and the beginning of the next topic.

A great many teachers of mathematics in Grades 4 to 8 work with heterogeneous groups of students and do not employ self-pacing or grouping for instruction. The purpose of this article is to describe and illustrate a strategy for accommodating individual differences in mathematics within such a classroom setting. The suggested strategy involves the use of multilevel performance problem-solving activities.

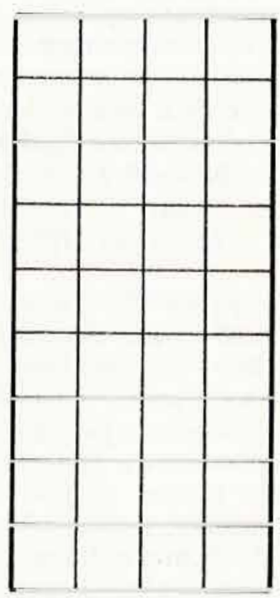
Most, if not all, concepts and problems in mathematics can be presented and investigated at a variety of levels of sophistication. Thus, in teacher classes composed of students of mixed ability, it is not only desirable, but often possible, to provide learning activities that develop or reinforce the important concept or skill of the day's lesson and that permit each student to work at his/her level of understanding. To be appropriate for slow learners, the problem or game setting must allow each student to begin working immediately at some level. For example, the task might initially involve the use of concrete materials or require only basic counting techniques. To accommodate the mathematically talented, the problem could be open to more than one interpretation or have several methods of solution, and solving the problem would suggest to the student new problems to be explored. Capable students should also be encouraged to discover why a result holds and to generalize beyond specific cases.

I'll describe four learning activities that possess these characteristics. The first is a mathematics laboratory activity; the second, a setting for computational practice; the third, a game; and the fourth, a simulated "real-life" problem-solving project. For each activity, a procedure for presenting the task to the class is suggested, possible learning outcomes are described, and ways in which the activity might accommodate the needs of less able and more able students are discussed.

Activity 1. Area and Perimeter.
Materials. Thirty-six square tiles for each pair of students.

Teacher presentation and instructions

Using an overhead projector, the teacher arranges the 36 tiles to form a rectangular region as indicated. Terminology is developed/reviewed using this example. The *base* of the rectangle is (by counting) four units long and the *height* is nine units. The *perimeter* or distance around the outside of the rectangle is (counting) 26 units. The area, which is the number of unit squares covering the region, is (again counting) 36 square units.



The students are then instructed to make other rectangular regions, using all 36 tiles each time, and to find the perimeter and area of each, recording findings in a table as illustrated.

base	height	perimeter	area
4	9	26	36

Learning outcomes and individual differences

All students should be able to use the tiles to form another rectangular region and find its perimeter and area. Low-ability students usually require extensive work at the concrete level and will likely continue to use the materials to form the various rectangular regions throughout the class period. Using basic counting techniques to find the perimeter and area will reinforce the meaning of these concepts and the difference between them. This experience will normally lead students to develop more efficient ways of finding the perimeter and area of a rectangle. They are often surprised, having "discovered" that area is base times height and perimeter can be found by doubling the sum of the base and the height (or taking double the base plus double the height), that these are the formulas they were previously taught. That the area is the same in all cases and the reason why this should be so are, for some students, non-trivial learning outcomes of this activity. Finding that the perimeter of the various rectangles is not constant surprises most students. It is usually noted that long, narrow rectangles have large perimeters and the square the smallest perimeter.

More able learners require less work at the concrete level and move quickly into abstract thinking. In this activity, capable students will likely put aside the tiles after using them to make one or two rectangles, and complete the table using symbolic procedures. The possible rectangles are related to the factors of 36, a formula is used to compute perimeter, and the area is seen immediately to be constant. These students may then be encouraged to draw graphs of the data; for example, they could plot base against height and perimeter as a function of base length. They could also go on to consider the perimeters of families of rectangles with areas of, say, 40, 26, and 31 square units, in order to generalize about the shape with the minimum perimeter for a fixed area.

An investigation of the areas of simple closed curves with a constant perimeter would be a natural follow-up activity. For example, if 20 m of fencing were to be used to enclose a garden plot, what shape would have the greatest area?

Activity 2. Practice and Patterns in Addition.

Teacher presentation and instructions
On the chalkboard, the teacher draws a 2 x 2 grid and writes a number in each of the four cells (Figure 1). With student participation, the numbers are added two at a time; first across, then down, and the sums written as indicated (Figure 2). Two loops are drawn on the upper corners of the grid, and in them are written the sums of the pairs of numbers along the diagonals (Figure 3). Next the two horizontal sums are added, and the answer is written underneath (Figure 4). The vertical and diagonal sums are also added, and it is noted (with appropriate flair) that the three answers are the same.

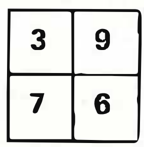


Figure 1

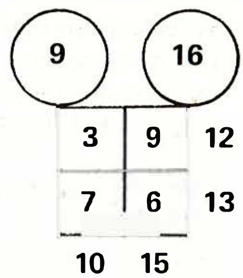


Figure 3

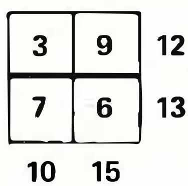


Figure 2

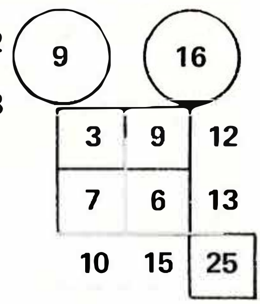


Figure 4

The question that should arise from this presentation is: "Does it always work?" Each student is asked to choose his/her own set of four numbers and try it again. The question "Why does it work?" should then be raised (by the class). After discussion, the setting is used to provide practice in addition. The numbers can be multi-digit whole numbers, decimals, fractions, or mixed numbers. Each student creates and checks his/her own set of exercises by selecting four numbers and following the procedure.

Learning outcomes and individual differences

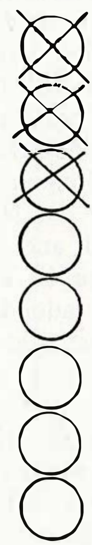
Working through this procedure requires the student to do nine addition questions, and each one must be done correctly in order for it to work. This task, however, appears less forbidding to the slow learner than a worksheet or textbook assignment of nine questions. Moreover the feedback is immediate and comes from the work itself rather than from an external authority. Pupils of lesser ability might at first choose easier numbers to work with, but having completed the task once, might be willing to try it again (nine more questions) with larger or more difficult numbers.

High-ability students can work with numbers requiring more complex computation and will perhaps do more sets of exercises in a given period of time than other class members. This setting, though, can be extended to provide worth-while mathematical activity beyond practice in addition, through consideration of "what if?" questions. More capable students can be encouraged to formulate and investigate such problems as "What if the operation is multiplication, or subtraction?" It still works for multiplication (and therefore provides a setting for practice in this topic). For subtraction, questions such as "Why doesn't it work?" "What does happen?" and "Under what conditions does a particular result (for example, two of the three differences are the same) follow?" can be considered. Thus the thrust of learning activity is shifted from computation to discovery, hypothesis-building, and verification.

Activity 3. Nim (a strategy game).
Materials. Eight counters for each pair of students.

Teacher presentation and instructions

The game with the eight counters is played by two people as follows. Taking turns, each player removes one, two, or three of the counters. The player who is forced to take the last of the eight counters loses.



Learning outcomes and individual differences

After varying amounts of experience in this game situation, students will discover that the player who has the first turn can always win by removing three counters on the first play. The logical strategy of considering all (in this case, three) possible moves by the second player and the corresponding subsequent moves by the first player can then by discussed as a "proof" for the winning strategy.

As previously stated, the point that the first player can *always* win and that chance is not a factor is understood by different students at different times during the activity, and it is clear to an observer when an individual actually gains an understanding of the situation. The teacher should not attempt to explain the logic to someone who has not discovered it, but should simply allow the student to continue to play the game until the idea is understood.

For those who discover and understand the winning strategy, the game is, of course, not a game. However, if any one of the variables is changed, then the process of determining a winning strategy must be repeated. Varying the game conditions as follows provides natural extensions of the learning activity.

1. The player who takes the last counter *wins.*
2. The number of counters is varied. For example, the game is played with nine, twenty-one, or fifty counters.
3. More than three counters (for example, up to four or five or nine) may be removed at each play.

Each combination of the above variables leads to a different winning strategy, but the logical precedure for determining the strategy is the same. Ask the class to establish conditions for one of these games, determine how to win it, and then demonstrate to the teacher that he/she can win every time. For example, a student might say to the teacher, "Start with 15 counters. Remove one to four counters each play. The person who takes the last counter wins. And you have to start."

Having determined the winning strategy under a variety of conditions, mathematically talented students could be challenged to begin to generalize the method. For example, if one to nine counters may be removed, and the person who takes the last counter loses, what should be the first move if you start with 60, 72, 87,... counters? Suppose three, four, five...counters may be removed on a play?

Activity 4. Shopping Spree.
Materials. Department-store catalogues, calculators.

Teacher presentation and instructions
The catalogues are distributed to the class—one to every student or pair of students. The students are told they may "spend" up to $400 to purchase clothing shown in the catalogue, assuming there is a 20% discount on prices listed and that a 6% sales tax must be paid. Since an implicit aspect of the task is to spend as close to $400 as possible (but not more), the procedure is repeated several times with different numbers by each student. In working on the task, two different problems emerge: (1) What is the maximum total list price? and (2) Can I find a desirable combination of items costing a sum close to this amount?

While this activity provides opportunities for practice in estimation, mental arithmetic, and computation, hand-held calculators should be permitted and their use encouraged. The problem-solving strategy of using successive approximations is encountered in exploring the first problem. Investigating the second problem provides students practice in formulating their own questions and identifying relevant data needed to answer them.

While doing this activity, students often generate alternative methods for calculating the final cost of the clothing. Consideration of these procedures can lead to some interesting mathematical questions. I'll list some of these:

1. Could you compute the final price (sale price plus sales tax) of each item separately and then add these figures? (Yes)
2. Could you simply subtract 14% (20%-5%) from the list price to get the final cost? (No)
3. Do you get the same final answer if you first add the 6% sales tax and then subtract the 20% discount as you get doing it the other way? (Surprisingly, yes.)
4. Could you compute the sale price in one step instead of two? (Yes, multiply by 0.8.)
5. Given the list price, could you compute the final cost in a single step? (Yes, multiply by 0.848.)

The equation, final cost = list price x 0.8 x 1.06, provides the answers to questions 2, 3 and 5 and also gives a way of finding the maximum list price directly:

$$\text{maximum list price} = \$400 \div 0.848 = \$471.70$$

To provide practice in solving verbal problems, have each student make up three story problems based on information in the catalogue. Encourage each student to create interesting and challenging questions, but to be able to answer them himself/herself from the information provided. After the students have prepared the problems and worked out an answer key, problem sets could be exchanged (between students of comparable ability). Solutions would be returned to the authors of the problems for marking.

Summary

The purpose of this article was to provide examples of multilevel performance problem-solving activities that can be used to accommodate individual differences among students in a heterogeneous whole-class instructional setting. Such experiences can enable slower learners to enjoy immediate success as they develop and practise basic concepts and skills. At the same time, more capable students may explore the topics in greater depth and engage in study involving higher-level mathematical processes. Participation in these class activities under the guidance of a teacher also provides valuable training that prepares the student for independent work on open-ended problems.

Many exercises and activities in school mathematics can be adapted by the teacher to allow for multilevel work. Two sources of appropriate materials are the "Ideas" section of the *Arithmetic Teacher* (Hirsch and Meyer, 1981) and the "Activities" section of the *Mathematics Teacher* (Hirsch, 1980). The use of multilevel performance problem-solving activities as a means of differentiating instruction in a whole-class setting can contribute in a positive way to the development of an instructional program that is *rich* for all students and *enriched* for more capable learners.

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Teach Nothing About Geometry

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Contrary to a likely interpretation of the title, I am not advocating the deletion of geometry from the mathematics curriculum. In fact, I am quite concerned about the near future of geometry in the curriculum and would not wish to see its position eroded any more than it is. I am concerned because the coming emphasis on and enthusiasm for computer literacy and microcomputer applications could easily push geometry further into the background, simply because geometry doesn't lend itself easily to micro-computer uses.

To return to the title, I am advocating the teaching of nothing about geometry in the sense of "no-thing." "No-thing" implies that we are not talking about a "thing." It is generally held that geometry instruction ought to include practice in space visualization, skills for organizing knowledge about space, attitudes favorable to local space exploration, and so on. But these are no-things that are about things. They are procedural skills, attitudes, or the seeing of relationships. The notion that no-things can be about things is crucial here, since the distinction between things and no-things is frequently the essence of arguments about the value of using geometric activities in the classroom. As an example, the "seeing of geometric relationships" might be acknowledged as an important mathematical goal, but none the less be slighted because it lacks a certain concreteness; for example, it is difficult to define as a teaching objective and is certainly difficult to test. None the less, a growing body of research indicates the existence of certain generalized skills and abilities that are important in problem-solving and applications. We ought to recognize these no-things of geometric activities and acknowledge their importance by insisting on their inclusion in the mathematics curriculum.

To further illustrate some of the points that I have been trying to make, I will describe and use a family of geometric activities. (Incidentally, these activities can easily be put into a game format if desired.) The activities will be defined, and references will be made to the no-things of geometry that they illustrate.

The game of "Turn a Pattern" (TAP)
(This is adapted from Marion Walter's *Boxes, Squares and Other Things.*) I will begin with a discussion of the rules for the two-dimensional version of the game:

1. Use line segments of the same length.
2. The line segments must be placed end-to-end with a right angle at every joint.
3. Play the game first with two line segments, and then with three, four, five segments, and so on.
4. The object is to generate as many "different" patterns as possible in each case.

Discussion
The following no-things would probably be exemplified in the activity above:

1. Two-dimensional space visualization skills would be exercised.
2. Inevitably, the process of defining "different" for Rule 4 above would include some no-things. For example, devising a rational decision rule for calling patterns different would probably be included.

(Are the three-segment patterns above different?)
3. Systematic methods for generating all possible patterns might emerge naturally or could be encouraged. E.g., from 

we can obtain either 

or 

4. Systematic record-keeping could also be practised so that number patterns might be explored.

These are just a few of the possible important no-things that could emerge in such a geometric activity.

This game has an obvious extension into three dimensions. One additional rule forbidding more than two sticks to come from each joint is necessary here. The rest of the rules are the same.

Conclusion
Some additional no-things could emerge in this setting:

1. Three-dimensional space visualization skills would be exercised, particularly when combined with some of the three-dimensional space transformations.
2. By permitting the variation of rules, it would be possible to set up natural comparisons between different systems.
3. The enjoyment of experiencing and exploring the familiar space around us could be enhanced. Most important, this can be done without the need for much formal knowledge of geometry.
4. That questions can be raised and problems posed is a recognition skill that would probably emerge naturally in these activities.

The activities and statements above are only suggestive of the importance of the no-things of geometry. Other activities and discussion points could be devised to illustrate these notions equally well.

As a final note, I would like to make a paradoxical plea that we recognize the possibility that teaching the no-things of geometry may be the most important thing that we can do in geometry.

Reference
Walter, M.I. *Boxes, Squares, and Other Things.* National Council of Teachers of Mathematics, 1970.

In the Red—An Integers Game for Junior Secondary

Reynold Redekopp
Manitoba Association of Mathematics Teachers

"In the Red" is a game that was devised to introduce and review the concepts of integers. It deals, of course, with one thing all students are familiar with—the gain and loss (emphasis on *loss*) of money. The game is set up so most students will end up in the "red" in their accounts.

When developing or reviewing integers, teachers can use this game because students are required to total (or sum), split (divide), and multiply (gains and losses. This knowledge can then be transferred to integer work.

Rules

1. Use groups of three or four (or 1,000?)
2. Each student must keep track of other group members' scores to guard against cheating (emphasize the cheating so that they do, in fact, keep track).
3. Decide which player will start—highest toss of die.
4. Each player in turn picks up a card from the face-down deck and follows the instructions. Scores are recorded with each student's turn, since the whole group is sometimes affected by the instructions.
5. A time limit (20-25 minutes—but quite variable) should be suggested to end the game.

Obviously, the game is very simple, but this contributes to its popularity. No special skill or knowledge is needed to play, and the game can be fiercely competitive (especially as your better students are dropping further into the hole.) Be prepared for some noise—the ecstasy of gain and the agony of loss.

Equipment
One die per group.
One set of game cards per group.
One scoresheet for each student.

Scoresheets can be organized in any numbers of ways. Two are illustrated:

Transaction	Balance		Gains	Losses	Balance
$ 650	$ 650		$650		$ 650
-$1500	-$ 850	or		+$500	-$ 850
-$ 300	-$1150			$300	-$1150

This uses more negative signs. Students could also keep track of the nature of transactions.

Sample cards

The following are some of the card transactions I have used. Many are outdated (but generate discussion). Feel free to use them and make up your own to suit your students' interests. The trick is to come up with a variety of ways of losing (mainly) money so the transactions don't become too repetitive. A sheet of cards can be made up in 15 or 20 minutes. (Use classes and friends for a variety of ideas.)

Note. Remind students before they play that having fun–smiling–should not be done in school and is strictly forbidden.

PAYDAY! Deposit $250.	OOPS! Color TV needs repair. Withdraw $150.
OOPS! Wrecked bike. Write a cheque for $125 for a new one.	INVEST! Buy an antique oak table and chairs for only $625.
SPLURGE! Buy a new stereo component system for $1100.	PAYDAY! Sold the most chocolate bars and won $100.
OOPS! Your first car accident. Deduct $50.	RACES! Your group bet $60 and came in last. Split the losses.
INFLATION! You own a car and spend $25 each week on gas. Roll die to see how many weeks.	INVEST! Invest in a jeans-and-tops shop. Borrow $1,200, which your partners will share with you.
PAY OFF! Your investment pays off. $1,000 times roll of die.	SLUMP! One of your trucks doesn't have any work. It loses $50 a day. Roll the die to see how many days.
WINNERS! Your hockey team is winning games. You make $150 for each game. Roll die to see how many games.	KAPUT! You are a florist, and your flowers are wilting. Roll die and multiply by $50 to see how much you lose.
? Donate $300 to the Munroe Outdoor Education Fund.	LOTTERY! You win $75 times roll of die, but share it with your playing partners.
CRASH! Business goes bankrupt. $150 times roll of die, but divide it with partners.	INVEST? Your gold mine lacks only one thing--gold! Split the loss of $210 times die roll with your partners.

The Tuesday Problems

John Barsby and Don Johnson

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In the Agenda for Action published by the NCTM, the number one recommendation for the 1980s suggested that problem-solving be the focus of the school mathematics program. St. John's-Ravenscourt School has had a formal problem-solving program since the fall of 1975. Here, in brief, is how our program is conducted.

At the Grade 8 and 9 level, we spend one day a week on problem-solving. Each week, we present the students with a set of eight problems that require various strategies to solve. Most of the problems are not directly related to the course work. The students have a week to solve them. When the week is up, we spend a period discussing them. The discussion period is a sharing of ideas, since many students find interesting and innovative ways of attacking some of the problems. It is important for them to realize that a given problem can be solved in a wide variety of ways.

The eight problems are carefully selected. One or two of them should be simple enough that every student will be able to solve them. The others should range from moderately easy to very difficult. Occasionally, there will be a problem that no student manages to solve. As long as students understand that perfection is not required, they do not object to this. In fact many of them tell us of the satisfaction that comes with finally cracking a problem they had been struggling with for days. To encourage the weaker students, we are generous with our praise for those who manage to solve even four out of the eight. It is also important, as the program advances, to recycle some of the ideas from the earlier problems. This allows the students to build on what they have learned.

We have whimsically named our problems after the days of the week. We started with *The Tuesday Problems* for the top stream of Grade 8s, and carried on the following year with *The Monday Problems* for the top stream of Grade 9s. Both problems series have been very successful and are still going strong. Our problem-solving at the senior secondary level has been less formal. The *Wednesday Problems* for Grade 12s were successful, but had to be cut back, since we could not afford to take one day a week from our regular curriculum. As a result, problem-solving in Grades 10 to 12 is done on a less regular schedule, usually two or three days at a time several times a year. We are also developing a problem series for our regular stream of Grade 8s.

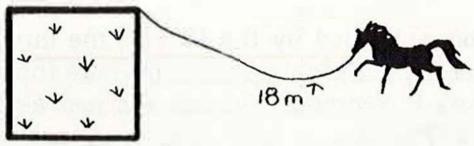
Several sample problem sets follow.

The Monday Problems

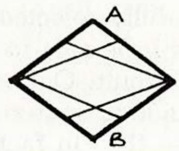
1. If $a*b = ab + 1$ and $a^\circ b = a + b + 1$, what is the value of $4*[(6^\circ 8)^\circ(3*5)]$?
2. Given the integers one to nine inclusive, how many ways can the sum 15 be obtained by taking them three at a time. Repetitions are not allowed.
3. Starting at point S on a circle, successive areas of $27\frac{1}{2}^\circ$ are marked off in one direction. If the first point obtained is called P_1, the second point P_2, the

thirds point P_3, and so on, what is the value of n if P_n is the first point to coincide with S?

4. Think of a number (this could be called x). Add 12. Multiply by four. Subtract 36. Divide by two. Write down two additional steps that will always get back to the original number.
5. A horse is tethered to a rope at one end of a square corral (outside the corral) 10 m to a side. The rope is 18 m long. What is the grazing area available to the horse?



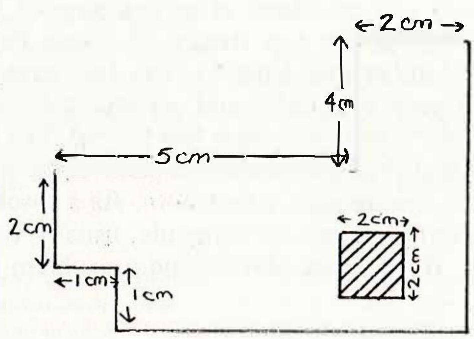
6. A contractor has a large number of pipes 2 m in diameter. He first makes a row of pipes side by side, and each is in contact with the next on a level surface. Then a second row is placed on top of this row so as to fit into the hollows between adjacent pipes. He continues this process until he has five rows. What is the total height of the pile of pipes? Express your answer in metres correct to one decimal place.
7. If only downward motion is allowed, find the number of paths from A to B in the following figure.



8. In the sequence 1, 3, 5, 2, 4, 6, 3, 5, 7, 4, 6, 8,...write the next few terms. Then find the sum of the first hundred terms.

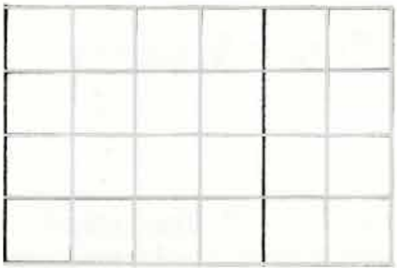
The Tuesday Problems

1. Find the area of the region shown below. The diagram is not drawn to scale. (All edges meet at right angles.)



2. Evaluate: $[\sqrt{.04} + \frac{1}{0.2} - \frac{6}{5}]^2$
3. a) Find the number half way between 1/3 and 4/5.
   b) Find the two numbers that divide the difference between ½ and 5/6 into three equal parts.
4. A 6 x 4 rectangle is divided into a number of equal-sized small squares as shown.

Find the total number of squares of all sizes in the diagram.



5. a) What is the value of: $100 - 99 + 98 - 97 + 96 - 95 + \ldots + 4 - 3 + 2 - 1$
   b) What is the value of: $100 + 99 + 98 + 97 + 96 + 95 + \ldots + 4 + 3 + 2 + 1$
6. How long will it take two water pipes to fill a 260 L tank if one pipe delivers 5 L in one minute and the other delivers 1 L in five minutes?
7. What is the least positive integer that 180 should be multiplied by if the product is to be a perfect square? What is the smallest positive integer we should multiply it by if we wish the answer to be a perfect cube?
8. If a student were to calculate the product of all the natural numbers from one to 25 inclusive (calculate $1 \times 2 \times 3 \times 4 \ldots \times 24 \times 25$), how many zeros would there be at the end of this result?

The Wednesday Problems

1. Factor: $x^2 + 5xy + x - 3y - 24y^2$.
2. If a and b are positive numbers such that: $\log_c \frac{1}{2}(a + b) = \frac{1}{2} \log_c a + \frac{1}{2} \log_c b$ show that $a = b$.
4. A sequence starts with three and ends with 71. There are five numbers in between. The law of the sequence is such that each term (after the first two) is the sum of the two preceding terms. Find the sequence.

   3, , , , , , 71

5. Graph the equation: $|y| + |x + 3| + |x - 3| = 10$
6. AB is the hypotenuse of a right triangle ABC. Medians AD = 7 and BE = 4. Find the length of AB.
   a) The base of a triangle is of length b and the altitude is of length h. A rectangle of height x is inscribed in the triangle with the base of the rectangle in the base of the triangle. Find the area of the rectangle in terms of b, h, and x.
   b) Prove that the largest rectangle that can be thus inscribed has area equal to one-half the area of the triangle.
7. A circle with centre C is given and eight distinct points are marked on the circumference so that no two points are on opposite ends of the same diameter. Prove that, no matter where the points are marked, it is possible to label two of them A and B so that three of the points lie on the smaller arc from A to B.
8. An escalator (moving staircase) of n uniform steps visible at all times descends at a constant speed. Two boys, A and Z, walk down the escalator steadily as it moves. A negotiating twice as many escalator steps per minute as Z. A reaches the bottom after taking 27 steps while Z reaches the bottom after taking 18 steps. Find the value of n.

Sines, Cosines and Ferris Wheels

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In *Using Advanced Algebra* (Travers *et al*, 1977), the textbook used in the Newfoundland Grade 11 matriculation program, a ferris wheel is presented in the introduction to the chapter on trigonometry. Such an analogy triggered a possible inductive approach to a discussion of trigonometric ratios, especially sine and cosine. This method has proven successful for me in three successive years of teaching, and its possibilities and objectives can vary. It could be oriented as a classroom demonstration or as group projects or as individual activities. Other circular analogs, besides the ferris wheel, could be considered. The length and depth of the project depend on your students' abilities and your own imagination. The suggestions below can be easily modified to meet your own teaching style.

The objectives of such a lesson may be to provide students with an opportunity to "discover" how the sine ratio (or cosine) can be developed; to investigate the need for constancy of these ratios; to experience "real" measuring events that ultimately lead to a rational conclusion.

The materials needed for the project:

1. A picture of a ferris wheel (if possible) or whatever circular rotating object you choose.
2. Bristol board (at least one sheet) to be used for modelling the ferris wheel.
3. Scissors, markers, and a thumbtack (or fastener).
4. Straightedge, protractor, large compass (board compass).
5. String (at least a metre); tape (masking is fine).

The only prerequisite for this activity is that students be familiar with the measuring tools (a metric ruler and a protractor) to be used.

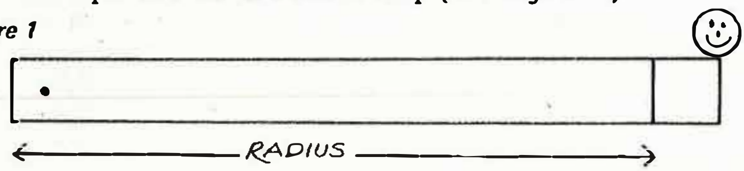
The preparation may be done by a group of students before the class, or during the class if desired. The purpose of the picture of the ferris wheel is to permit familiarization with it; not all students in Newfoundland have seen one. It should be posted in the classroom, if possible.

To build a model of a ferris wheel, follow these instructions:

1. Using the chalkboard compass (or a thumbtack, string and marker), draw a large circle on the sheet of Bristol board. Its size is limited only by the dimensions of the material.
2. Using a marker, indicate the centre point, and draw a diameter.
3. Cut out the circle.
4. From the remaining Bristol board, cut a strip one cm wide (at most) and let the strip's length be the same as the radius of the circle. Extend it "a little" to allow for later fastening.
5. To represent a rider on the ferris wheel, cut a small square with a circle atop,

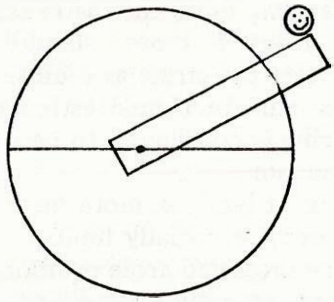
and tape it to the end of the strip (See Figure 1.)

Figure 1



6. Using the thumbtack or fastener, attach the "free" end of the strip to the centre point of the circle, so that the rider touches the edge of your circle. Tape the thumbtack in place to permit the strip to rotate (see Figure 2). The strip should rotate freely around the circle. If desired, it could be temporarily taped in a position.

Figure 2



A description of the inductive activity is presented below. The comments made are suggestions only.

1. Place the model of the ferris wheel in a convenient spot. The edge on the blackboard may serve useful. Be sure the diameter is in the horizontal position.
2. You might wish to demonstrate the rotation ability of the strip and discuss the analogy a little. For example, students may not realize that on the real ferris wheel, the seat of the rider adjusts, so that he/she is never upside down as our rider may be in certain positions.
3. Prepare for measurements. I have used the metre stick and a piece of string. The string should be cut the same length as the radius, since no measurement will exceed this. A protractor (for demonstration purposes, the larger board protractor) should be used. Other measuring tools such as a ruler or one's hands may also help cement the relationship to be developed.
4. Once the measuring tools are available, the induction should begin. A brief explanation may be necessary. Students should be told that they are about to measure the vertical distance of the rider in different positions on the ferris wheel. The vertical height will be measured from the fixed horizontal diameter, and the position will be recorded in angle measures. We will be looking for patterns.
5. Choose a beginning angle—this could be suggested by students or the teacher. Say 30° was chosen. Rotate the rider who should initially lie on the fixed diameter, to 30° above that line. Check the rotation measure with the protractor. Indicate, or have a student indicate, what would be considered the vertical distance. The idea of a perpendicular distance should come up and may need to be discussed.
6. Before making measurements, attach the rider in this position and briefly describe ways of recording data. A table of measurements is helpful in organizing the data. A partial table is given in Figure 3.

Angle	Verticle measure in cm	Verticle measure in string units
30°	7.5	0.50
50°	11.5	0.75
etc.	etc.	etc.

Figure 3

7. Now, measurements can begin. Ask a student to measure the vertical distance of the rider above the diameter using the metre stick. The centimetre will be the unit of measurement. Next, a student should measure the same distance with the string. Here, one sees the string as a unit, and the measures will be a fraction of the string. The student should estimate the fraction of the string needed. Remember the string is considered to be one unit or one radius. Now record your measures in the table.
8. Continue this procedure for at least six more measures. Alternate measures so all quadrants will be covered, or initially limit yourself to the first quadrant. The 60°, 45°, 90°, are some suggested angle positions.
9. Once the data are collected, an examination and a discussion are necessary. The teacher should permit students to guess and speculate as much as possible as to any possible relationships. If none is indicated, suggest measuring the radius of the circle with each instrument; record this at the top of each column. Now, suggest that this measure may help indicate a relation. It is hoped that the discussion will lead to the idea that the ratio of the vertical distance to the radius is the same, whichever unit of measurement is used. To develop the notion of cosine, use the horizontal distance from vertical diameter.
10. Check suggestions or guesses with further measures. Lead into the naming of such a measure as the sine function.

Activities such as presented above require at least one full class period. The use of this amount of time in the short term saves time in the long term, since students usually develop a better understanding of the concepts being taught. Have a good ride on your ferris wheel.

Reference

Travers, K.J. *et al. Using Advanced Algebra*, Doubleday, Canada Limited, Toronto 1977.

Finding Irrational Roots of Polynomials

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In the honors course in Grade 11, as well as in Mathematics 1010 at Memorial, one spends considerable time teaching how to find, when they exist, rational roots of a polynomial with integer coefficients. Several worth-while topics are usually taught or recalled at this time, namely the Fundamental Theorem of Arithmetic to prove the main theorem about possible rational roots, the relationship between roots and linear factors, and synthetic division as a fast way of checking the possible rational roots. (The next natural followup to this topic at the university is the more general algorithm known as Kronecker's Theorem for factoring completely into irreducible (prime) factors any polynomial with integer (or rational) coefficients.

In addition to spending time finding rational roots of a few very carefully selected polynomials, one should spend more time finding *all* the real roots of polynomials with real coefficients. This is possible now that all students have easy access to the calculator.

As many of you know, it is impossible, in general, to write a formula for the roots of a polynomial in terms of arithmetic operations of the coefficients and extraction of roots for a polynomial of degree greater than or equal to five. For lower degrees, there are formulas for the roots. In fact, in 1830 a brilliant young mathematician, Evariste Galois, two years before being killed in a gun duel over a girl before his 21st birthday, gave a very beautiful, necessary and sufficient condition that a given polynomial equation be solvable in radicals (in terms of arithmetic operations and extraction of roots) and hence showed that, in general, fifth and higher degree polynomial equations cannot be solved in radicals. (See the historical note on pages 392 and 393 of Johnson *et al*., 1975). In particular, he showed that, $f(x) = a_p x^p + \ldots + a_1 x + a_0 = 0$, where a_i are rational numbers, cannot be solved in radicals provided p is a prime greater than or equal to five, f is irreducible (prime) over Q, and f has two complex (conjugate) roots only. Two such examples are $f(x) = x^5 - 6x + 3$ and $f(x) = 3x^7 - 7x^6 - 7x^3 + 21x^2 - 7$. By elementary calculus, one can show that the first polynomial has three, and the second, five real roots. A very easy sufficient condition for a polynomial with integer coefficients to be irreducible, known as Einstein's criterion, states that if we can find a common prime divisor of each of the coefficients of f except the leading one and it does not divide the constant term any more than once then the polynomial is irreducible. For the first polynomial, we see that the prime 3 satisfies the conditions of irreducibility; for the second, the prime 7, satisfies the conditions for irreducibility. Hence, we are guaranteed for the above two polynomials there are no explicit formulas for the roots.

The method for finding irrational zeros described by Johnson *et al* on pages 379 to 381 is a good one and is a place where linear interpolation can be reviewed. Let us consider the example, $f(x) = x^5 - 6x + 3$, the problem being to find the three real zeros to within some prescribed accuracy. Let us assume for now that we know the most efficient way to evaluate f(x) on our calculator. We find that $f(-1) = 8$ and $f(-2)$

= -17; hence, one root r lies between -2 and -1. We may wish to find the next digit by trial and error since, if the initial guesses are far apart or near a relative, extrema for the function using linear interpolation may not save us any time. So guessing intelligently, we see that f(-1.7) = -1 and f(-1.6) = 2.11 so our root r lies between -1.7 and -1.6. By comparing slopes where r = -1.6 - a, we have a/.1 = 2.11/(2.11 + 1) and hence a = .07, so we guess that r is either between -1.66 and -1.67 or between -1.67 or -1.68. We find that f(-1.68) = -0.3 and f(-1.67) = 0.03; hence r is between -1.67 and -1.68, and so we continue. Depending on the accuracy of our calculator, we could ask for r accurate up to eight or nine places after the decimal. In fact, the real roots correct to eight places are -1.67093526, 1.40164188, and 0.50550123.

For the polynomial $f(x) = 3x^7 - 7x^6 - 7x^3 + 21x^2 - 7$, one of the irrational zeros correct to eight places is -1.23707354, since one can check that f(-1.237073543) = 7.6×10^{-8} and f(-1.237073544) = -6.5×10^{-8}. Noting that f(-2) = -699, f(-1) = 11, f(0) = -7, f(1) = 3, f(2) = -43 and f(3) = 1451, we know approximately where all the zeros are. In fact, the points (-1, 11), (0, -7), (1, 3) and (2, -43) are the relative extrema of the function.

For the information of the secondary school mathematics teachers, we record here some detailed solutions to some of the problems in section 8-10 of the Johnson text. The three real zeros of $x^3 - 5x + 1$ are -2.330058739, 2.128419066, and 0.201639674. For exercises 6, 7, and 8 on page 382, the unique real zeroes of each of $x^3 + x - 3$, $x^3 - 3x^2 - x - 6$ and $x^3 + 3x^2 - 3x + 2$ are 1.213411662, 3.706527954, and -3.900571874 respectively.

The next problem is that of evaluating f(x) on our calculators in as efficient a way possible. For an arbitrary polynomial:

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ of degree n, the first method of evaluating is

the straightforward one of going from left to right evaluating term by term. For the purposes of comparison, let us assume that entering a number, storing a number, or recalling a number from memory requires *one* key operation each. Evaluating f(x) in this way on a calculator with true algebraic logic requires 6n + 1 key steps. Even for the Hewlett-Packard (HP) calculator with reverse polish arithmetic, this method requires 6n + 2 key steps. For the average student, this may well be a suitable method for finding f(x). However, there is one major problem if students have Texas Instrument (TI) calculators. The TI calculators (as well as others such as Canon), as contrasted for example with the Casio Calculators, cannot raise a negative number to an integer power, hence, cannot evaluate f(x) by the above method for negative x and degrees greater than two. (Try evaluating $(-2)^3$ on the TI calculators.)

A second, faster method of evaluating f(x) is rewrite f(x) in the nested form:

$$f(x) = (\ldots(((a_n \cdot x + a_{n-1}) \cdot x + a_{n-2}) \cdot x + a_{n-3})\ldots + a_1) \cdot x + a_0$$

and start evaluating from the inside out. The calculators with true algebraic logic will hinder us here. Instead of using the bracket feature on your calculator, use the equal button (if your calculator has true algebraic logic). If f(x) is evaluated by the key steps: enter a_n, times, enter x, store x, plus, enter a_{n-1}, equals, times, recall x, plus, enter a_{n-2}, equals, etc., then 5n+2 by steps are required. However, on the HP calculators, only 3n+5 steps are required. If we had used the brackets, then 6n+1 key steps are required, so we might as well have evaluated f(x) in the more straightforward way described earlier.

In any case, you should get to know your particular calculator, because you may be able to discover shorter methods of evaluating f(x). For example, using a TI-35, one can evaluate fourth- and lower-degree polynomials by writing f(x) in the form: $x \cdot (x \cdot (x \cdot (x \cdot a_4 + a_3) + a_2) + a_1) + a_0$, then the argument x need not be stored, since it remains in the register as "times" and "left bracket" are pressed. No higher degrees can be evaluated in this way, since there can be, at most, four pending operations on the TI-35.

On some calculators, e.g., Casio College fx-100, "times" can be omitted before "left bracket."

Using synthetic division to evaluate f(x) will not result in fewer key steps, since this method is exactly the same as the nested form above, requiring 5n+2 key steps.

Students should be given the opportunity to approximate irrational roots of polynomials. Finding rational roots requires learning some very nice algebra, but it is very limited in the polynomials selected. It would also help us in Calculus I at university if students could solve the equation $f^1(x) = 0$, where $f^1(x)$ is a polynomial of degree three or more, and where the roots need not be rational.

References

Johnson, R.E., LL. Lendsey, W.E. Slesnick, G.E. Bates, *Algebra and Trigonometry*. Addison-Wesley, Don Mills, Ontario, 1975.

Probability and Statistics Corner

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This first article for the year of 1982 coincides with another first, the first issue of the *Canadian Mathematics Teacher*. I hope that PS Corner will become a regular feature of the journal, as it has in *Vector*, the journal of the B.C. Association of Mathematics Teachers.

At the recent Leadership Conference on Statistics in the Classroom, organized by the ASA/NCTM Joint Committee on the Curriculum in Statistics and Probability, I had the pleasure of meeting teachers from British Columbia, Alberta, Ontario, and Nova Scotia who were involved at some level of teaching statistics. All of those provinces are moving strongly toward a greater emphasis on statistics and probability in all levels of the curriculum. The majority of those present at the leadership conference were working at, or had a major interest in, Grades K-8. Now that it is generally accepted that future revisions of the mathematics curriculum must "recognize statistics and probability as important and indentifiable topics of the curriculum" (from the recommendations following the recent mathematics assessment in B.C.), we must determine the components of those topics and their place in the curriculum.

The following ideas are offered as the starting point of a discussion that will, I hope, produce a curriculum in statistics and probability. That curriculum should capture the investigative spirit of the subject and avoid the pitfalls that have produced the present situation of topics being on a syllabus, but avoided by most teachers.

The essential elements of an elementary school program:
Collecting numerical information.
Exploring the visual representations of such data, pictographs, stem and leaf graphs, indeed any kind of visual presentation that assists with asking and answering questions.
Developing skills in asking questions of such data; such skills will also develop the use of such concepts as *greater than, greatest, how many are less than...*, *range* (greatest–least), and variability.
Basic concepts of probability, laying the framework for the later development of the idea of a simulation. Use of a variety of games to develop the idea of a fair game as opposed to an unfair game.
Two co-ordinate graphs are introduced at this level and can be used to develop ideas of variability through an examination of such graphs as height vs. weight or span vs. height, etc.

Two emphases could predominate a program for Grades 7-9:
The concept and use of simulation. This is a powerful problem solving tool and valuable for the reinforcement of the ideas of probability and variability.
Exploring data, coupled with a project approach. Calculation of means, etc., is a pointless exercise if the process stops at the calculation. Tying these skills to a topic of interest to the student (sports would be of interest to a large number of students) gives some point to the exploration of data. Interest in the subject is very important if this area is to be learned successfully. It should, perhaps, be obvious to a number of you that these two topics lend themselves to integration with computer literacy. Computers should play an important role at this level, if not earlier levels. The computer will make possible tasks that would otherwise be tedious or time consuming.

Other areas that are appropriate for this level are precision of measurement and errors; simple sampling experiments, sample bias and biased sampling, reading tables; and reading and interpreting newspaper articles.

In a recent editorial by Peter Holmes in *Teaching Statistics*, he makes a relevant point: Where in the curriculum should we develop the skills necessary for students to be able to answer questions such as "Should seat belts be made compulsory?"
4(a) In th

In the United Kingdom at the moment there is a growing debate about numeracy. How numerate are our pupils? The first reference to the word "numeracy" that I can trace is in the Crowther report (1959) where it was used in the context of the ability to reason quantitatively and to have some understanding of the scientific method. In this sense to be numerate is to be able to use numbers in drawing inferences, the very essence of statistics. More recent definitions have drawn the difference between algorithmic skills and informed numeracy. Having algorithmic skills means having the ability to do sums; informed numeracy means the ability to know what sums to do, when to do them and how to use the answer. Another recent definition of numeracy talks of "the ability to use number in practice." Clearly all these definitions relate closely to the statistical education we should be incorporating into the main secondary school curriculum. Perhaps we need to rethink carefully our basic statistical material for schools, asking ourselves, "Does this help pupils develop the ability to use number in practice?" As an example consider the following question from a national examination in English to 16-year-old pupils of average ability.

Read this extract from a booklet published by the West Yorkshire Metropolitan Police:

USE OF SAFETY BELTS
drivers and front seat passengers in cars and light vans

	CASUALITIES			
SAFETY BELT	KILLED	SERIOUS	SLIGHT	TOTAL
Fitted and worn	4	72	447	523
Fitted but not worn	62	612	2543	3217
Not fitted or not known	2	43	132	177
Total casualities	68	727	3122	3917

...and those figures tell their own story, don't they?

Discuss whether the wearing of seat belts should be compulsory by law.

How well has our teaching enabled pupils to answer questions like this? Do they know what to look for in the table, what they might find, how to allow for variability in interpreting their answer, and so on? We owe it to our pupils to prepare them for this type of use of number in practice.

The program for Grade 10-12:
In my experience, the project or experimental approach is the most effective way to convey the idea that statistics is experimental and investigative. The subject matter at this level therefore develops investigative skills. It includes such things as survey sampling, the idea of a reference distribution applied to YES/NO samples, measurement samples, and samples of the chisquared statistic. The chisquared material should not be a formula that is interpreted using a table, but a means of developing the concepts of expectation and a reference distribution. The elements of interpretation of scatter plots is also appropriate at this level.

So much for a start as to what should be in the curriculum. I have not addressed the issue of content for the range of courses at the senior levels, consumer math, statistics courses, computer science, etc., nor have I addressed the issue of what should be omitted. Some people might note certain omissions, such as standard deviation and the normal distribution. Such things will, I hope, be considered in the ensuing discussion.

Teaching Problem-Solving

Dr. Don Kapoor

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The history of mathematics education shows that problem-solving has always been an issue of concern. Most recently, the National Advisory Committee on Mathematics Education (NACOME) and the National Council of Supervisors of Mathematics (NCSM)[1] have indicated that problem-solving is one of the most important basic skills that every student should master in order to survive in our society. The National Council of Teachers of Mathematics (NCTM) has also taken a position supporting the infusion of problem-solving into the school mathematics curriculum for the 1980s.[2] However, there is a great deal of misunderstanding over what problem-solving is and even what constitutes a problem. Jeremy Kilpatrick[3], in his article entitled "Stop the Bandwagon, I Want Off," decries the "use of 'problem-solving' as an empty vessel that we can fill with our own meanings." In many cases, mathematics teachers teach solving problems with the mistaken idea that they are teaching problem-solving.

This article has basically two purposes: to illuminate and reflect on problem-solving and to illustrate, by way of an example, how to teach problem-solving skills in a classroom.

What is a problem?

One common concept of a problem is that of a question proposed for an answer or solution. The teacher has this concept in mind when he/she says to a mathematics class, "Your assignment for tomorrow is to work problems one to ten on page 164." The question that may be either *explicit* or *implicit* in each problem is, "What is the answer?"

A second concept of a problem considers the existence of a question to be necessary, but, unlike the first concept, existence of the question is not regarded as sufficient. The additional conditions pertain to the individual who is considering the questions. What may be a problem for one individual may not be a problem for another. A problem for a particular individual today may not be a problem tomorrow.

The necessary conditions for the existence of a problem for a particular individual are:[4]

1. The individual has a clearly defined goal of which he/she is consciously aware and whose attainment he/she desires.
2. Blocking of the path toward the goal occurs, and the individual's fixed patterns of behavior or habitual responses are not sufficient for removing the block.
3. Deliberation takes place. The individual becomes aware of the problem, defines it more or less clearly, identifies various possible hypotheses (solutions), and tests them for feasibility.

This concept of "problem" holds that when these three necessary conditions are met, a problem exists for the particular individual. Moreover, Cronbach[5] points out, "...it is not posing the question that makes the problem, but the person's accepting it as something he must try to solve." The second concept of a problem appears to be the more useful concept in most educational contexts.

Problem-solving and problem-solving skills

Problem-solving is a process. It is the means by which an individual uses previously acquired knowledge, skills, and understanding to satisfy the demands of an unfamiliar situation. When students solve various "types" of textbook problems (age problems, coin problems, motion problems, etc.), they are simply applying a previously learned algorithm or model to a familiar situation–they are merely solving problems. Only when a student synthesizes what he/she knows and applies it to a NEW situation is the student problem-solving.

Problem-solving is more than a single skill–it is a set of skills that includes the ability to:

Read and understand.
Explore. (That is, play around with a problem, trying different approaches.)
Select an appropriate strategy.
Carry through the strategy.
Look back and extend.

Skills are the building blocks used in solving a problem. Some of the skills are unique to mathematics; others are interdisciplinary. The categories used by the Lane County Mathematics Project (LCMP) are:[6]

1. Problem discovery, formulation
2. Seeking information
3. Analyzing information
4. Solving–putting it together–synthesis
5. Looking back–consolidating the gains
6. Looking ahead–formulating new problems

These categories are further broken down into 46 different skills.

According to Polya[7], the process of problem-solving has four phases. First, we have to understand the problem; we have to see clearly what is required. Second, we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a plan. Third, we have to carry out our plan. Fourth, we have to look back at the completed solution, review and discuss it. An awareness of these skills being used to solve a problem is probably the most important step in the development of a pupil's problem-solving abilities.

George Polya contends that the technique for teaching problem-solving has two aspects: abundant experience in solving problems and serious study of the solution process. He expresses the need for the first in this way: "Solving problems is a practical art, like swimming or skiing or playing the piano; you can learn it only by imitation and practice." However, he warns that imitation and practice are not sufficient. Not only must problems be solved, but the learner's attention must be directed to the methods used. These must be general enough so that they become available for use in solving similar problems in the future. Polya uses the term *heuristics* to describe this way of teaching problem-solving in mathematics.

Implications for mathematics education

Solving problems is human nature itself. We may characterize humans as "problem-solving animals." If education fails to contribute to the development of the intelligence, it is obviously incomplete. Yet intelligence is essentially the ability to solve problems in everyday life. The student develops his/her intelligence by using it, he/she learns to do problems by doing them. Which kind of problems should a secondary school student do to develop his/her ability to solve problems? What is the teacher's role in teaching problem-solving? George Polya[8] in his article originally published in 1949 and later appearing in NCTM 1980 Yearbook sums up the thoughts as follows:

"A boy or girl of high school age and average ability can solve on a scientific level mathematical problems, but no other kind of problems. An average boy of fifteen can obviously not acquire the technique or knowledge or judgment needed in treating on a scientific level a problem of biology or history or physics. Yet, if he has a good teacher, he can, after a while, solve a problem of geometric construction or invent by himself the proof of a simple theorem on the level of Euclid, and Euclid's level is fully scientific.

"This is the great opportunity of mathematics: mathematics is the only high school subject in which the teacher can propose and the students can solve problems on a scientific level. This is so because mathematics is so much simpler than the other sciences. Because of this simplicity, the individual, just as the human race, can arrive so much earlier to a clear view in mathematics than in the other sciences.

"In my opinion, the first duty of a teacher of mathematics is to use this great opportunity: he should do everything in his power to develop his students' ability to solve problems.

"First, he should set his students the right kind of problems: not too difficult and not too easy, natural and interesting, challenging their curiosity, proportionate to their knowledge. He should also allow himself some time for presenting the problem appropriately, so that it appears in the proper light.

"Then, the teacher should help his students properly. Not too little, or else there is no progress. Not too much, or else the student has nothing to do. Not ostentatiously, or else the students get disgusted with the problem in the solution of which the teacher had the lion's share. Yet, if the teacher helps his students just enough and unobtrusively, leaving them some independence or at least some illusion of independence, they may experience the tension and enjoy the triumph of discovery. Such experiences may contribute decisively to the mental development of the students."

However, here Polya reiterates the first condition for discovery: No teacher can impart to his/her students the experience of discovery if he/she has not got it. Therefore, future teacher-education programs should capitalize on discovery approaches much more than they have in the past with emphasis on the practical ability to solve not too advanced problems and the methods of solution.[9]

How to teach problem-solving skills—an illustration

This problem is designed to help teachers teach specific problem-solving skills. Four common but powerful problem-solving skills[10] are:

1. Guess, check, and refine.
2. Look for and/or use a pattern.
3. Make a systematic list.
4. Make and use a drawing or model.

Pupils might use other skills to solve the problems (for example, working backwards, etc.). They can be praised for their insight, but it is usually a good idea to limit the emphasized list of skills directly taught during the first few lessons.

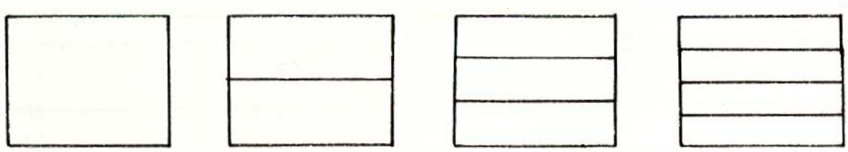
Aim. Looking for pattern.
Level. Grades 7-12.
Problem. Ten people are at a party. Each person shakes hands with each of the others. What is the total number of handshakes?
Heuristics. Start with easier cases. Complete the table. Look for patterns.

Number of people	**Number of handshakes**
2	1
3	3
4	6
5	
6	
7	
8	
9	
10	
•	
•	
•	
n	

Can you generalize the problem?

Variations of the problem

1.



1 3 6

Notice that:
The first drawing contains one rectangle.
The second drawing contains three rectangles.
The third drawing contains six rectangles.

How many rectangles are in the fourth drawing?

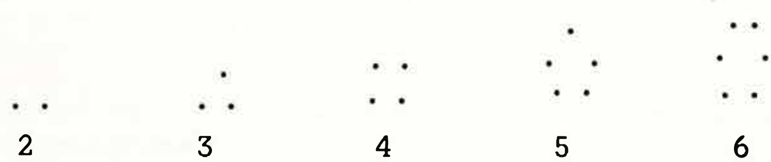
If we continued the pattern, how many rectangles would be in:
The fifth drawing?
The sixth drawing?
The tenth drawing?

Complete the following table.

Drawing	**Number of rectangles**
first	one
second	three
third	six
fourth	
fifth	
etc.	

Can you generalize the problem?

2. Two points determine at most one line segment.
   Three points determine at most three line segments.
   Four points determine at most six line segments.
   Five points determine at most ten line segments.
   Six...
   Seven...
   Eight...
   •
   •
   n



Generalize and check.

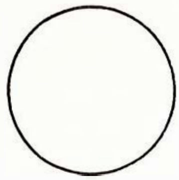
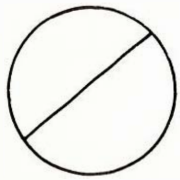
3. Detect the regularity:

1	= 1
1 + 2	= 3
1 + 2 + 3	= 6
1 + 2 + 3 + 4	= 10
1 + 2 + 3 + 4 + 5	= 15

Generalize and check.
1 + 2 + 3 + ... + n

Extension of the problem
Find the maximum number of parts into which the interior of a circle can be divided by a given number of lines in the plane.

Complete the following table:

Number of lines in plane (n)	Drawing	Maximum number of regions (n)
0		1
1		2
2		
3		
4		
n		

Can you generalize the problem?

Can you check to see that the formula works?

Footnotes
[1] *National Council of Supervisors of Mathematics Position Paper on Basic Mathematical Skills.* Minneapolis, Minnesota: National Council of Supervisors of Mathematics, 1977.
[2] *An Agenda for Action: Recommendations for School Mathematics of the 1980s.* Reston, Virginia: National Council of Teachers of Mathematics, 1980.
[3] Jeremy Kilpatrick, "Stop the Bandwagon, I Want Off." *The Arithmetic Teacher*, 28(8), April 1981, p. 2.
[4] NCTM: 21st Year Book, pp. 230.
[5] Lee J. Cronbach, "The Meaning of Problems," *Arithmetic, 1948.* Supplementary Educational Monographs, No. 66. Chicago: University of Chicago Press, 1948, pp. 32-43.
[6] LCMP: Introduction to the LCMP Mathematics Problem-Solving Programs, (Oregon ESEA Title IV–C Program).
[7] George Polya, *How To Solve It.* New York: Doubleday and Company, Inc., 1957.
[8] NCTM: *1980* Yearbook.
[9] *Ibid.*, p. 2.
[10] LCMP, *Op. Cit.*

[illegible]