be represented by the region S in three-dimensional space described by the inequalities $x > 0$, $y > 0$, $z > 0$ and $1 - x - y - z > 0$; that is, by the tetrahedron bounded by the co-ordinate planes and the plane $x + y + z = 1$. The volume of this tetrahedron is $1/6$.

The four pieces can be joined to form a quadrilateral if and only if the length of each piece is less than ½. Thus the set of selections favorable for the formation of a quadrilateral is represented by a region F consisting of those points that belong to S and whose co-ordinates satisfy the inequalities $x < ½$, $y < ½$, $z < ½$ and $1 - x - y - z < ½$.

The region F may be visualized as follows: Imagine the cube bounded by the co-ordinate planes and the planes $x = ½$, $y = ½$ and $z = ½$. F consists of the points inside this cube and between the parallel planes $x + y + z = ½$ and $x + y + z = 1$. The volume of the cube is $(½)^3 = 1/8$. The corner of the cube cut off by the plane $x + y + z = ½$ is a tetrahedron with volume $1/6\,(1/2)^3 = 1/48$ and the corner of the cube cut off by the plane $x + y + z = 1$ is also a tetrahedron with volume $1/48$. The volume of F is thus $1/8 - 1/48 - 1/48 = 1/12$.

The required probability is therefore given by:

$$\frac{\text{(The volume of F)}}{\text{(The volume of S)}} = \frac{(1/12)}{(1/6)} = ½.$$

For a solution of the Broken-Stick Problem for n + 1 pieces see: G.A. Heuer, "Solution to Problem E1480," *The American Mathematical Monthly*, 1962, 235-236.

A Mathematical Valentine

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One of the most familiar of the polar curves is the cardioid. The shape of the cardioid with equation $r = a\,(1 - \sin\Theta)$ is shown in Figure 1.

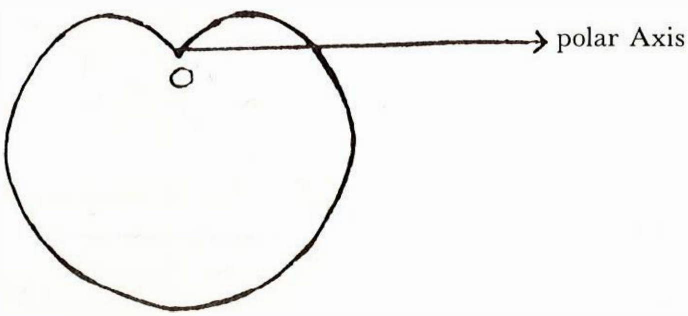

Figure 1

The heart-shaped figure, however, is flat at the "bottom"; whereas the standard Valentine heart, seen each February, is not.

In this article, the equation and construction of a Valentine heart (see Figure 2) will be discussed.

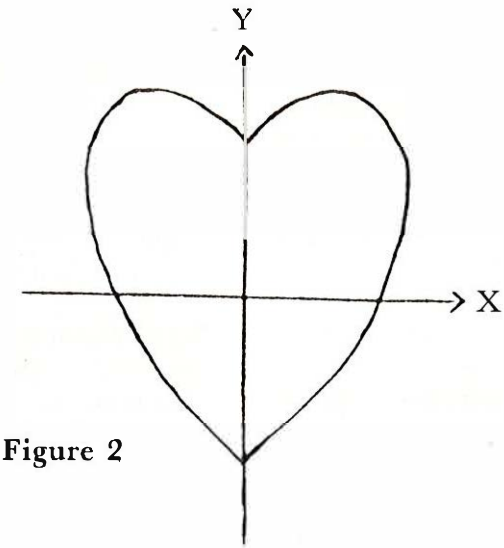

Figure 2

Consider first an ellipse of the form $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, $A > B$, and a straight line of the form $y = mx$, $m > o$. If the graphs of these two curves are superimposed on the same co-ordinate system (see Figure 3), it can be seen that the part of the ellipse on either side of the straight line looks like "half of a Valentine heart."

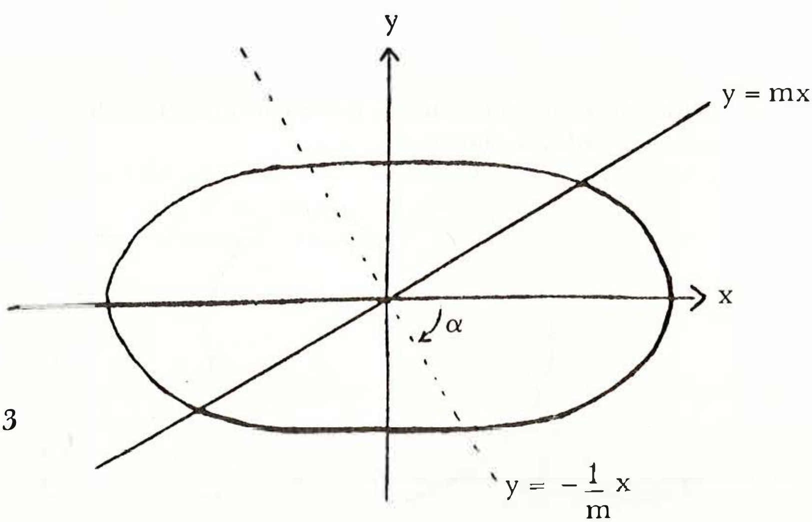

Figure 3

Thus motivated, now rotate the x, y axes through an angle α such that $\tan\alpha = -\frac{1}{m}$.

If the new axes are denoted by X and Y, then the formulae connecting the old and new co-ordinates are $x = X\cos\alpha - Y\sin\alpha$; $y = X\sin\alpha + Y\cos\alpha$, and the equation of the ellipse relative to the X, Y co-ordinate system is:

$$\frac{(X\cos\alpha - Y\sin\alpha)^2}{A^2} + \frac{(X\sin\alpha + Y\cos\alpha)^2}{B^2} = 1 \qquad (1)$$

Substituting $\sin\alpha = -\frac{1}{\sqrt{1+m^2}}$, $\cos\alpha = \frac{m}{\sqrt{1+m^2}}$ into equation (1) and

simplifying gives $aX^2 + 2hXY + bY^2 = 1$ where

$$a = \frac{A^2 + B^2m^2}{A^2B^2(1+m^2)}, \quad h = \frac{m(B^2 - A^2)}{A^2B^2(1+m^2)}, \quad b = \frac{A^2m^2 + B^2}{A^2B^2(1+m^2)}.$$

This equation is quadratic in Y. Solve for Y to get

$$Y = -\frac{h}{b}X \pm \frac{\sqrt{b - (ab - h^2)X^2}}{b}.$$

The graph of this equation is shown in Figure 4.

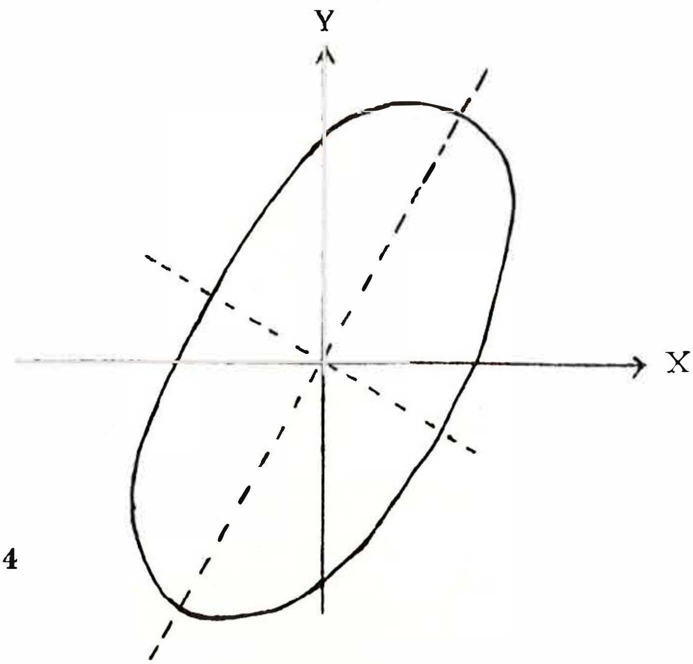

Figure 4

Then using an absolute value, the equation of a Valentine heart (see Figure 2) is:

$$Y = -\frac{h}{b}|X| \pm \frac{\sqrt{b-(ab-h^2)X^2}}{b}.$$

For example, if $b = 1$, $h = \frac{-3}{4}$ and $a = \frac{25}{16}$, then the equation becomes

$$Y = \frac{3}{4}|X| \pm \sqrt{1-X^2}$$, with points as shown in Figure 5.

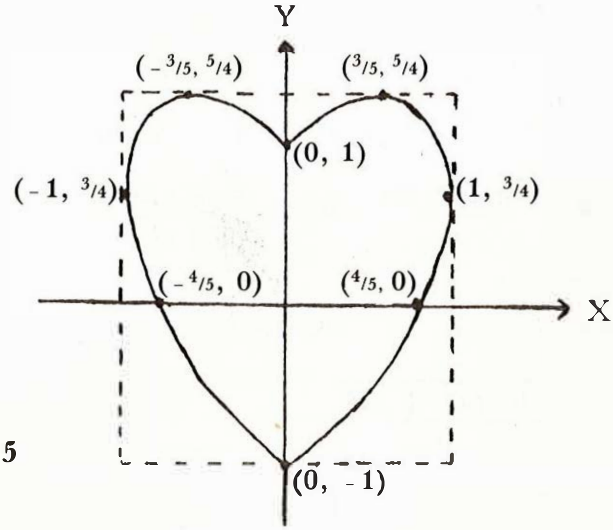

Figure 5

To draw such a Valentine heart, fix two drawing pins P and Q to a sheet of paper on a drawing board a distance 2c, where $c < A$, apart and attach to them the ends of a piece of string of length 2A. With a pencil in the loop of the string and keeping the string taut, the point R of the pencil will trace an ellipse (with foci P and Q and eccentricity c/A). See Figure 6. Cut out this ellipse.

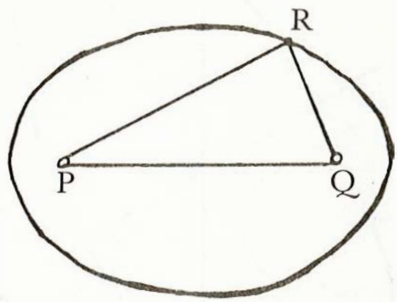

Figure 6

Finally, draw any straight line through O the midpoint of PQ (except the line POQ or the line perpendicular to POQ) and cut along this line. This yields two halves of a Valentine heart which, if joined correctly, produces the desired heart.