IRRATIONAL NUMBERS

ACTIVITY A

Are you still puzzled about $\sqrt{2}$? Let's see if we can solve the problem by examining the following decimal numbers:

2.236...	$.\overline{14257}$
$.625\overline{0}$	$38.\overline{12}$
$.3\overline{3}$	.01001000100001...
$9.\overline{0}$	.3149765389...
4.1232332333...	17.634979779777...

If you were to place the 10 decimal numbers above into two distinct groups, how would you group them? Try placing each of the above numbers into Group A or B according to your own definition.

Group A	Group B

What's the difference between your two groups of numbers?

____

____

Do your two groups agree with your partner's?

Check the way your teacher grouped them. Do you agree with him/her?

Whether or not you agreed with your teacher, you should notice that only 5 of the 10 numbers are rational numbers. Remember, any rational number can *always* be expressed by a repeating decimal, if you include 0 as a repetend. (If necessary, review pages 217-221 of your text.) Remember also that *any* rational number can be expressed as a fraction.

BUT, what about numbers like 4.1232332333... that don't have a repetend. Are they rational numbers? ______ No, and since they are not rational, we call them ir rational numbers. Not rational - irrational, get it?

We can also conclude that irrational numbers can't be expressed as fractions. Right? Try changing an irrational number to a fraction--when you have lots of time!

ACTIVITY B

Classify the following numbers as rational or irrational:

(a) $0.75\overline{0}$	(e) $-3.1121112...$
(b) $2.\overline{31}$	(f) $-2.\overline{1423}$
(c) $0.\overline{7}$	(g) $3.9763674...$
(d) 0.721374	(h) $.7565565556...$

A rational number can be defined as a repeating infinite decimal. The symbol R denotes these numbers.

An irrational number can be defined as a non-repeating infinite decimal. The symbol $\bar{R}$ denotes these numbers.

What about the $\sqrt{2}$? Which set do you think it belongs to: R or $\bar{R}$?

If you picked $\bar{R}$, you are correct. No matter how hard you try, you will never find a repeating pattern in the decimals. Using the computer, some mathematicians have calculated $\sqrt{2}$ to over a thousand decimal places--and still no repeating pattern.

The number π is also an irrational number. Wrong, you say! What about 3.14 or 3 1/7? Those are just approximations. If you don't believe this, ask your teacher to show you the computer's calculation of π to two thousand decimal places. See if you can find a repeating pattern.

There are some very interesting stories about the number π. Check your library to see if it has any information about this irrational number.

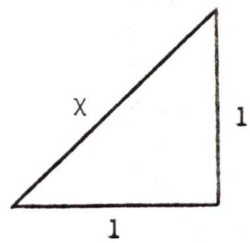
Those of you who like to prove statements might be interested in a proof that an irrational number like $\sqrt{2}$ is not rational. Ask your teacher for a copy of this.

ACTIVITY C

Read the following story:

THE PYTHAGOREAN MAFIA!*

Over two thousand years ago, there was a group of extremely dedicated and brilliant mathematicians called the Pythagorean Brotherhood. Their founder, as you may have guessed, was the famous Greek mathematician, Pythagoras. One of their favorite activities was geometry and one day while working with the pictured triangle, they were trying to express the side χ in the form of a fraction. They knew that $\chi^2 = 2$, so all they had to do was find a number times itself that equalled 2. They tried, and tried, and tried. They tried 1 2/5 with no success. They tried 1 41/100, but it didn't work. They tried 1 207/500, but again no luck.



They kept on trying until finally Pythagoras discovered a proof that showed side χ could <u>not</u> be expressed as a fraction. This meant that <u>χ was not a rational number</u>.

Pythagoras and the rest of the members of the brotherhood were so deeply shocked about this discovery that they swore all the members to the utmost secrecy. Apparently they believed that knowledge about a number that was not rational would cause a scandal in the rest of the community and possibly the world. Alas, one of the members, Hippasus, couldn't keep the secret. Legend says that the other Pythagoreans were so angry that they took Hippasus for a walk along a steep cliff by the sea and he never returned.

ACTIVITY D

Which of the following are rational numbers?

(a) $\sqrt{1}$	(e) $\sqrt{30}$	(i) $\sqrt{76}$
(b) $\sqrt{5}$	(f) $\sqrt{9/26}$	(j) $-\sqrt{16/49}$
(c) $\sqrt{16}$	(g) $-\sqrt{9}$	(k) $\sqrt{60}$
(d) $\sqrt{4/9}$	(h) $-\sqrt{29}$	(l) $-\sqrt{81}$

---

*Modified after Howard Eves, An Introduction to the History of Mathematics (New York: Holt, Rinehart and Co., Inc. 1953); and Isaac Asimov, Chap. 7 "Digging for Roots," Realm of Numbers (Boston: Houghton Mifflin Company, 1959), pp. 114-132.

Perhaps you have observed that the square root of any number is irrational unless:

1. the number is a perfect square.
2. the number is a fraction composed of perfect squares.

ACTIVITY E

Make up 4 irrational numbers of your own. Are they non-repeating and non-terminating?

(a)

(b)

(c)

(d)

Just for curiosity, see what kind of irrational numbers your partner made up.

ACTIVITY F

Generate your own irrational number:

1. Select one blue and one green cube from the game of TUF.*
2. Write down a decimal point. (Ignore the decimal point, ⊙, on the green cube.)
3. Shake and roll the two cubes or dice and record after the decimal point the two numbers on the top face, in any order, i.e., .37.
4. Roll the cubes again and record the next two numbers after the preceding two: i.e., .3749.
5. Repeat step #4, about six more times.

Is your number an irrational number?

Could you continue this process indefinitely?

Make up another one if you have time.

---

*TUF, (Baltimore: Avalon Hill Co., 1969).

Compare your number(s) with your partner's.

ACTIVITY G

Do (a) <u>or</u> (b) - not both:

(a) An ancient approximation for Pi (π) was 335/113. Is this value correct to seven decimal places? (Check your answer with the computer's and if they differ, state by how much.)

____

(b) We often use $\pi = 22/7$ as an approximation today. Calculate this fraction to 6 decimal places and compare your answer with the computer's. By how much do you differ?

____

ACTIVITY H

REAL NUMBERS:

A <u>REAL NUMBER</u> is any number that can be named by an infinite decimal.

Are all the numbers we have been studying to date, real numbers?

____

The set of real numbers is the union of the rational numbers and the irrational numbers, or the union of the repeating infinite decimals and the non-repeating infinite decimals.

Your text uses the symbol D to name the set of real numbers. (See page 263.)

Does $R \cup \bar{R} = D$?

Examine the Venn diagram on page 262 of your text. Read over exercises R to V (page 263) if you have difficulty understanding it.

Do questions 1-12 on page 265 of your text.

ACTIVITY I

A GAME OF REAL NUMBERS*

This numbers game can be played by 2 or more players. To win the game, you must be able to detect and write down on a piece of paper more numbers (of a particular kind) than any other player. The numbers to be detected and written down are those that can be made from what appears on the top faces of the cubes or dice shaken out.

To play the game, select one blue, one green, one red, one yellow, and one blank cube from the game of TUF (five cubes in all). The blank cube represents the square root symbol ($\sqrt{\ }$).

A game is finished after every player has had one shake. Play proceeds according to the following rules.

1. TIME RULE - players should agree on a time limit for each shake. e.g., 1, 2, 3, 5, 10 minutes or any other time the players agree on.

2. SCORING - when time is up, each player receives one point for each correct appropriate number. Players lose one point for each number that shouldn't be on the list.

3. TYPE OF NUMBER - Before each player shakes the cubes, he must specify which kind or type of number is to be written down. This could be any one of the following:

   (a) The Integers (I) or the subsets I_p or I_N

   (b) The Rational numbers (R) or the subsets R_p or R_N

   (c) The Irrational numbers ($\bar{R}$)

   (d) The Real Numbers (D)

   Use the Venn Diagram on page 262 of STM2 to help you imagine the kinds of numbers and their relationships.

   IMPORTANT: Since this section is based on the study of irrational numbers, one player must call irrational numbers for one shake of each game.

4. ORDER OF SHAKING - each player rolls a blue and a green cube. The player with the highest total begins, next highest total

---

*Modified after Layman E. Allen, The Real Numbers Game (Downsview, Ont.: WFF'n PROOF Publishers).

goes second and so on.

5. Ignore the $\odot$ symbol on the green cube.

6. Each symbol can be used once only in making up numbers.

OPTIONAL RULES:

7. The $\odot$ symbol on the green cube may be used to denote exponentiation. e.g.,

$$2 \odot 3 = 2^3 = 8$$

8. BIG BOASTER'S RULE

If a player thinks he has written down <u>all</u> of the specified numbers possible on the shake before the time limit has expired, he can stop the game immediately by declaring "BIG BOAST." For his bravery, he will get a bonus if right, a penalty if wrong.

If a "Boaster" is correct and has all possible numbers written down for that shake and no incorrect ones, he <u>gains</u> one point for each correct number listed and the other players receive a score of 0.

If a "Boaster" has missed one or more numbers, he <u>loses</u> one point for each number he missed or for each incorrect one and scores 0 for the shake. The other players score points as stated in rule 2.

HERE IS AN EXAMPLE:

Suppose there are four players:

Suppose the following symbols appeared on the top faces of the cubes rolled on the table -

$$4 \quad 7 \quad 1/2 \quad - \quad \sqrt{\ }$$

If the set of Integers (I) had been called, each of the following numbers could be listed since they can be formed from the symbols above:

4, 7, $3(7-4)$, 47, 74, $2(\sqrt{4})$, $5(7-\sqrt{4})$, -4, -7,

$-3(4-7)$, -47, -74, $-2(-\sqrt{4})$, $-5(\sqrt{4-7})$

Total = 14 numbers

If the set of positive Rational Numbers (R_p) had been called, each of the following numbers could be listed in addition to the positive integers above:

1/2, 3 1/2(4-1/2), 6 1/2(7-1/2), 7 1/2, 4 1/2,

47 1/2, 74 1/2, 1 1/2($\sqrt{4-1/2}$), 46 1/2(47-1/2),

73 1/2(74-1/2)

Total = 17 numbers

If the set of Rational Numbers (R) had been called, each of the following numbers could be listed in addition to the numbers listed for the sets I and R_p:

-1/2, -47 1/2, -74 1/2, -3 1/2(1/2-4), -6 1/2

(1/2-7), -1 1/2(1/2-$\sqrt{4}$), -46 1/2(1/2-47),

-73 1/2(1/2-74)

This would make a possible total of 35 numbers.

If the set of Irrational Numbers ($\bar{R}$) had been called, each of the following numbers could be listed:

$\sqrt{7}$, $\sqrt{1/2}$, $\sqrt{3}(\sqrt{7-4})$, $\sqrt{47}$, $\sqrt{74}$, $\sqrt{46\ 1/2}(\sqrt{47-1/2})$,

$\sqrt{6\ 1/2}(\sqrt{7-1/2})$, $\sqrt{3\ 1/2}(\sqrt{4-1/2})$, $\sqrt{73\ 1/2}(\sqrt{74-1/2})$,

$\sqrt{7\ 1/2}$, $\sqrt{4\ 1/2}$, $\sqrt{47\ 1/2}$, $\sqrt{74\ 1/2}$, $-\sqrt{7}$, $-\sqrt{1/2}$, $-\sqrt{3}(-\sqrt{7-4})$,

$-\sqrt{47}$, $-\sqrt{74}$, $-\sqrt{7\ 1/2}$, $-\sqrt{4\ 1/2}$, $-\sqrt{47\ 1/2}$, $-\sqrt{74\ 1/2}$

Total = 22 numbers.

If the set of Real Numbers (D) had been called all of the above numbers could have been listed for a total of 57 numbers.

The winner would be the one with the highest point total.

SPECIAL CHALLENGE:

After you have had a few games, maybe you can think of some new rules to make a different game. Or maybe you can make up a completely new one. If you do, let your teacher know and try it out with your friends. Lots of luck.

ACTIVITY J*

Do you remember the property that stated that there is at least one rational number between any two given rational numbers? This property gave the impression that the number line was continuous and was called the Density Property.

For example take two rational numbers like 2.4 and 2.5. Between these we have 2.45. Or take two rational numbers like 2.45 and 2.46. Between these we have 2.455--and so on. But is $\sqrt{2}$ a rational number? Is the number 2.45373773777. . . a rational number? The answer is no and you must be thinking that there are a bunch of holes or empty spaces in the number line. You are right. And the only way to plug the holes is to fill in with irrational numbers. Then our number line (the real number line) is continuous and we won't "fall through" anywhere.

Let us try fitting an irrational number like 2.45373773777. . . into the number line. Try it on your own first.

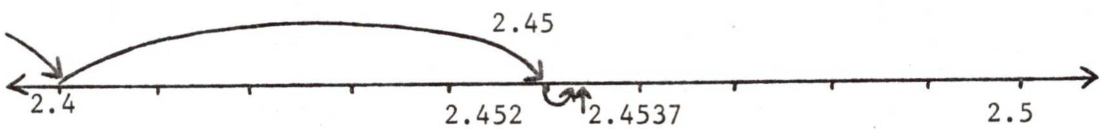
You should see that it would lie

to the right of 2.4

to the right of 2.45

to the right of 2.453

to the right of 2.4537 and so on.



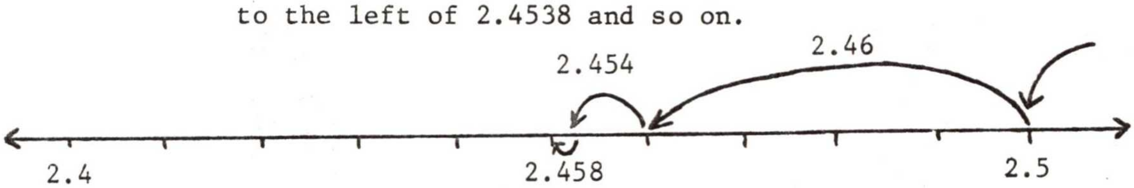
From the other side you should see that it would lie

to the left of 2.5

to the left of 2.46

to the left of 2.454

to the left of 2.4538 and so on.



---

*Modified after M. P. Dolciani, *et al.*, *Structure and Method 8*, Modern School Mathematics Series, (Boston: Houghton Mifflin Company, 1967), pp. 235-236.

Of course you could continue this process for the rest of your life. Interested? The important idea here is that there is room (lots of room) on the number line for irrational numbers. When we combine the irrational numbers with the rational numbers to get the real numbers we can say that there is exactly one point on the number line corresponding to any given real number, (completeness property).

Another way of stating this property is that no matter where you locate a point in a line, there is a real number that can be associated with that point. That is, there are no "holes" in the real number line.

A. If you are interested, try fitting an irrational number of your own on the number line or try one of these:

(1) 0.237927779237779. . .

(2) 3.141601637947. . .

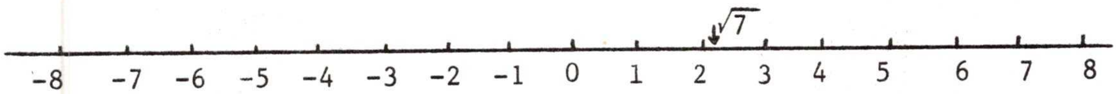
B. Copy the number line segment below:

On your copy, indicate where the graph of π lies in relation to the graphs of 3.14, 3.141, 3.1415 and 3.15, 3.142, 3.1416.

ACTIVITY K

To give you an idea of how the values of some irrational numbers compare to those of integers, try placing the following irrational numbers on the real number line as best as you can. (Use arrows--e.g. $\sqrt{7}$.)

(a) $\sqrt{12}$	(e) 1.673733. . .	(i) -6.4987342
(b) $\sqrt{1}$	(f) $-\sqrt{2}$	(j) $-\sqrt{31}$
(c) $-\sqrt{7}$	(g) $\sqrt{40}$	(k) $\sqrt{5}$
(d) $\sqrt{50}$	(h) π	(l) $-\sqrt{13}$



ACTIVITY L

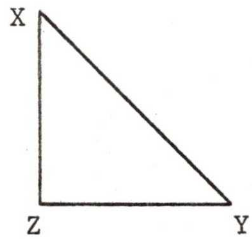
$\sqrt{2}$ on the Number Line*

The problem in this activity is to measure a line equal to $\sqrt{2}$ inches in length. The method used will be based on the formulas for the area of a right triangle and the area of a square. Do you remember the formula?

Area of a right triangle =

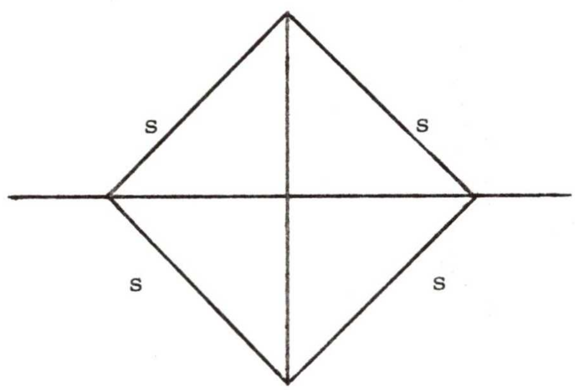
Area of a square =

Suppose we have a right triangle like Δ XYZ



$$\begin{aligned} \text{Area of } \Delta\ XYZ &= 1/2\ ab \\ &= 1/2\ (1)\ (1) \\ &= 1/2 \text{ sq. in.} \end{aligned}$$

Now if we take four of these triangles and fit them together, like the pieces of a jigsaw puzzle, we notice that they form a square. Since each triangle had an area of 1/2, and there are four such triangles in the square, the area of the square is 4(1/2), or 2.

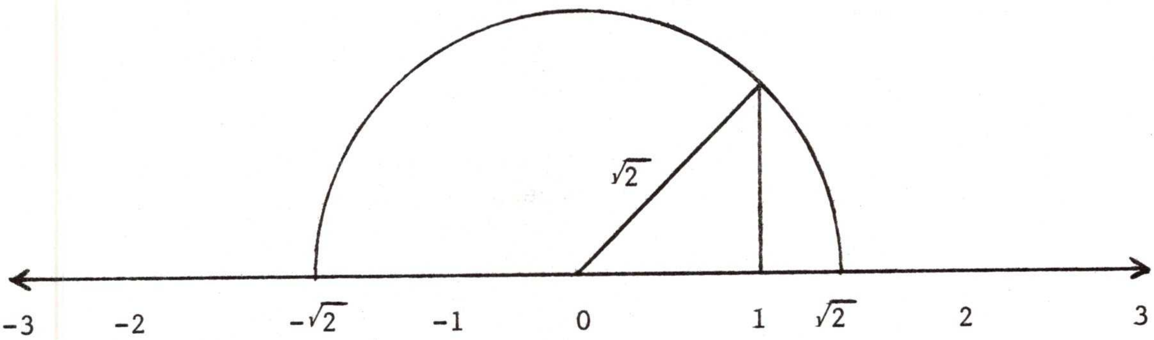


But the area of a square is equal to the length of its side multiplied by itself, is equal to 2(s^2=2). From our previous work we know s = $\sqrt{2}$.

---

*School Mathematics Study Group (SMSG), "Part III: An Experimental Approach to Functions," rev. ed. Mathematics Through Science (Stanford, California: Leland Stanford Junior University, 1964).

Now let's find $\sqrt{2}$ on the number line using this method. All we have to do is construct a right triangle with the two sides of length one inch and transfer the length of the third side to the number line.



This we can do by drawing a circle whose centre is at point 0 on the number line and whose radius is the same length as the third side of the triangle (use your compass). This circle cuts the number line in two points: $\sqrt{2}$ and $-\sqrt{2}$.

NOTE: This is still an approximate value and the accuracy depends on your construction. The idea is to find a point on the number line that corresponds to an irrational number and not a rational number.

Using the same number line if you wish, try and find points on the number line for the following coordinates.

(a) $2\sqrt{2}$

(b) $-2\sqrt{2}$

(c) $3\sqrt{2}$

(d) $\sqrt{2} + 1$

(e) $\sqrt{2} - 1$

(f) $-(\sqrt{2} + 1)$

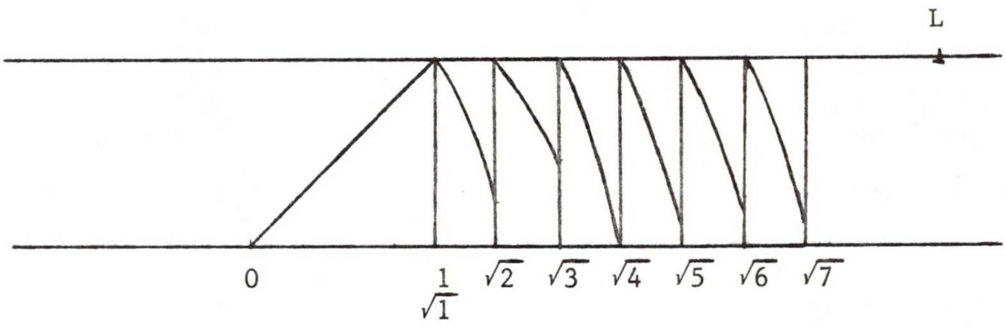
ACTIVITY M*

Irrationals on the Number Line

This activity is a special challenge and while you may be able to follow the construction below you should know the Pythagorean Theorem to understand this method.

---

*J. F. Pearcy and K. Lewis, Experiments in Mathematics, stages 1, 2 and 3 (London, Ont.: Longmans, Green and Co., Ltd., 1966).

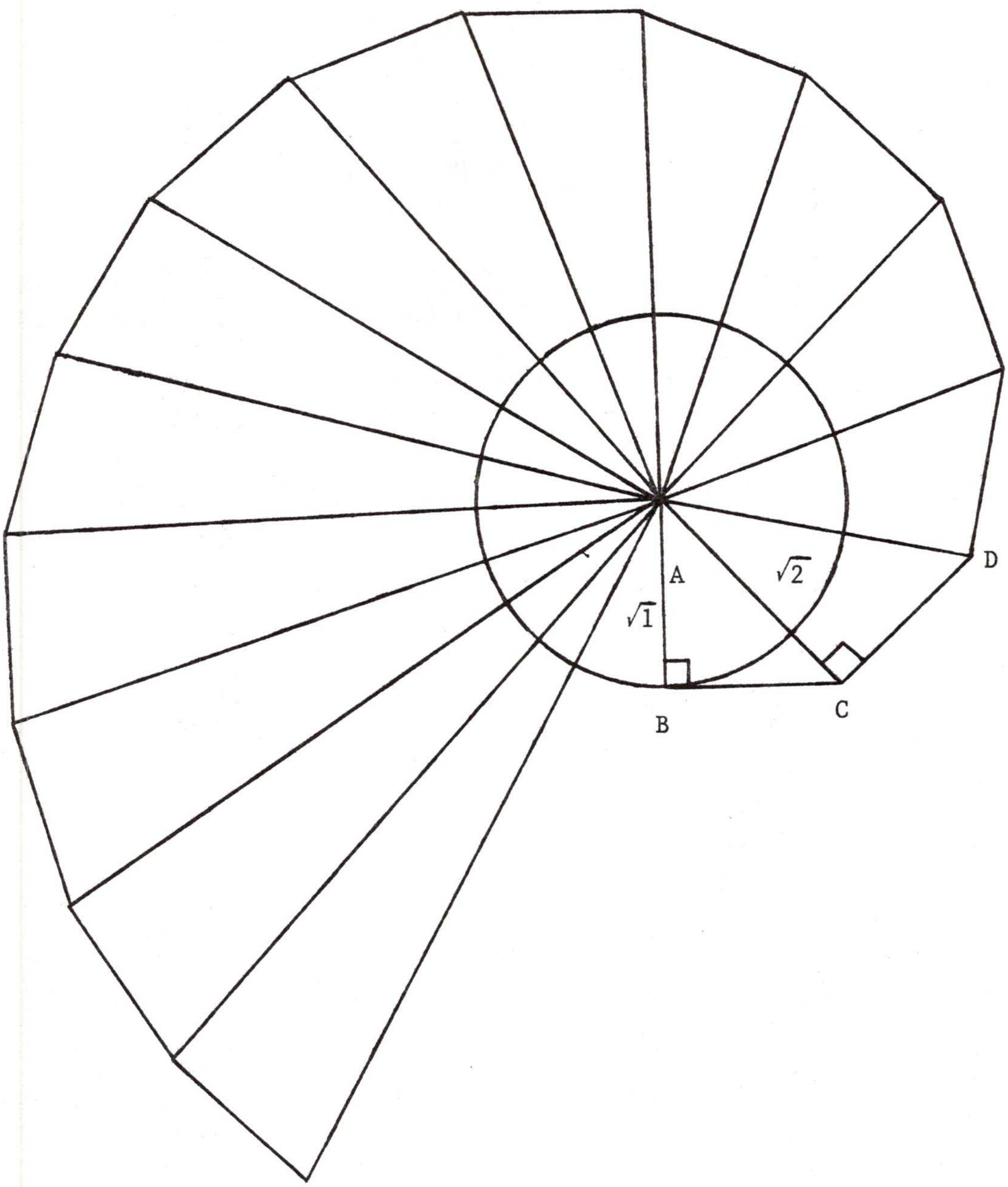


The method you used for graphing $\sqrt{2}$ (ACTIVITY L) can be extended to give a method for determining successively $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, etc. In the above construction let L be a line parallel to the number line and one inch away from it. Construct $\sqrt{2}$ as in ACTIVITY L. Draw or construct a perpendicular to $\sqrt{2}$ which meets L at point M. Using a compass with radius OA, and centre O, draw an arc that meets the number line at $\sqrt{3}$. Applying the same technique to $\sqrt{3}$, you can locate $\sqrt{4}$. You can continue this technique for $\sqrt{3}$, $\sqrt{6}$, etc.

How would you graph $-\sqrt{3}$? $-\sqrt{7}$?

ACTIVITY N

Archimedes Spiral*



*E.D. Ripley and G. E. Tait, Mathematics Enrichment (Toronto: Copp Clark Publishing Co., 1966).

CONSTRUCTION

Draw $\triangle$ ABC

$\angle$ ABC = 90°

$\overline{AB}$ = $\overline{BC}$ = 1 inch

$\overline{AC}$ = $\sqrt{2}$ inches

At C draw a 1 inch segment CD perpendicular to AC.

$\overline{AD}$ = $\sqrt{3}$. Do you know why?

You can continue constructing right-angled triangles in this fashion to obtain line segments equal to $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, etc.

The pattern forms the figure called Archimedes' Spiral.

TEACHERS' GUIDE

IRRATIONAL NUMBERS

LESSON OBJECTIVES:

1. Students learn about irrational numbers by trying to find the square root of certain rational numbers and discovering they cannot be expressed by repeating decimals and/or by observing and discovering differences in decimal patterns between a mixture of rational and irrational numbers.

   NOTE: Understanding and appreciating the existence of irrational numbers is fundamental to the development of the Real Number System and, I believe, the most important new concept that is developed in this unit.

2. Students learn that the set of real numbers (D) is the union of the set of rational numbers (R) and the set of irrational numbers ($\bar{R}$).

3. Students learn that there is a one-to-one correspondence between the set of real numbers and the set of all points in the number line. (The completeness property.)

4. Students learn a technique(s) of locating certain irrational numbers on the number line.

ACTIVITY A

This is essentially a simple discovery approach to irrational or non-repeating decimal numbers. Note that I have not distinguished between terminating and non-terminating repeating decimals. Unless some students become confused here, I don't believe this distinction is important as long as they realize that both types are repeating decimals.

ACTIVITY B

See the answer key for the computer calculation of π.

A proof that $\sqrt{2}$ is an irrational number follows on page 140. Show this to your top students or any others interested. As a special challenge they might try proving $\sqrt{3}$, is an irrational using a proof similar to the one shown.

ACTIVITY D

This activity should help students understand the real number game.

ACTIVITY E

This is an opportunity to let the students create their own irrationals and show their knowledge and understanding.

ACTIVITY F

This activity should provide some amusing and active reinforcement and understanding. Don't let students spend too much time on this one; e.g., 5 - 15 minutes - ?

ACTIVITY H

A good student may not need to do this activity. With most students it should help prepare them for the real numbers game.

ACTIVITY I

Note that I have stressed each game must include some player specifying irrational numbers on one shake. To save class time, you could display the cubes (dice) the students will use in the game and have them study the rules and example at home. Perhaps some might wish to make up their own game from sugar cubes or wooden blocks.

I don't think it is particularly important that players succeed in finding all possible numbers for a particular set as it is that they specify the correct ones for each set. Your better students might be challenged to find all possibilities.

This game will vary in motivation and activity with each student of course. I think all students should play it at least once since it not only provides understanding and reinforcement for the study of irrational numbers, but all number systems the students have studied to date and therefore is an excellent summary of their study of number systems.

ACTIVITY J

Good students might skip this activity in favour of activities M and N.

ACTIVITY K

Stress that all these locations on number line are approximate.

ACTIVITY L

There are many different constructions for locating an irrational number on the number line, but most depend on an understanding of the Pythagorean Theorem. This activity is the simplest I could find and should not prove too difficult to most of your students, I hope. You might be wise to go over the construction with some, using more accurate measurements and a more accurate number line.

Exercises(a) to (f) can be assigned at your discretion depending on the students' capabilities. Note these can be located without any further constructions.

ACTIVITY M and N

For your top and very interested students.

Students should try the activity enclosed on the Pythagorean Theorem to really understand the theory and construction for these activities, even though they may be capable of repeating the construction.

Suggested Time Limit: 3 - 5 days.

Optional Activities

H and J might not be necessary for some of your better mathematicians.

M and N are not recommended for your less talented mathematicians.

Proof for an irrational number ($\sqrt{2}$)--see Activity B

$\sqrt{2}$ - NOT RATIONAL?*

How can we prove $\sqrt{2}$ is irrational? A computing machine can quickly give us thousands of decimal digits of its numeral, and we can see that no repeating pattern has emerged. But perhaps $\sqrt{2}$ is a rational number whose simplest fraction has a very great numerator and a very great denominator. It might easily require millions of digits, instead of thousands, before a repeat begins. We can never prove the number is irrational by looking at digits, no matter how many of them we produce.

Let us try and prove $\sqrt{2}$ is rational--That is, write $\sqrt{2}$ as a fraction. But first we should review the following facts:

"A. All whole numbers are either even or odd. No number is both even and odd. (The even whole numbers are 0, 2, 4, 6, . . .: the odd ones are 1, 3, 5, 7, . . .)

B. Any even whole number can be written as two times another whole number. For example,

$$10 = 2 \times 5,$$

$$12 = 2 \times 6.$$

Any number that is twice a whole number is even.

C. The square of an even number is even. The square of an odd number is odd. (Try some examples to convince yourself.)

D. Any fraction is equivalent to the fraction changed to its simplest form--that is, the form whose numerator and denominator have 1 as their greatest common divisor. For example, 30/42 is equivalent to 5/7, which is in simplest form.

E. If a fraction is in simplest form, then its numerator and denominator are not both even. (For then, according to statement B, we could divide numerator and denominator by 2.) At least one of them is odd.

Now let us try to write $\sqrt{2}$ as a fraction. This fraction (if there is one) can be in simplest form, by statement D. The denominator is some counting number, q, and the numerator is a whole number, p.

---

*National Council of Teachers of Mathematics, More Topics in Mathematics (30th yearbook, Washington, D.C.: NCTM, 1968).

Then we have

$$p/q = \sqrt{2}, \quad \text{or} \quad p/q \times p/q = \sqrt{2} \times \sqrt{2},$$

$$\text{or} \quad p^2/q^2 = 2, \text{ by the definition of } \sqrt{2}.$$

Now, multiplying both sides of the equation by q^2, we get

$$p^2/q^2 \times q^2 = 2 \times q^2,$$

$$\text{or} \qquad p^2 = 2 \times q^2.$$

Thus p^2 is even, by statement B; and statement C implies that p is even.

Then q is odd, by statement E. Further, using the result that p is even and applying fact B again, we see that there is some whole number r such that $p = 2 \times r$. Therefore, we can substitute $2 \times r$ for p, in the equation $p^2 = 2 \times q^2$, so that

$$(2 \times r)^2 = 2 \times q^2, \quad \text{or} \quad (2 \times r) \times (2 \times r) = 2 \times q^2.$$

Now the associative and commutative properties of multiplication permit us to regroup the expression on the left as follows:

$$(2 \times 2) \times (r \times r) = 2 \times q^2$$

$$\text{or} \qquad 4 \times r^2 = 2 \times q^2,$$

or, when we divide both sides by 2,

$$2 \times r^2 = q^2$$

Now statements B and C imply that q^2 and q are even.

Hence, q is both even and odd. But, by fact A, this is impossible; q does not exist and there is no fraction for $\sqrt{2}$. This completes the proof--but if you have found it difficult, then perhaps you should start again to appreciate its subtleties."

TWO THOUSAND DECIMALS OF PI CALCULATED ON IBM 704

3.14159	26535	89793	23846	26433	83279	50288	41971	69399	37510
58209	74944	59230	78164	06286	20899	86280	34825	34211	70679
82148	08651	32823	06647	09384	46095	50582	23172	53594	08128
48111	74502	84102	70193	85211	05559	64462	29489	54930	38196
44288	10975	66593	34461	28475	64823	37867	83165	27102	19091
45648	56692	34603	48610	45432	66482	13393	60726	02491	41273
72458	70066	06315	58817	48815	20920	96282	92540	91715	36436
78925	90360	01133	05305	48820	46652	13841	46951	94151	16094
33057	27036	57595	91953	09218	61173	81932	61179	31051	18548
07446	23799	62749	56735	18857	52724	89122	79381	83011	94921
98336	73362	44065	66430	86021	39494	63952	24737	19070	21798
60943	70277	05392	17176	29317	67523	84674	81846	76694	05132
00056	81271	45263	56082	77857	71342	75778	96091	73637	17872
14684	40901	22495	34301	46549	58537	10507	92279	68925	89235
42019	95611	21290	21960	86403	44181	59813	62977	47713	09960
51870	72113	49999	99837	29780	49951	05973	17328	16096	31859
50244	59455	34690	83026	42522	30825	33446	85035	26193	11881
71010	00313	78387	52880	58753	32083	81420	61717	76691	47303
59825	34904	28755	46873	11595	62863	88235	37875	93751	95778
18577	80532	17122	68066	13001	92787	66111	95909	21642	01989
38095	25720	10654	85863	27886	59361	53381	82796	82303	01952
03530	18529	68995	77362	25994	13891	24972	17752	83479	13151
55748	57242	45415	06959	50829	53311	68617	27855	88907	50983
81754	63746	49393	19255	06040	09277	01671	13900	98488	24012
85836	16035	63707	66010	47101	81942	95559	61989	46767	83744
94482	55379	77472	68471	04047	53464	62080	46684	25906	94912
93313	67702	89891	52104	75216	20569	66024	05803	81501	93511
25338	24300	35587	64024	74964	73263	91419	92726	04269	92279
67823	54781	63600	93417	21641	21992	45863	15030	28618	29745
55706	74983	85054	94588	58692	69956	90927	21079	75093	02955
32116	53449	87202	75596	02364	80665	49911	98818	34797	75356
63698	07426	54252	78625	51818	41757	46728	90977	77279	38000
81647	06001	61452	49192	17321	72147	72350	14144	19735	68548
16136	11573	52552	13347	57418	49468	43852	33239	07394	14333
45477	62416	86251	89835	69485	56209	92192	22184	27255	02542
56887	67179	04946	01653	46680	49886	27232	79178	60857	84383
82796	79766	81454	10095	38837	86360	95068	00642	25125	20511
73929	84896	08412	84886	26945	60424	19652	85022	21066	11863
06744	27862	20391	94945	04712	37137	86960	95636	43719	17287
46776	46575	73962	41389	08658	32645	99581	33904	78027	59009

---

*D. G. Seymour, and R. Gidley, EUREKA (Palo Alto, California: Creative Publications, 1968).

You might be interested to know that it took the computer about 70 hours to perform this calculation. If you were to sit down with paper and pencil to work this out, you would have to live longer than Methuselah before you would be finished.

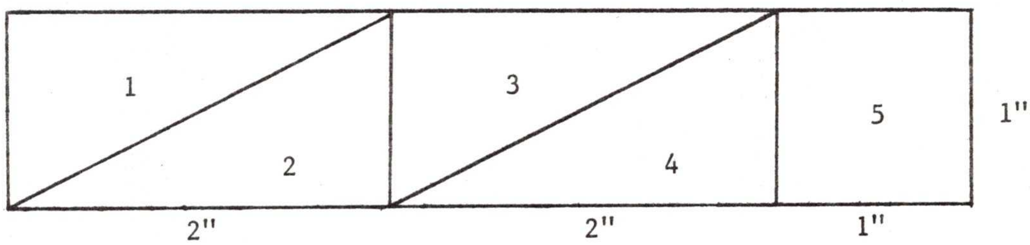
As mentioned in your lesson notes, this discovery by the Pythagoreans was very shocking. Another legend states that Pythagoras swore his associates to secrecy and slaughtered one hundred oxen as a sacrifice.

PYTHAGORAS' THEOREM*

PART 1

Equipment: Card, scissors, and glue.

Take a rectangle of card 5 inches × 1 inch and mark it out:

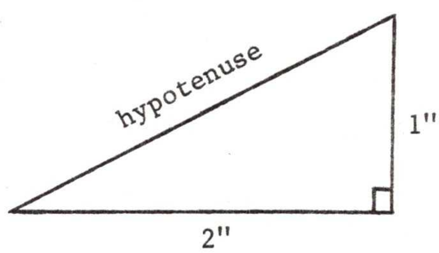


Cut out the five pieces then:

(a) Assemble pieces 1, 2, 3 and 4 to make a square.

(b) Assemble all five pieces to make a square.

(c) Draw the following triangle in the centre of a page.



Place the square formed in (b) on the hypotenuse of this triangle.

Now cut up another strip in exactly the same way as before, place piece number 5 on the shortest side of the above triangle and the square

*J. F. Pearcy and K. Lewis, Experiments in Mathematics, stages 1, 2, 3. (London, Ont.: Longmans, Green and Co., Ltd., 1966).

formed in (a) on the 2 inch side. Stick these three squares down around the triangle.

The square on the hypotenuse is made up of the same pieces which go to make the other two squares on the other sides of the right angled triangle.

This shows that the area of the square on the hypotenuse is equal to the area of the squares on the other two sides of this triangle.

Pythagoras' Theorem proves that this result is true for every right angled triangle.

This is but one dissection to demonstrate Pythagoras' Theorem. Many others exist and it is left as an extra exercise for you to find some of them.

ANSWER KEY

ACTIVITY A

Group $.625\overline{0}$, $.3\overline{3}$, $9.\overline{0}$, $.\overline{14257}$, $38.\overline{12}$

Group 2.236 ... , 4.123233 ... , .010010001... ,

.3149765389... , 17.634979779...

It does not matter whether each group of five numbers is in group A or B as long as you have the same five numbers in each group.

The basic difference between the two groups is that one group has a repetend or repeating decimal while the other group does not have a repetend or repeating decimal.

ACTIVITY B

Rational numbers: (a), (b), (c), (f)

Irrational numbers: (d), (e), (g), (h)

ACTIVITY D

Rational numbers: (a), (c), (d), (g), (j), (l)

ACTIVITY F

You could go on rolling the cubes until you are old and grey. After the first 2 or 3 rolls it would be next to impossible to roll a repeating decimal or rational number.

ACTIVITY G

(a) π (335/113) = 2.9646017

π (computer) = 3.1415926

Difference = 0.1769909

(b) π (22/7) = 3.142857

π (computer) = 3.141592

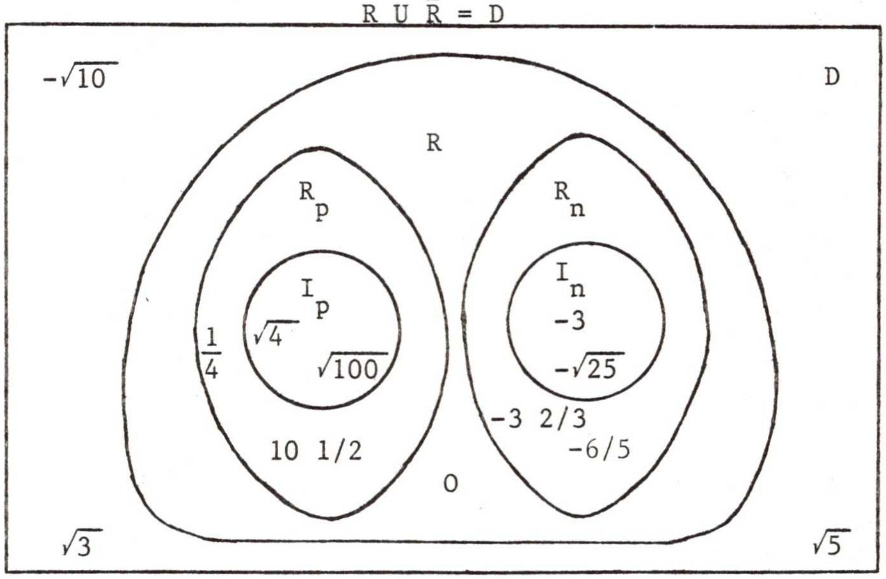
Difference = .001265

22/7 > π by 1/2500 or .04%

ACTIVITY H

Yes, all the numbers you have studied to date (both R and $\bar{R}$) are real numbers since both sets can be named by infinite decimals.

$R \cup \bar{R} = D$



ACTIVITY I

I'm glad I didn't holler "Big Boast."

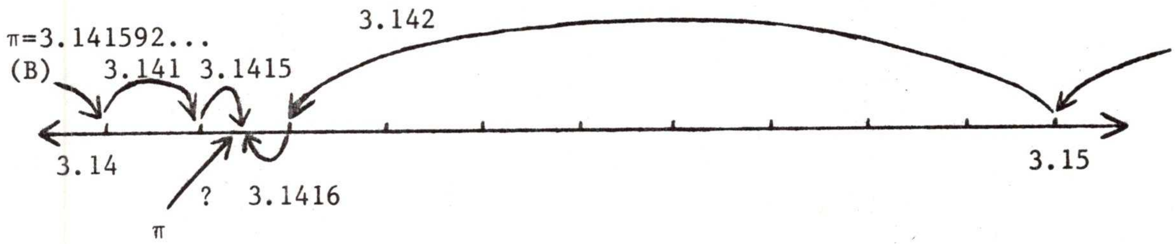
Here are some more irrational numbers:

$(4-\sqrt{7})$, $(\sqrt{7}-4)$, $(\sqrt{1/2}-7)$,

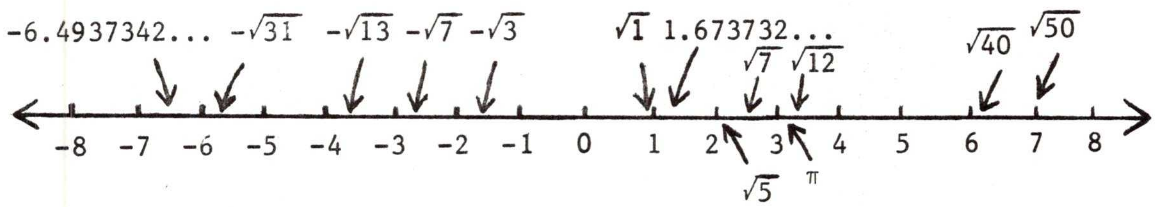
$(4-\sqrt{1/2})$, $(\sqrt{1/2}-4)$, $(\sqrt{7}-1/2)$

Did you find any others not listed in the example?

ACTIVITY J

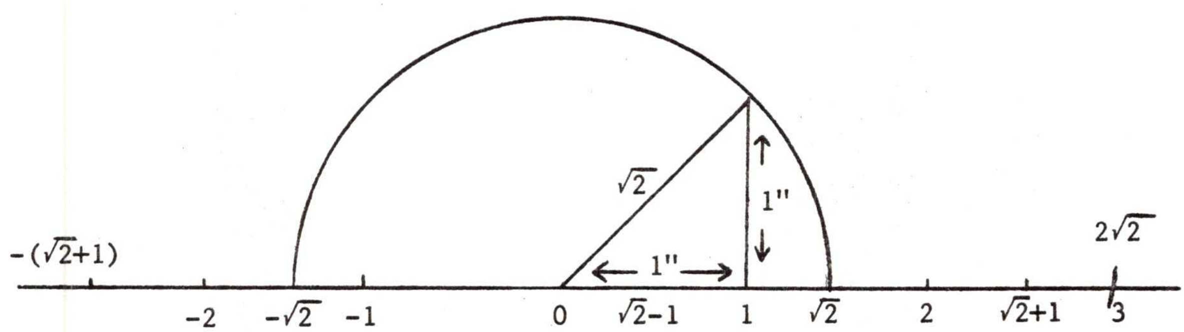


ACTIVITY K



ACTIVITY L

All these numbers can be located on the number line using your original construction for locating $\sqrt{2}$ and a compass and/or ruler.



ACTIVITY M

To graph the negative irrationals - e.g. $-\sqrt{3}$, $-\sqrt{7}$, etc.

Simply rotate the compass, with centre at 0, to the other side (negative side) of the number line. See how $-\sqrt{2}$ was located on the number line in Activity L.

ACTIVITY N

Note: For all of activities L, M, and N the unit of measure need not be restricted to one inch. It could be 1 foot, 1 centimeter or any unit segment.

Labelling the hypotenuses (hypoteni?) of Archimedes Spiral from $\sqrt{1}$ to $\sqrt{16}$ might help some students understand this construction. If they use 1 inch as their unit measure, their construction can be checked at $\sqrt{4}$, $\sqrt{9}$, or $\sqrt{16}$.