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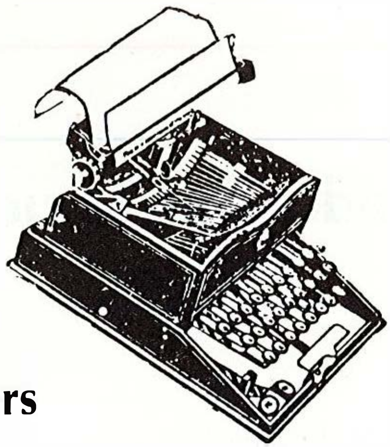
Inside This Issue



From the Editors

Les Dukowski

Volume 2 of *The Canadian Mathematics Teacher* is a co-operative venture of the associations of mathematics teachers in Alberta, British Columbia, Manitoba, New Brunswick, Newfoundland, and Saskatchewan. That the number of provincial associations contributing to the journal has increased over last year is encouraging. We hope that before too long, all of the provinces will be represented.

You will find in this issue articles dealing with assessment of mathematics programs, problem-solving, applications, and classroom activities. If you have an article you would like to publish in this journal, submit it to the editor of your provincial math teachers' journal. That person is the one who chooses the article from your province for publication. *The Canadian Mathematics Teacher* also welcomes letters. Please send them to me at the address below.

Since this journal is a co-operative effort, I thank the following contributing editors:

Dale Drost
The Mathematics Council of the Newfoundland Mathematics Teachers' Association

Gary Hill
The Alberta Teachers' Association Mathematics Council

Don Kapoor
The Saskatchewan Mathematics Teachers' Society

Judy MacKnight
The New Brunswick Teachers' Association Mathematics Council

Alan Wells
The Manitoba Association of Mathematics Teachers

The editors of *The Canadian Mathematics Teacher* hope that you find this issue interesting and thought-provoking.

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Preliminary Results from the Second International Mathematics Study in British Columbia

David F. Robitaille

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During the 1980-81 school year, several thousand Grade 8 and Grade 12 mathematics students from British Columbia, along with their mathematics teachers, participated in the Second International Mathematics Study. More than 20 countries took part in various phases of the study, with Canada being represented through the participation of British Columbia and Ontario. A list of participants is given below:

Australia	Israel
Belgium (Flemish)	Ivory Coast
Belgium (French)	Japan
Canada (British Columbia)	Luxembourg
Canada (Ontario)	Netherlands
Chile	New Zealand
England	Nigeria
Finland	Scotland
France	Swaziland
Hong Kong	Sweden
Hungary	Thailand
Irish Republic	United States

The Second International Study of Mathematics is a project of the International Association for the Evaluation of Education Achievement (IEA). IEA is an association of educational research organizations and ministries of education whose primary goals are to conduct educational research on an international level and to assist member-states in undertaking co-operative research projects. IEA has conducted international surveys in the past, including the first mathematics study (Husén, 1967) and the Six-Subject Survey (Peaker, 1975; Walker, 1976). Twelve countries participated in the first mathematics study, but Canada was not among them.

The international study had a three-fold purpose:

- to investigate the mathematics being taught in those countries participating in the study.
- to compare the ways mathematics was being taught,
- to study the effects of that teaching on students' learning of and attitudes toward mathematics.

The data collected in the study will enable researchers to construct an international portrait of mathematics education.

Two populations of students were identified for investigation in the international study. The first, *Population A,* was defined as consisting of all those students enrolled in the grade where the majority of students have reached the age of 13 by the middle of the school year. In British Columbia, this corresponds to Grade 8. The international definition for the second group, *Population B,* was intended to encompass all students in the last year of secondary school who were studying mathematics as a significant part of an academic program. In other words, students who were studying college-preparatory mathematics in Grade 12 in British Columbia were to be included in Population B, but those who were not taking any mathematics or who were taking a terminal course were to be excluded. Population B was therefore defined to consist of all students in the province who were taking Algebra 12. It included approximately 40% of the Grade 12 population.

Because British Columbia was one of the first of the participants in the study to complete its data collection and to publish a local report (Robitaille, O'Shea, Dirks, 1982), few international results were available for comparison, except for a few concerning the curriculum. In addition, it has not yet been possible to examine the B.C. data for relationships between the strategies employed by teachers and the achievement levels attained by their students.

The following is a brief description of some of the major findings to date.

CURRICULUM ANALYSIS

• At the Population A (Grade 8) level, virtually all students in all of the participating countries are still studying mathematics. Moreover, the B.C. curriculum at that level coincides fairly well with the international grid developed for the study. In other words, the B.C. mathematics curriculum at the Grade 8 level is similar in many respects to the curricula of other countries.

• In some countries, only a very small proportion of the age group is still studying mathematics in the final year of secondary school; less than 10% in some cases. In British Columbia, for the Algebra 12 course, this figure is just over 30%. This fact has obvious implications for the content of the Algebra 12 course, since it must be made suitable for a comparatively large proportion of the student population.

• From the data obtained regarding the amount of time available in the school program for mathematics, it is clear that in many countries, students at the senior secondary level are able to specialize in mathematics to a much greater degree than is possible in British Columbia.

• A major difference between the Algebra 12 curriculum in British Columbia and the comparable curricula in other places is the importance given to the study of calculus. In B.C., calculus is not part of the Albegra 12 curriculum, except as one of a sequence of optional topics for the enriched format of the course. On the other hand, the study of calculus is a major component of the mathematics curriculum at the senior secondary level in other countries, notably in Europe.

• At both levels in British Columbia, little or no importance is given to the study of probability and statistics. This contrasts sharply with the importance given to these topics at either level or both levels in most other countries. More important, given the importance of these topics in contemporary society, this situation should be examined and discussed as soon as possible.

THE TEACHING OF MATHEMATICS IN GRADE 8

• Over 20% of Grade 8 teachers in B.C. teach students the British system of measurement as well as the metric system in spite of the facts that the Curriculum Guide calls for only the latter to be taught and that the prescribed textbooks refer only to the metric system. Teachers may have decided that this generation of students needs to be familiar with both systems, and, therefore, that both systems need to be taught. That the United States seems farther away from converting to the metric system than at any other time in the recent past lends additional weight to such an opinion, and this matter should be discussed as part of the curriculum renewal process.

• Teachers rarely assume that students have mastered concepts or skills in previous grades to the point where no re-teaching or review is needed. Even topics such as concept of a fraction, which is introduced in the primary grades, are reported as having been taught or reviewed in almost 90% of the classrooms. This may be partly due to a certain lack of familiarity on the part of Grade 8 teachers with the content of the mathematics curriculum at the elementary school level. On the other hand, there is evidence that students perform relatively poorer on items dealing with content teachers have not reviewed.

• There are indications that in teaching computation techniques such as multiplication of decimals, teachers emphasize the sequence of steps to be followed in using the algorithm rather than an understanding of why the algorithm works the way it does. For example, 98% of teachers agree that they teach students how to multiply decimals by referring back to multiplication of whole numbers and giving students a rule for placing the decimal point in the answer. Similar, rule-oriented approaches to teaching are apparent for topics such as operations with positive and negative numbers, and the development of formulas.

• The amount of teaching time devoted to topics is highly variable. While every Grade 8 teacher spends some time teaching about integers, 10% say they spend, at most, eight class periods on the topic, while another 10% report devoting more than 20 periods to the same topic.

• In the opinion of teachers, the principal factors that account for students' lack of satisfactory academic progress in mathematics are students' indifference, absenteeism, and lack of ability. Few feel that unsatisfactory progress can be attributed to a shortage of appropriate resources for teaching or to a lack of teachers' expertise.

• According to teachers, whether or not a given approach to teaching a particular concept or skill is easy for students to understand is the single most influential factor in determining which of a number of alternative teaching approaches to use. This factor is substantially more important than whether or not students enjoy the approach or whether that approach is used in the textbook. That a given approach is seen as being difficult for students to understand is a powerful factor in decisions not to use a particular teaching approach.

• While teachers do not select teaching approaches primarily because they are present in the textbook, many are reluctant to follow approaches other than the ones in the textbook. This indicates that, if certain topics, approaches, or orientations are to be implemented in the mathematics curriculum, care must be taken to ensure that these are included either in the prescribed textbooks or in other instructional materials made available to teachers.

• The majority of teachers emphasize only one approach to teaching a given topic. While they might use others or refer to them occasionally, they tend to emphasize only one.

• Over 25% of Grade 8 teachers of mathematics report that, in their teaching, they seldom follow the treatment given to a topic in the textbook. Since a large number also report that they rely primarily on locally developed worksheets, many teachers may not be utilizing the prescribed textbooks to any significant degree.

• Although most Grade 8 teachers agree that students should be permitted to use calculators, they are also strongly of the opinion that students should master computational skills.

THE TEACHING OF MATHEMATICS IN GRADE 12

• There are very substantial differences in the amounts of time teachers spend on the different components of the Algebra 12 course. For analytic geometry, the range is from 6 to 45 periods, with 21 periods as the mean. In the case of trigonometry, the range is from 10 to 56 periods, with the mean being 29. For algebra, the range is from 7 to 96 periods, with the mean being 49.

• Approximately 40% of Algebra 12 teachers agree with a statement that there should be a separate, semester-long course in trigonometry. Only 3% disagree with this opinion; the remainder are undecided.

• As in the case of Grade 8, Grade 12 teachers spend substantial amounts of time reviewing material covered in earlier courses.

• Relatively little time is spent on developing students' ability to construct graphs of functions.

• Despite the fact that virtually all Algebra 12 teachers permit their students to use calculators, 50% of them emphasize the computational application of logarithms in their teaching.

• In presenting various formulas, identities, or generalizations to their students, almost all of the Algebra 12 teachers provide some form of proof or informal justification. However, they seldom expect their students to be able either to replicate such proofs or to produce them on their own. Instead, they expect students to recall the formulas and apply them.

• The use of calculators in class is permitted by 97% of Algebra 12 teachers. Moreover, 16% of the teachers also use microcomputers in some manner.

• Algebra 12 teachers adhere more closely to the course outline in the Curriculum Guide than do teachers of Mathematics 8, probably because the course outline for Algebra 12 is much more detailed than is the one for Mathematics 8.

• Algebra 12 teachers are more likely to use a particular teaching approach because they feel it will be easy for students to understand than because they think students will enjoy it. That a given approach is seen as being difficult is a stronger factor in deciding not to use a given approach to a topic than is the belief that students will not enjoy it.

• In a typical Algebra 12 class, correction of homework occupies 25% of the time, lecturing by the teacher an additional 40%, and individual seat work by students, 20%. Little time is allocated for having students work together in small groups.

• Results show that Algebra 12 teachers typically do not differentiate among levels of ability in their teaching. Most teachers teach the same content in the same way and give the same assignments to all students.

ACHIEVEMENT RESULTS GRADE 8

• Over 80% of the test items developed for use in the Second International Mathematics Study were judged to be appropriate for B.C. students in Grade 8. Students were told that the test contained some items with which they would not be familiar, and they were advised to skip those items.

• All Grade 8 students were administered a 40-item test at the beginning of the school year, and the test was re-administered near the end of the course. Results on the posttest were 4.4 points higher than those on the pretest, an improvement of about 25%.

• The range of growth scores on the 40-item core test was from – 1.8 to 10.1. That is, in one class the posttest score was 1.8 points lower than the pretest score, and, at the other extreme, the performance in one class improved by 10.1 points.

• Gains in achievement from pretest to posttest were greatest on the algebra and geometry subtests, the two areas that consisted primarily of content presented for the first time in Grade 8.

• In a number of cases, classes performed poorer on one or more of the subtests on the posttest than on the pretest. The losses occurred mainly on the subtests dealing with content from earlier grades; for example, fractions, measurement, and ratio and proportion.

• Teachers were asked to indicate whether or not the mathematics content required to respond to each item had been presented to students. Overall, there was only a moderate correlation between this Opportunity-to-Learn measure and the achievement results. This may mean that teachers underestimate their students' knowledge or that they are unfamiliar with the curriculum of earlier grades. Alternatively, it may be that teachers did not interpret the Opportunity-to-Learn question in the way that had been intended. Further analyses of these results should clarify this matter.

ACHIEVEMENT RESULTS GRADE 12

• No pretesting was done with Grade 12 students. Moreover, since the test items were distributed among eight rotated forms, there was no common set of items all students wrote.

• Of the total of 136 items on the eight test forms, 45 were judged to be inappropriate for B.C. students because they dealt with content outside the Algebra 12 curriculum.

• In constructing items for the tests, four levels of cognitive behavior were used: computation, comprehension, application, and analysis. In general, results show that achievement decreases as cognitive level increases. Similar results were found at the Grade 8 level.

• On the 91 items judged to be appropriate for the B.C. curriculum, the average percent correct was 42%. Given the level of difficulty of the items, this seems a satisfactory result.

STUDENTS' ATTITUDES

• When asked to react to a list of 15 topics and activities in mathematics, students at both the Grade 8 and Grade 12 levels indicated that they enjoy them, feel they are important, and do not find them too difficult.

• Teachers gave approximately the same ratings as their students to most of the items. However, teachers feel that problem-solving is more important, more enjoyable, and more difficult than their students do.

• Teachers and students agree that mathematics is a dynamic, growing field in which new developments are continually being made and not a static, unchanging one. Teachers are more positive in this regard than their students, and Grade 12 students are somewhat more positive than Grade 8 students.

• Students indicate that their parents consider mathematics to be an important area of study, and that their parents want them to do well in mathematics. Students at both levels report that, while both parents encourage them to do well in mathematics and believe that learning mathematics is important, their fathers enjoy mathematics more than their mothers and are more likely to be able to assist them with assignments.

• Overall, Algebra 12 students' attitudes toward mathematics are slightly more positive than those of Grade 8 students.

• A clear majority of students at both levels agree that it is important to know mathematics in order to get a good job; indeed, 32% of Grade 8 students strongly agree with that opinion.

• On a subscale consisting of four items, Grade 8 and Grade 12 students indicate that both males and females are equally capable in mathematics, have the same need for careers, and have the same need to know mathematics.

• Over two-thirds of students at both levels believe that it is important for them to know something about computers, and that computers can be used in beneficial ways.

The B.C. data have been submitted to the two international processing centres located at the University of Illinois and the New Zealand Department of Education. Most of the international analyses will be conducted at those two locations, although persons at a number of other European and North American institutions will also be involved in various aspects of the analyses as well as in the preparation of the international reports.

Three volumes are scheduled for publication over the next 15 months. The first will consist of a comparative analysis of the mathematics curricula in the participating countries. The second will deal with teaching practices and their relationship to student outcomes, and the third will deal primarily with achievement and attitudes. A conference to discuss the preliminary findings on an international level will be held in May 1983 at the University of British Columbia.

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Can Manitoba Students Add?

A Note on the Manitoba Mathematics Assessment Program

Alan Wells

Alan Wells is editor of *The Manitoba Mathematics Teacher*

Yes, say the results of the Manitoba Mathematics Assessment Program 1982 carried out by the Manitoba Department of Education as an ongoing program to provide accurate and current information on the performance of Manitoba students.

Students in Grades 3, 6, 9, and 12 were assessed, using tests that were written by Manitoba faculty of education members Lars Jansson, Betty Johns, William Korytowski, Murray McPherson, and Clare O'Neill.

The objectives of the mathematics curriculum include a wide range of skills and thinking processes on a number of topics: number systems, operations, and properties; measurement; geometry; algebraic concepts; graphing and statistics; and consumer applications.

The following is a summary of the recommendations that were included in the report made under the title *Mathematics Assessment Program 1982* published by the Manitoba Department of Education.

SUMMARY OF RECOMMENDATIONS

On the basis of student test results and interpretations and of the teacher survey summaries, a large number of recommendations were generated. Some of the highlights are:

• That topics in geometry and measurement receive more attention in professional development of educators and also by teachers in their classrooms.

• That educators pay more attention to the learning of operations with decimals, particularly in division.

• That, at each grade level, concepts and skills from earlier grade levels be reviewed, in part by embedding them in a problem-solving context.

• That calculators be encouraged as a supportive tool in areas other than basic arithmetic.

Among the recommendations specific to a particular grade level are:

• That at the Grade 3 level, subtraction and division receive appropriate attention by teachers, and that they be taught as inverse operations of addition and multiplication respectively.

• That at the Grade 6 level, the topic of place value be reviewed with students and that more attention be given to work with fractions.

• That at the Grade 9 level, teachers cover the core topics designated in the curriculum guide before going on to optional topics. In particular, emphasis should be placed on basic geometry and certain aspects of algebra, such as equation-solving.

• That for both Mathematics 300 and Mathematics 301, the objectives of the courses be closely examined in the context of teacher expectations of students, university entrance requirements, availability of calculators and computers, and time allotments and organization (for example, the semester system).

Cueing into Word Problems

Walter Ryan

Walter Ryan teaches for the Roman Catholic School Board in Brevin, Newfoundland.

Why can't children solve word problems? "They can't read," respond some teachers. "They can't think," reply others. "They can't decide what to do," answer still others.

Although these reasons may be true, they do not adequately describe why so many children fail to solve word problems. To identify specific reasons why students cannot solve word problems, one needs to look at the basic steps used in the process of solving word problems.

1. Reading and comprehending the word problem.
2. Translation from the English into symbolic mathematical expressions and equations.
3. Applying the necessary computational skills to carry out the calculations.
4. Looking back and evaluating the solution process.

Research has shown that the majority of students are able to read most word problems and are able to perform the necessary computations to solve them. But the translation skill is poorly developed in many students.

One approach that is often used to teach the translation skill is to teach students to recognize certain cue words embodied in the problem statement. This approach does not equip students with translation skills that are applicable over a variety of problem situations.

CUES AND DISTRACTORS

Webster defines a *cue* as a "hint or suggestion as to what to do or when to act." In mathematics, terms such as *more than, less than, altogether* and *average,* are considered cue words by many teachers of mathematics because they appear to have a singular mathematical meaning. But all of these terms can possess other meanings different from the assumed in certain contextual circumstances.

A cue word used out of context is called a distractor. That is, the cue word is used, but does not imply the assumed mathematical meaning. Consider the following two examples:

EXAMPLE (A): On Sunday, the milkman brought 4 bottles of milk **more than** on Monday. On Monday he brought 7 bottles. How many did he bring on Sunday?

EXAMPLE (B): The milkman brought 11 bottles of milk on Sunday. That was 4 **more than** he brought on Monday. How many bottles did he bring on Monday?

In Example (A) **more than** is a cue word for addition. In Example (B) **more than** is a distractor, since it implies the operation of addition while the correct procedure is subtraction.

CUEING TECHNIQUE— A CRITICISM

Teaching students to concentrate on cue words without consideration of the syntactic context can only lead to confusion and frustration for the students when they are confronted with the cue word in a different context. Consider, for example, students who are taught to use the cueing technique on word problems containing expressions such as "five **more than** three" (5 + 3). Later when these students deal with inequalities, they will encounter expressions such as "five **is more than** three" (5 > 3). Students whose previous learning has emphasized the cueing technique will have difficulty accommodating this new idea to their existing framework, namely that **more than** "equals" addition.

The best approach to teaching translation skills is to have the student concentrate on the underlying relationships that exist in word problems. Consider, for example, the following three word problems.

1. What number is eight **less than** 20?

2. The length of the Brown's house is 20 metres. That is 8 metres **longer than** the White's house. What is the length of the White's house?

3. Don won $20. That is $8 **more than** Jack won. How much money did Jack win?

All three of these word problems can be described mathematically by the expression: 20 − 8.

An attempted solution based on cue words would produce only one correct answer. If students are to grasp the relationship that exists in these problems, they must disregard the cue words **less than, longer than, more than** and concentrate on the type of relationship that underlies the problem. The student must be able to perceive that these three word problems are structurally the same mathematically.

SUMMARY

When teaching students how to solve word problems, emphasizing cue words without consideration of the contextual situation can give students a crutch of limited value. Students will be successful at solving the word problems containing the cue words in the anticipated form, but they will have undue difficulty with the word problems containing the cue words in a different context. A better approah is to teach students to concentrate on the mathematical relationships that exist in the problem. This can be accomplished by following some basic rules:

1. Present the word problem with a cue word in anticipated form.

2. Present the word problem with the same cue word, but this time, use the cue word as a distractor.

3. Present the word problem with neither cue words nor distractors.

For example:

1. On Tuesday the milkman brought 6 bottles of milk fewer than on Wednesday. On Wednesday he brought 13 bottles. How many did he bring on Tuesday?

2. The milkman brought 13 bottles of milk on Tuesday—6 fewer than he brought on Wednesday. How many bottles did he bring on Wednesday?

3. The milkman delivers milk on Tuesday and Wednesday. In the two days he delivered 13 bottles. He delivered 6 bottles on Tuesday. How many did he deliver on Wednesday?

By discussing the three word problems above, the students should perceive that a complete reliance on cues will prevent them from getting all three problems correct. A better approach would be to consider the context in which the cue word appears. That is a consideration of the basic structure of the problem.

If word-problem assignments contain word problems of each type, they should prevent students from relying totally on cues contained within the problem.

"Face" Values

Barry Witkze

Barry Witkze teaches in Russell, Manitoba.

Here is a set of activities for students in Grades 6 to 8. Try them out. How good are your students at finding patterns? Answers and hints appear at the end of the activities.

A. Complete the last four

1 ☺ = 1 + 0 = 1

2 ☺ = 2 + 1 = 3

3 ☺ = 3 + 2 + 1 = 6

4 ☺ = 4 + 3 + 2 + 1 = 10

5 ☺ = 5 + 4 + 3 + 2 + 1 = 15

6 ☺ = 6 + 5 + 4 + 3 + 2 + 1 = 21

7 ☺ =

8 ☺ =

9 ☺ =

10 ☺ =

B. Complete the chart (use the above information)

1⊙	2⊙	3⊙	4⊙	5⊙	6⊙	7⊙	8⊙	9⊙	10⊙	11⊙	12⊙	→	100 ⊙	1000 ⊙	n⊙
1	3	6	10	15	21							→			

C. "Sign" Values (study group A, and then try group B)

GROUP A

STOP = 2 ⊙ 1

MEN WORKING = 4 ⊙

STEEP HILL = 3 ⊙ 3

DETOUR = 3 ⊙

SLIPPERY WHEN WET = 5 ⊙

YIELD = 2 ⊙ 2

DEER CROSSING = 4 ⊙ 2

BUS STOP = 3 ⊙ 1

LANDSLIDE = 4 ⊙ 3

ENTRANCE = 3 ⊙ 2

GROUP B

DEADEND =

MERGE =

NO PARKING =

ONE WAY =

FALLING ROCKS =

EXIT =

PLAYGROUND =

LOOSE GRAVEL =

NO U TURN =

D. Think of other road signs that equal:

3 ⊙ 4 ⊙ 5 ⊙

E. Counting in "face" values

Count up to 20 (or 5 ⊙ 5) in "face" values.

F. Basic skills (give answers in "face" values)

2☺ + 3☺ =	5☺ − 4☺ =	1☺ × 2☺ =
3☺ + 4☺ =	4☺ − 3☺ =	2☺ × 3☺ =
3☺2 + 2☺2 =	6☺2 − 3☺3 =	2☺2 × 3☺ =

G. Ratios and comparisons

Write 3☺ / 4☺ in simplest terms.

Which is larger 6☺ / 5☺ or 5☺ / 4☺, and by how much?

H. "Math" values match (place numbers 1 to 12 in the blanks)

2 ☺ 2	______	sum
3 ☺	______	circumference
4 ☺ 2	______	difference
4 ☺ 4	______	prime
2 ☺	______	product
3 ☺ 1	______	quotient
3 ☺ 3	______	area
4 ☺ 1	______	mathematics
2 ☺ 1	______	multiplication
4 ☺	______	factor
4 ☺ 3	______	denominators
3 ☺ 2	______	functions

COMMENTS AND ANSWERS

My Grade 7s and 8s found this activity both interesting and challenging. Read 5 ☺ as "five face"; read 4 ☺ 3 as "four face three."

A. This is a good sequence activity
Answers (in order): 28, 36, 45, 55

B. The last three parts of this chart are a real challenge and good for discussion.
Answers (on the chart):

28	36	45	55	66	78	$\longrightarrow$	5050	500500	$\frac{n(n+1)}{2}$

C. Any related words or groups of words may be used.
(e.g. "Food" values such as "pizza = 2 ☺ 2.")

Students may need the following hints:
The number of words is irrelevant.
The total number of letters is relevant.
STOP = 2 ☺ 1 actually means STOP = 2 ☺ + 1.

The answers are derived simply by counting the total number of letters and representing that number by a "face" value.
(e.g. STOP = 2 ☺ 1 means STOP has four letters: 2 ☺ = 3 + 1 or 4)

Answers (for Group B):

3 ☺ 1

2 ☺ 2

3 ☺ 3

3 ☺

4 ☺ 2

2 ☺ 1

4 ☺

4 ☺ 1

3 ☺ 1

D. Students simply try to think of other road signs that have a total of 6, 10, or 15 letters. My students came up with these:

3 (∵) bridge, school, one way, no exit, police

4 (∵) fire escape, school zone, picnic site, no left turn

5 (∵) railway crossing, reserved parking, men working ahead

E. Represent the numbers 1 to 20 in "face" values.

Answers:

1 (∵)	3 (∵)	4 (∵)	5 (∵)
1 (∵) 1	3 (∵) 1	4 (∵) 1	5 (∵) 1
2 (∵)	3 (∵) 2	4 (∵) 2	5 (∵) 2
2 (∵) 1	3 (∵) 3	4 (∵) 3	5 (∵) 3
2 (∵) 2		4 (∵) 4	5 (∵) 4
			5 (∵) 5

F. Answers:

3 (∵) 3

5 (∵) 1

4 (∵) 3

3 (∵) 2

2 (∵) 1

4 (∵) 4

2 (∵)

5 (∵) 3

7 (∵) 2

G. Answers:

3/5

5 (∵) / 4 (∵) by 1/10

H. Answers:

5, 11, 10, 1, 6, 12, 9, 8, 4, 2, 3, 7

Applications of Mathematics—Some Aspects

Don Kapoor

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Secondary mathematics, besides being a thing of many beauties and essential background for many of the students who "go on for higher studies," has direct applications to trades and occupations, personal finance, consumerism, homemaking, sports, and hobbies. Potential applications of secondary mathematics range from direct use of a single mathematical idea or technique to a fairly extensive theory-building of mathematics. However, applications of secondary mathematics are hard to teach, since they rarely occur in secondary mathematics. Teachers are uncomfortable with classroom applications of mathematics. Moreover, applications in mathematics texts are often represented by artificial "story problems," which normally appear at the end of every chapter in a mathematics text and are basically designed for reinforcement or enrichment of concepts already learned. Their use hardly convinces the student that mathematics has its applications in everyday life. Last, there is a great dearth of suitable instructional materials; so applications can become outdated faster than topics in "pure" mathematics.

Pamela Ames, of University of Chicago Laboratory Schools, has categorized mathematical applications into four broad levels:

Level Zero
These are a large collection (mostly mental) of very short statements consisting of references or allusions to uses of mathematics. They are used in class when we are dealing with a particular idea. Just refer to uses even if no applications are actually being considered or planned in a lesson.

Examples
(Lesson on vector sum) Airplane pilots use vector sums for every day trip.
(Lesson on congruence) Congruence is the basis of all mass production.

Most of the level zero statements are complete in themselves; once you have mentioned them, there is nothing more to say or do. The purpose is to keep the class in touch with the real world. Such statements do not arise automatically; you have to look for them and plan their appearance during instruction. Resolve to make one remark per unit or lesson. With effort and time, you can develop a large enough bank of these statements to make at least one such remark a day.

Level One
These are the so-called story or word problems. They are short, self-contained, artificial, real-world situations that usually pose a question that has a single solution or an easily obtainable definite number of solutions. Normally a "search model" is a linear or a quadratic equation which is readily available.

Textbooks and teachers do a fairly adequate job in this area though most of the problems are too artificial and remote.

Level Two

These are entire lessons built around a single real-world situation, and they may take from one to five class periods to complete. A single situation is investigated in depth using many different mathematical techniques. This level of application is the most important and the most neglected.

In this application stage, the plan is to work long enough within one situation to see mathematics as a resource to build understanding of the situation rather than mathematics as a tool to carve out answers to specific questions. An example of Level Two application may be:

—Building a garage or a swimming pool
—Making cost estimates or drawing floor plans.

Level Three

These are open-ended investigations. They are simply fertile ideas that I have not really dealt with yet. I have used few of these with a class, because I rarely take a whole class on an open-ended investigation. I save them for individuals who are ready to strike out alone or in small groups.

As a teacher, I place the situations that I would like to do something with, but when the ideas do get investigated, I get some Level Zero, One, and Two material from the results.

The purpose of this paper is to focus on *Level Two Applications,* since they are the most important and the most neglected in secondary mathematics. We can all accomplish this goal if we take a problem-solving strategy with a wide variety of applications and uses of mathematics. You have to look constantly for ideas. The two problems given below show how mathematics can be applied, problem-solving skills developed, and incidental learning in another area incorporated. As a by-product of this approach, students gain awareness of some of the environmental issues that confront our society today. Moreover, teaching secondary mathematics through applications will assure students that mathematics is all around us and there is no reason to fear mathematics.

PROBLEM #1

How to estimate the size of wildlife populations.

Some realizations of this problem

How many fish in a pond?
How many trees in a given forest area?
How many chips in an envelope?

Some ways to solve the problem

Count—This method is tedious, time consuming, and impractical.
Example: Catch all the live fish in a pond an count them. One way is to drain the water and count all the fish. The result? Dead fish. The end would not justify the means.

Reduce the problem to a situation that is solvable (closer to reality).

Defining parameters of the problem

How many chips in this envelope?
What is n? (n = no. of chips)

Do the following tasks:
Select 10 chips, mark them, and return them to the envelope.
Shake the envelope, select any 15 chips; record the number of marked ones, and return all of them to the envelope.
Shake the envelope and repeat (10 times in all).
Tabulate your observations.

A suggested solution
Defined variables:
n: Total number of chips in an envelope
y: Number of chips marked
q: Number of chips taken out each time
$\bar{y}$: Mean of the marked chips for x trials

Given y, q, and $\bar{y}$, we can calculate n using: $\frac{y}{n} = \frac{\bar{y}}{q}$

The table below the values for y, q and $\bar{y}$ for 10 trials.

TRIAL	NUMBER MARKED: y = 10	NUMBER SELECTED: q = 15
1	3	15
2	2	15
3	4	15
4	2	15
5	1	15
6	0	15
7	1	15
8	0	15
9	1	15
10	2	15
	TOTAL 16	
	$\bar{y}$ = 1.6	

Using $\frac{y}{n} = \frac{\bar{y}}{q}$, we have

$$\frac{10}{n} = \frac{1.6}{15}$$

$$\Rightarrow n = 93.75 \cong 94$$

The exact number of chips in the envelope was 100. The answer of 94 is a reasonable estimate. However, if the experiment is repeated a number of times, the answer will be closer to 100.

Can this problem help us to solve the original problem?
Surely, applying the same strategy on a pond problem or taking samples of the various regions of a forest can give us a reasonable estimate of the number of fish in a pond or the size of the wildlife populations in a forest.

PROBLEM #2

Mathematics of water conservation
Are we wasting part of the water supply? Let's look at the average daily water use per person. (See Table 1.)

TABLE 1
Average Daily Water Use Per Person

Purpose	Gallons	Litres
Flushing toilets	24.6	82
Washing, bathing	22.2	74
Kitchen use, drinking	6.6	22
Cleaning house, clothing	4.2	14
Washing car, watering garden	2.4	8
TOTAL	60.0	200

Adapted from Thomas R. Brehman. *Environmental Demonstrations, Experiments and Projects for the Secondary School,* West Nyack, New York: Parker Publishing Co., 1973, p. 141. (This table uses 1 litre = 0.3 gal.)

PART A
Some questions for class discussion:

1. What per cent of the total daily use of water is spent for each item?
   Which has the largest per cent?
   (Flushing toilets—41%)

Which has the smallest percent? (Washing cars, gardening—4%)

2. Make a circle graph to represent the daily water use per person.

3. How could you conserve water?

4. How much water would be used daily in U.S.A., Canada, North American Continent, Europe, Asia, or world?

5. What per cent of the world's daily use of water is taken up by Canada, U.S.A., and North America?

Answers to some of these questions give information about one person or nation or country. Applied on a national or world level, the numbers are staggering and begin to give students a sense of the enormity of the problem.

PART B

Table 2 shows the estimated daily water use in the following categories from 1940-1980 in U.S.A.: Irrigation, Utilities, Domestic, Industrial.

TABLE 2
Estimated Water Use 1940-1980
(Billions of Litres)

Year	Irrigation	Utilities	Domestic	Industrial	Total
1940	269	126	12	110	516
1950	379	227	17	144	767
1960	511	457	23	232	1222
1970	451	560	16	211	1239
1980*	514	859	18	284	1675

*Projected

From *Statistical Abstract of the United States*. 96th Annual Edition. Washington, D.C.: Bureau of the Census, 1975, p. 179.

Some questions for discussion

1. What per cent of the water used in the U.S.A. in 1940-1980 was for various categories?

2. How do you account for the increase or decrease in water consumption? Why is the increase in utilities the greatest? What category has the least consumption?

3. If domestic users save water, will the total amount used be off sharply? Why?

Table 2 shows how much water is used for various purposes each day in the U.S.A. in billions of litres. Students can see from this table that the huge amounts of water tallied in considering daily water use per person are only a small part of the total water use!

PART C

Where does water supply come from? Table 3 gives the facts needed to compute each person's share of Earth's water. Are we using more water than our share?

TABLE 3
Facts Needed To Compute Each Person's Share of Earth's Water

Earth's water supply:
109,000,000,000 gal. = 4.12×10^{11}L.

World population:
3,616,000,000 people

Area of Earth covered by oceans:
139,356,000 mi.^{2} = 3.60×10^8 km^2

Average ocean depth:
12,451 ft. = 3.79 km

Per cent of Earth's water in ocean:
92%

The data in each of these tables is estimated and will vary with the source, which enables us to emphasize another important concept: Question the sources of information closely before using data to draw conclusions.

What concepts do students learn besides getting a feel for these vital issues of natural resource?

Multiplication and division of large numbers.
Ratio, per cent, and proportion.
Interpreting and using data.
Making and interpreting graphs.
Sampling techniques.
Problem-solving.

Additional thoughts for projects

Project #1
Locate a dripping faucet.

Measure in millilitres the amount of water wasted in 10 minutes.

Predict how much water is lost in an hour; in a day.

Find out what the water rates are in your community. With this information, calculate how much the drip costs in one year. Find out the repair costs. How long will it take for the washer to pay for itself by the water it saves?

Project #2
How much water do you waste while brushing your teeth or taking a shower?

Devise a measuring scheme.

Figure out how much water you waste in a week, a month, a year, for the whole family over the same time period.

Make a table and a graph of this information.

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Strategies of Problem-Solving

Bruce Stonell

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Students should learn and be able to choose procedures for solving problems. The procedures are easy to state and recognize. However, teachers have difficulty teaching problem-solving, because, unlike teaching computational skills or concepts, no specific content is involved. In problem-solving, an individually acquired set of processes is brought to bear on a situation that confronts students.

Four procedures (steps) are inherent in problem-solving. These procedures, their descriptions and associated strategies have been compiled and adapted from a variety of sources and authors (George Polya, J.F. LeBlanc, Ohio Department of Education, Math Resource Project, 1980, NCTM Yearbook) and are listed below:

STEPS IN PROBLEM-SOLVING

Understand the Problem

What is the problem? What are you trying to find? What is happening? What are you asked to do?

Strategies:

Paraphrase the problem or question (Students restate the problem in their own words to internalize the problem.)

Identify wanted, given, and needed information (Students focus on what has to be determined from the problem statement, and they list information so that they may discover a relationship between what is known and what is required.)

Make a drawing (Students depict the information of a problem, especially situations involving geometric ideas.)

Act it out (Students can picture how the problem actions occur and how they are related, thereby gaining a better understanding of the problem.)

Check for hidden assumptions (What precisely does the problem say or not say? Are students assuming something that may not be implied? Beware of mistaken inferences.)

Devise a Plan To Solve The Problem

What operations should students use? What do students need to do to solve the problem? How can students obtain more information or data to seek the solution?

Strategies

Solve a simpler (or similar) problem (Students momentarily set aside the original problem to work on a simpler or similar case. The relationship of the simpler problem should point to the solution for the original problem.)

Construct a table (Organizing data in tabular form makes it easier for students to establish patterns and to identify missing information.)

Look for a pattern or trend (Does a pattern continue or exist? In connection with the use of a table, graph, etc., patterns or trends may be more apparent.)

Solve part of the problem (Sometimes a series of actions, each dependent upon the preceding one, is required to reach a solution. Certain initial actions may either produce a solution or uncover additional information to simplify solving the problem.)

Make a graph or numberline (Students may organize information in such a way that it makes the relationship between given information and desired solution more apparent.)

Make a diagram or model (When using the model strategy students select objects or actions to model those from the actual problem that represents the situation accurately and enables them to relate the simplified problem to the actual problem. May be used in connection with, or in place of similar strategies; i.e., acting out the problem.)

Guess and check (Students shouldn't associate guessing for a solution with aimless casting about for an answer. The key element to this strategy is the "and check"; the problem solver checks his/her guesses against the problem conditions to determine how to improve his/her guess. The student repeats this process until the answer appears reasonable. This "guess and check" strategy gets the student involved in finding a solution by establishing a starting point from which he/she can progress. Used constructively with a table or graph this strategy may be a valuable tool.)

Work backwards (Frequently, problems are posed in which the final conditions of an action are given, and a condition that occurred earlier or that caused the final outcome is asked for. Under such circumstances, working backward may be valuable.)

Change your point of view (Some problems require that a different point of view be taken. A student with a "mind set" or certain perspective of the problem may have difficulty discovering a solution. Frequently, if the first plan adopted is not successful, the student returns to the point of view and adopts a new plan. This may be productive, but it might also result in continuous failure to obtain a solution. Students should try to discard previous notions of the problem and redefine the problem in a completely different way.)

Write an open sentence or equation (Often in conjunction with other strategies—using a table, diagram, etc., students select appropriate notation and attempt to represent a relationship between given and sought information in an open sentence.)

Carry Out the Plan

For some students, the strategies selected may not lend themselves to a solution. If a plan does not work, the problem solver should revise the plan or try another plan or combination of plans.

Look Back at the Steps Taken (Consolidate Gains)

Is the result reasonable and correct? Is there another method of solution? Is there another solution? Is obtaining the answer the end of the problem?

Generalize (Obtaining an answer is not necessarily the end of a problem. Reexamination of the problem, the result, and the way it was obtained, will frequently generate insights far more significant than the answer to the specific situation.

It may enable students to solve whole classes of similar and even more difficult problems.)

Check the solution (The very length of a problem or the fact that symbolic notation is used may make students lose sight of the original problem. Does the answer appear reasonable? Does it satisfy all the problem requirements?)

Find another way to solve it (Can you find a better way to confront and deal with the problem? The goal of problem-solving is to study the processes that lead to solutions. Having discovered a solution, students should search the problem for further insights and unsuspected ideas and relationships.)

Find another solution (Students approach many problems expecting only one correct solution. In many practical, daily-life situations, there may be many answers that are correct and acceptable.)

Study the solution process (Studying the process of solution makes problem-solving more than answer-getting and can expand an individual problem into a meaningful total view of a family of related problems.)

The four steps of the above model are not necessarily discrete. For example, students may move without notice into devising a plan while attempting to generate more information to understand the problem better.

If the four-step model is used, the key is to select an appropriate strategy or strategies to help answer the questions each step suggests. The strategies listed, and those students devise, should alter the problem information, organize it, expand it, and make it more easily understood. Strategies are the tools of problem-solving, and the four-step model, the blueprint.

The Broken-Stick Problem for Four Pieces — A Geometric Solution

Diane Johnson

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Consider the problem: A stick of length 1 is broken into four pieces. What is the probability that the four pieces can be joined to form a quadrilaterial? (Assume that the three points of breakage are independent random variables, each uniformly distributed on the interval [0, 1].

In this solution of the problem, the sample space is represented as a region in three-dimensional space.

Let the lengths of the four pieces be x, y, z and $1 - x - y - z$. Since each of these lengths is positive, the sample space may be represented by the region S in three-dimensional space described by the inequalities $x > 0$, $y > 0$, $z > 0$ and $1 - x - y - z > 0$; that is, by the tetrahedron bounded by the co-ordinate planes and the plane $x + y + z = 1$. The volume of this tetrahedron is $1/6$.

The four pieces can be joined to form a quadrilateral if and only if the length of each piece is less than ½. Thus the set of selections favorable for the formation of a quadrilateral is represented by a region F consisting of those points that belong to S and whose co-ordinates satisfy the inequalities $x < ½$, $y < ½$, $z < ½$ and $1 - x - y - z < ½$.

The region F may be visualized as follows: Imagine the cube bounded by the co-ordinate planes and the planes $x = ½$, $y = ½$ and $z = ½$. F consists of the points inside this cube and between the parallel planes $x + y + z = ½$ and $x + y + z = 1$. The volume of the cube is $(½)^3 = 1/8$. The corner of the cube cut off by the plane $x + y + z = ½$ is a tetrahedron with volume $1/6\,(1/2)^3 = 1/48$ and the corner of the cube cut off by the plane $x + y + z = 1$ is also a tetrahedron with volume $1/48$. The volume of F is thus $1/8 - 1/48 - 1/48 = 1/12$.

The required probability is therefore given by:

$$\frac{(\text{The volume of F})}{(\text{The volume of S})} = \frac{(1/12)}{(1/6)} = ½.$$

For a solution of the Broken-Stick Problem for n + 1 pieces see: G.A. Heuer, "Solution to Problem E1480," *The American Mathematical Monthly,* 1962, 235-236.

A Mathematical Valentine

Joanne Harris

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One of the most familiar of the polar curves is the cardioid. The shape of the cardioid with equation $r = a\,(1 - \sin\Theta)$ is shown in Figure 1.

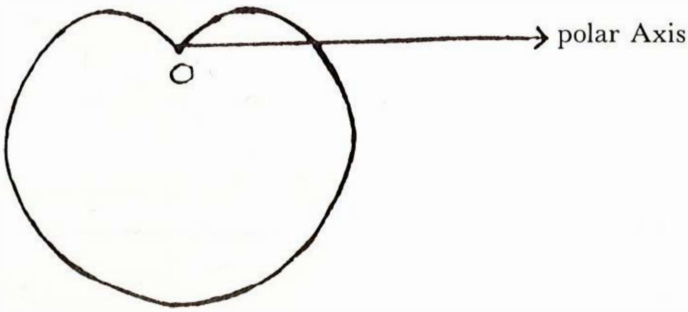

Figure 1

The heart-shaped figure, however, is flat at the "bottom"; whereas the standard Valentine heart, seen each February, is not.

In this article, the equation and construction of a Valentine heart (see Figure 2) will be discussed.

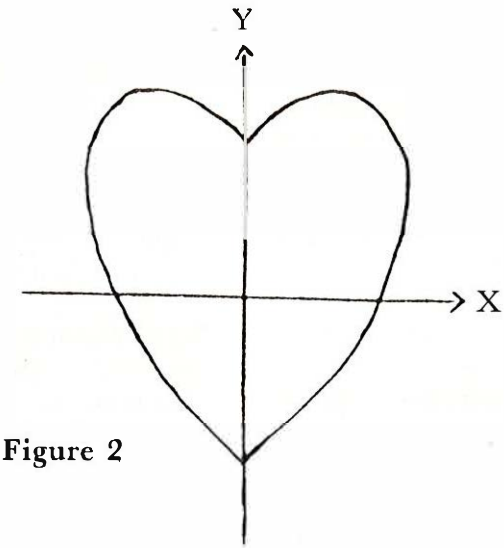

Figure 2

Consider first an ellipse of the form $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, $A > B$, and a straight line of the form $y = mx$, $m > o$. If the graphs of these two curves are superimposed on the same co-ordinate system (see Figure 3), it can be seen that the part of the ellipse on either side of the straight line looks like "half of a Valentine heart."

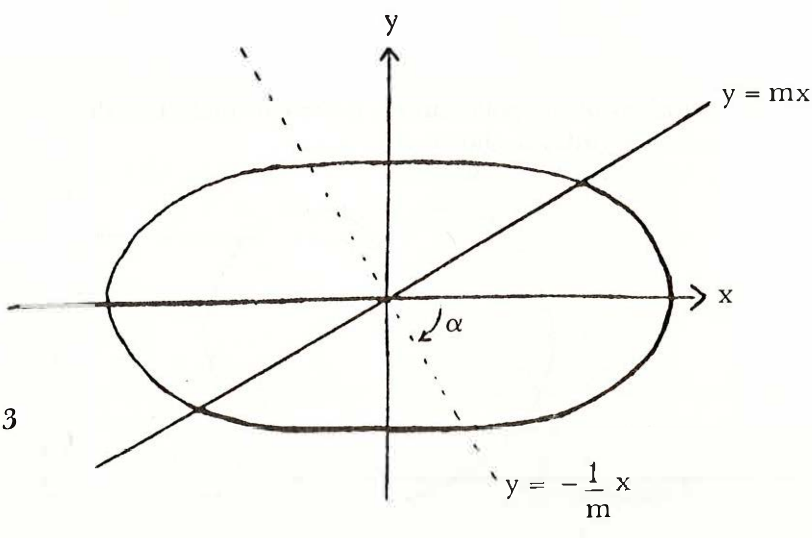

Figure 3

Thus motivated, now rotate the x, y axes through an angle α such that $\tan\alpha = -\frac{1}{m}$.

If the new axes are denoted by X and Y, then the formulae connecting the old and new co-ordinates are $x = X\cos\alpha - Y\sin\alpha$; $y = X\sin\alpha + Y\cos\alpha$, and the equation of the ellipse relative to the X, Y co-ordinate system is:

$$\frac{(X\cos\alpha - Y\sin\alpha)^2}{A^2} + \frac{(X\sin\alpha + Y\cos\alpha)^2}{B^2} = 1 \qquad (1)$$

Substituting $\sin\alpha = -\frac{1}{\sqrt{1+m^2}}$, $\cos\alpha = \frac{m}{\sqrt{1+m^2}}$ into equation (1) and simplifying gives $aX^2 + 2hXY + bY^2 = 1$ where

$$a = \frac{A^2 + B^2m^2}{A^2B^2(1+m^2)}, \quad h = \frac{m(B^2 - A^2)}{A^2B^2(1+m^2)}, \quad b = \frac{A^2m^2 + B^2}{A^2B^2(1+m^2)}.$$

This equation is quadratic in Y. Solve for Y to get

$$Y = -\frac{h}{b}X \pm \frac{\sqrt{b - (ab - h^2)X^2}}{b}.$$

The graph of this equation is shown in Figure 4.

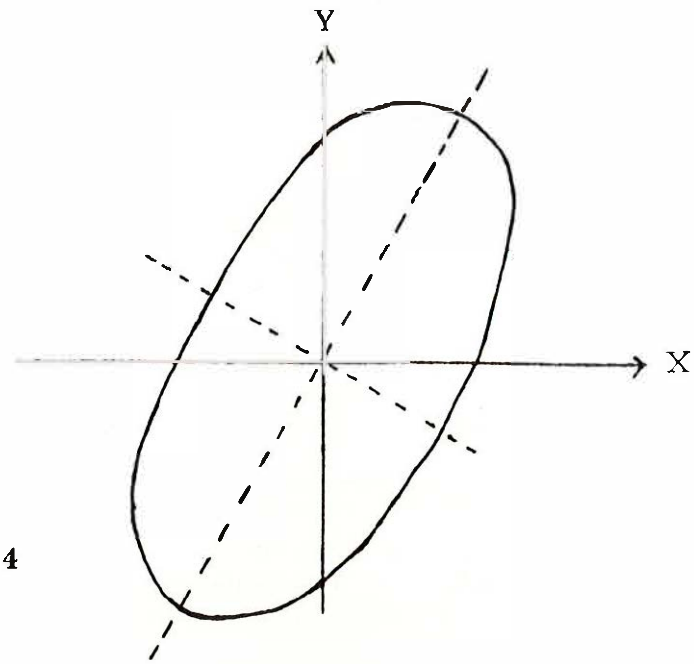

Figure 4

Then using an absolute value, the equation of a Valentine heart (see Figure 2) is:

$$Y = -\frac{h}{b}|X| \pm \frac{\sqrt{b-(ab-h^2)X^2}}{b}.$$

For example, if $b = 1$, $h = \frac{-3}{4}$ and $a = \frac{25}{16}$, then the equation becomes

$$Y = \frac{3}{4}|X| \pm \sqrt{1-X^2},$$ with points as shown in Figure 5.

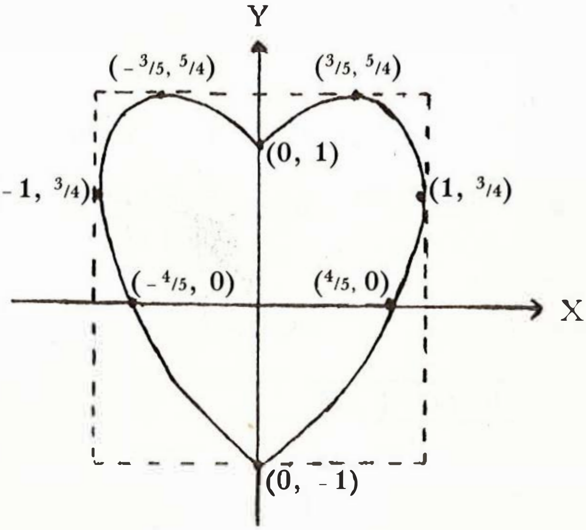

Figure 5

To draw such a Valentine heart, fix two drawing pins P and Q to a sheet of paper on a drawing board a distance 2c, where $c < A$, apart and attach to them the ends of a piece of string of length 2A. With a pencil in the loop of the string and keeping the string taut, the point R of the pencil will trace an ellipse (with foci P and Q and eccentricity c/A). See Figure 6. Cut out this ellipse.

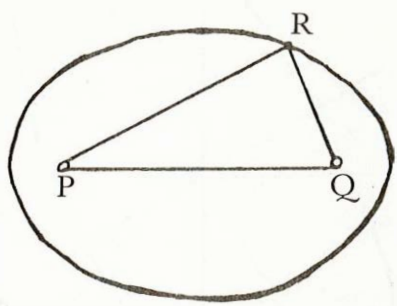

Figure 6

Finally, draw any straight line through O the midpoint of PQ (except the line POQ or the line perpendicular to POQ) and cut along this line. This yields two halves of a Valentine heart which, if joined correctly, produces the desired heart.

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