THE REAL NUMBER FIELD

Hello, Sports Fans!

You have all heard of the soccer field and the football field. Have you heard of the real number field? It might be called a mathematical game--so let's compare it to a game of, say, soccer!

GAME	SOCCER	REAL NUMBER FIELD
EQUIPMENT	boots, uniform, ball, goal posts, etc.	2, -4, 1/3, 3 5/8, $\sqrt{11}$, etc. (All the real numbers - D)
RULES	Point is scored by kicking ball into net. Ball cannot be passed with hands during play, etc.	commutative, associative, closure, distributive, inverse, identity, etc.
PLAYS	Centre passes to left forward. Right half back passes to centre, etc.	adding, multiplying, subtracting, dividing, finding square root, etc.

Do you recognize the "equipment" for the real number field? You should from many of the previous activities.

You should be quite familiar with most of the "plays" also.

What about the rules? These are very important. Sometimes it is easier to learn these by applying them to different situations. Let's try.

ACTIVITY A

SOME MATHEMATICAL "RULES" OR PROPERTIES*

I. If two things can be done in any order with the same results, then the operation is commutative. e.g., a + b = b + a

For example, if Irving puts on his left sock first and then puts on his right sock, the result is that both socks are on. If Irving puts on his right sock first and then his left sock, the result is the same. (In other words, it does not matter which order he puts his socks on.) Therefore, the operation of putting on the socks illustrates which rule or property?

____

Which of the following show this property?

(a) to go through a doorway and then open the door.

(b) to put on a coat and hat.

(c) to put on your shoes and socks.

(d) to put on a swimming suit and to jump in the pool.

(e) to add 7 to 19.

(f) to cook your dinner and to eat it.

(g) to multiply 4 by 13.

(h) to wash your face and to put on make-up.

Make up a couple of examples of your own and try them out on your friends.

II. If three things are done in a particular order, the result is the same no matter which two are done first. e.g. (a + b) + c = a + (b + c)

Which "rule" or property is this? ________________________

Does this rule hold for:

(a) the operation of putting on your boots and your hat and then your coat?

(b) the operation of mixing three different colors of paint to produce a new color?

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*Modified after Properties of Operations with Numbers UNIT 2, Experiences in Mathematical Discovery Series (Washington, D.C.: (NCTM) National Council of Teachers of Mathematics, 1966).

(c) the operation of making a right turn followed by a left turn and then a right turn?

(d) the operation of adding 11, 17, and 31?

(e) the operation of multiplying 2, 4, and 6?

(f) the problem of $(16 \div 8 \div 4)$?

(g) the problem of $16 - 8 - 4$?

Can you make up a couple of examples of your own?

III. Which of the "rules" or properties combines two different kinds of plays in a particular order?

Maybe this example will help:

3 boys and 2 girls are going to the zoo.
The bus fare is 10 cents each.
How much will it cost for the five children to go to the zoo on the bus?

Two ways to solve:

1. It will cost 3×10, or 30 cents for the boys.
   It will cost 2×10, or 20 cents for the girls.
   It will cost $30 + 20$, or 50 cents for the girls and the boys.

$$(3 \times 10) + (2 \times 10) = 50$$

2. The number of children is $3 + 2$, or 5.
   It will cost 5×10, or 50 cents to take the boys and the girls.

$$(3 + 2) \times 10 = 50$$

   Therefore $(3 \times 10) + (2 \times 10) = (3 + 2) \times 10$

Which method is the easiest? ______________

Which "rule" or property does this illustrate? __________

Use this property to determine which of the following is true.

(a) $2 \times (5 + 6) = (2 \times 5) + (2 \times 6)$

(b) $4 + (5 \times 3) = (4 + 5) \times (4 + 3)$

(c) $a \cdot (b + c) = (a + b) + (b + c)$

(d) $12 \times (10 - 9) = (12 \times 10) - (12 \times 9)$

(e) $4a \times 3a = (4 + 3)a$

Try out a couple of examples or a problem of your own on your partner.

IV. Which property states that an operation on any two members of a set results in a member of the same set?

Suppose we have a set A of integers from 0 to 500. 300 and 410 are members of this set. 300 + 410 = 710. Is this a member of the set A?

Which property does not hold for set A for the operation of addition?

Does this property hold for the following operations?

(a) the set of even numbers and addition.

(b) the set of even numbers and multiplication.

(c) the set of odd numbers and addition.

(d) the set of paints, red, yellow, and blue, and the operation of mixing the paints.

(e) the numbers 0 and 1 and addition.

(f) the numbers 0 and 1 and multiplication.

V. Doing something and then "undoing" it is an example of the

________________________ "rule" or property.

For example, the <u>inverse</u> of putting your hat on is taking it off.

1. Complete the following:

ACTIVITY	INVERSE
e.g. sitting down	standing up
e.g. going to sleep	waking up
(a) putting on shoes	
(b) making a pencil mark	

ACTIVITY	INVERSE
(c)	subtracting 3
(d) walking 3 blocks south	
(e)	dividing a number
(f)	walking down 5 flights of stairs
(g) squaring a number	
(h) subtracting 17	
(i) multiplying by 8	
(j) a right turn	

2. Which of the following have no inverses?
   (a) reading a book.
   (b) turning on a light.
   (c) putting on a coat.
   (d) scrambling eggs.
   (e) cutting the grass.
   (f) driving from Calgary to Edmonton.

VI. The ____________________ is a certain element of the set which remains unchanged when combined with any other member of the set.

1. For ordinary addition what is the element? Answer this by completing the statements below.
   (a) $5 + \_\_\_\_\_ = 5$
   (b) $\sqrt{6} + \_\_\_\_\_ = \sqrt{6}$
   (c) $\_\_\_\_\_ + 100 = 100$
   (d) $x + \_\_\_\_\_ = x$

2. What is the element that fits the property for ordinary multiplication? ________________ Make up a couple of examples.

(a)

(b)

(c)

ACTIVITY B

FINITE MATHEMATICAL SYSTEMS

Experiment 1*

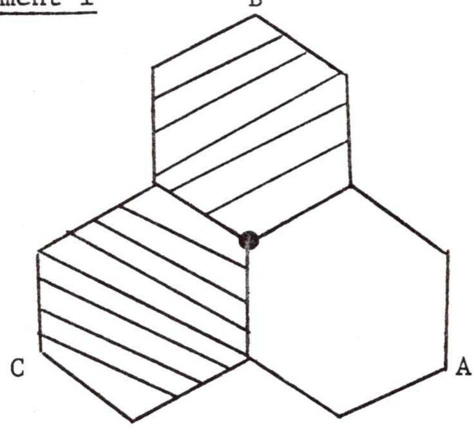

Figure X

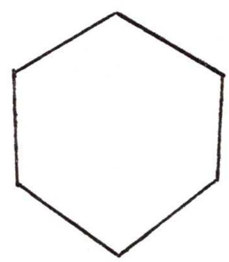

Figure Y

Trace Figure Y on a piece of paper and cut it out. This is your hexagonal counter.

Instruction A means: leave your counter where it is.

Instruction B means: rotate your counter about the black dot through 1/3 turn anti-clockwise.

Instruction C means: rotate your counter about the black dot through 1/3 turn clockwise.

Starting each time with your counter on the white hexagon in Figure X, find out where your counter would land after each of the following pairs of instruction:

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*Modified after The School Mathematics Project (SMP), Book D (Cambridge, Mass.: Cambridge University Press, 1970), pp. 1-2.

(a) B followed by C.

(b) C followed by B.

(c) A followed by C.

After B followed by C your counter lands on the white hexagon. This result could be obtained more simply by using instruction A.

Then we could say instead:

B followed by C gives A

Can you replace the other two pairs of instructions by a single instruction? (Always start from the white hexagon.)

See if you can complete the table:

"Followed by"		Second Instruction A	B	C
First	A			C
Instruc-	B			A
tion	C		A	

Now, let's see if the set {A,B,C} and the operation "followed by" have the properties of some of the mathematical systems you have studied.

CLOSURE

In Experiment 1, the members of the set {A,B,C} can always be combined by the operation "followed by" to give a result which belongs to this set. There are no results in the table which do not belong to the set {A,B,C}. Therefore the set {A,B,C} is "closed" under the operation "followed by."

COMMUTATIVE

(i) A followed by B =

(ii) B followed by A =

(iii) Does A followed by B = B followed by A?

(iv) Does A followed by C = C followed by A?

(v) Does B followed by C = C followed by B?

Therefore the set {A,B,C} is commutative under the operation "followed by."

ASSOCIATIVE

Step (a) C followed by B = A

Step (b) A followed by C = C

Does C followed by B followed by C = C?

Using parentheses we can write this expression as:

(C followed by B) followed by C = C

This means we did step (a) first.

What would happen if we did step (b) first?

e.g. C followed by (B followed by C) = ___________

B followed by C = A

C followed by A = C

Note we get the same result no matter which way we group them?

What is the result of:

(i) A followed by (B followed by C)

(ii) (A followed by B) followed by C

If we always get the same result no matter which way we group any three members in the same order of set {A,B,C} then the set {A,B,C} is associative when "followed by."

IDENTITY

Look again at the operation table from Experiment 1:

"Followed by"		Second Instruction A	B	C
First	A	A	B	C
Instruc-	B	B	C	A
tion	C	C	A	B

You can see that:

A followed by A gives A.
A followed by B gives B.
A followed by C gives C.
B followed by A gives B.
C followed by A gives C.

Notes that any member of the set remains unchanged when combined with A.

Therefore we say that A is the identity of the set.

What is the identity when rational numbers are:

(i) added? ____________________

(ii) multiplied? ____________________

INVERSES

Look again at the operation table.

The identity is A and three pairs of instructions combine to give the identity.

A followed by A gives A.
B followed by C gives A.
C followed by B gives A.

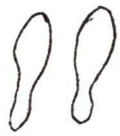
When two members of a set can be combined in either order to give the identity, they are called inverses of each other.

Note that every member of the set has an inverse.

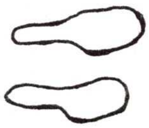
Each of the five properties we have discussed holds for the set {A,B,C} and the operation "followed by."

Now let's see if you can determine whether these five properties hold for the following system.

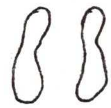
Experiment 2: "Command Arithmetic"*



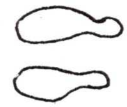
FRONT FACE (F)



RIGHT FACE (R)



ABOUT FACE (A)



LEFT FACE (L)

To obey each of these commands, start with your feet pointed toward the front of the room.

FRONT FACE (F): Do not move.

RIGHT FACE (R): Make a quarter (90°) turn to the right so that your feet face in the direction shown.

ABOUT FACE (A): Make a half (180°) turn so that you point in the opposite direction.

LEFT FACE (L): Make a quarter (90°) turn to the left.

Take turns giving and obeying some of these commands with another student.

The operation in the system of drill commands is "followed by."
Use $\oplus$ to represent "followed by."
Example: What single command is equivalent to $R \oplus A$?

Solution: The command $R \oplus A$ means Right Face followed by About Face. Starting with the feet pointed to the front, you should end up in the same position as the single command Left Face (L).

Therefore: $R \oplus A = L$.

What single command is the same as:

$A \oplus L =$

$L \oplus R =$

$R \oplus F =$

$A \oplus A =$

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*Modified after Ray W. Cleveland, unpublished manuscript, 1969.

Complete the table:

		Second Command			
	⊕	F	R	A	L
First	F				
Com-	R	R		L	
mand	A			F	R
	L		F		

(Check your table with the correct answer before proceeding.)

TESTING FOR MATHEMATICAL PROPERTIES:

CLOSURE

Is every element in the table a member of the set {F,R,A,L}?

Does the closure property hold for Experiment 2?

COMMUTATIVE

Test the operation ⊕ for commutivity by checking to see if the expressions below result in the same position.

(a) F ⊕ L and L ⊕ F.

(b) A ⊕ R and R ⊕ A.

(c) R ⊕ L and L ⊕ R.

(d) F ⊕ A and A ⊕ F.

Do you think ⊕ for the set of commands is commutative?

ASSOCIATIVE

Test the operation ⊕ for associativity by checking to see if the expressions below result in the same position.

(a) (A ⊕ R) ⊕ R and A ⊕ (R ⊕ R).

(b) (A ⊕ L) ⊕ F and A ⊕ (L ⊕ F).

(c) (R $\oplus$ L) $\oplus$ R and R $\oplus$ (L $\oplus$ R).

(d) (F $\oplus$ R) $\oplus$ L and F $\oplus$ (R $\oplus$ L).

Does the operation $\oplus$ for set of commands appear to be associative?

IDENTITY

Examine the operation table.

Is there a command which does not change the position occupied?

Which command is it?

INVERSE

Look again at the operation table.

The identity is F.

Is there a command which will make each of the following statements true?

(a) A $\oplus$ ______ = F.

(b) F $\oplus$ ______ = F.

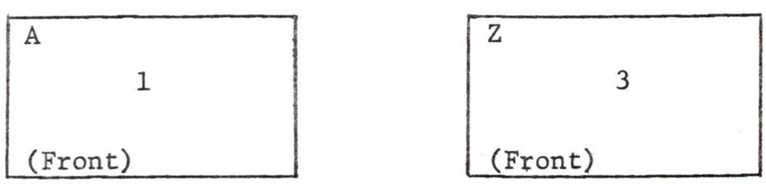
(c) L $\oplus$ ______ = F.

(d) R $\oplus$ ______ = F.

Is there an inverse for each element of the set {F,L,A,R} under $\oplus$?

<u>Experiment 3</u>*

Cut out two rectangles and mark one A and the other Z. Mark them with numbers front and back as shown below.



*Modified after The School Mathematics Project (SMP), Book D (Cambridge, Mass.: Cambridge University Press, 1970), pp. 12-13.

A
2
(Back)

Z
4
(Back)

Place them in the position below to start each move.

A
1
(Front)

Z
3
(Front)

Move M: leave the cards as they are.

Move N: turn both cards over.

Move O: turn card A over only.

Move P: turn card Z over only.

Move M gives the number 13.

Move N gives the number 24.

What numbers do you get with moves O and P?

* means the operation "followed by."

O * P gives the number 24.
The number 24 is equal to Move N.
Therefore O * P = N.

What number is given by M * N? O * M?

Try some other combinations:

*	M	N	O	P
M		N		
N				
O				N
P		O		

Can you complete the table above?

Using the above table, test the set {M,N,O,P} and the operation *, for the following properties:

(a) Closure.

(b) Commutativity.

(c) Associativity.

(d) Identity Element.

(e) Inverse for each Element.

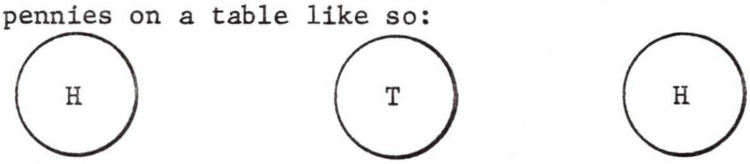
Can you think of some different moves and operations which you can make an operation table for and test for the above properties?

What about one card only and some different moves?

What about different shaped cards?

<u>Experiment 4</u>*

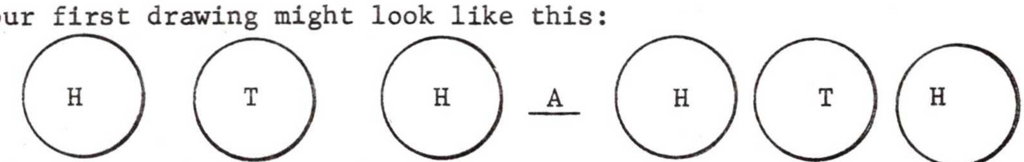
Place three pennies on a table like so:



You are allowed to turn any two pennies over at one time. See if you can do this so that all three pennies are heads up.

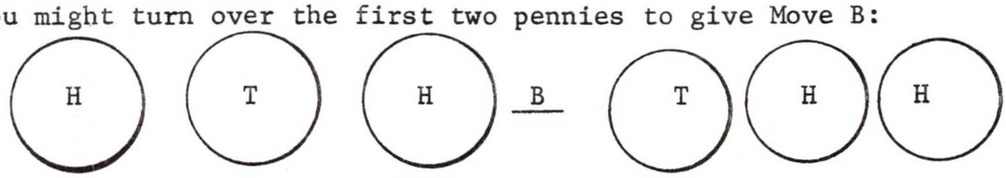
Make a drawing of each of the possible situations. Remember-- turn over only two pennies at a time.

Your first drawing might look like this:



This is Move A. No pennies were moved.

You might turn over the first two pennies to give Move B:



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*Modified after The School Mathematics Project (SMP), Book D (Cambridge, Mass.: Cambridge University Press, 1970), pp. 15-16.

You should find two more moves. Call them C and D. Remember to start from A each time. Make diagrams of moves C and D.

Complete the table below for your moves under the operation followed by, ⓧ .

	ⓧ	Second Move A	B	C	D
First Move	A		B		
	B				
	C				
	D				

Determine whether:

(a) your table is closed.

(b) your set of moves has an identity.

(c) your table is commutative.

(d) your table is associative.

(e) each member of {A,B,C,D} has an inverse.

Can you think of some different moves you might make? What would happen if you used only two coins? four coins?

Make an operation table and see if the properties hold for this new set and operation.

Experiment 5*

Pick one each of the white, red, green, and purple colored rectangular rods plus a black rod (a one-unit square piece of black paper) out of the sack of colored rods.

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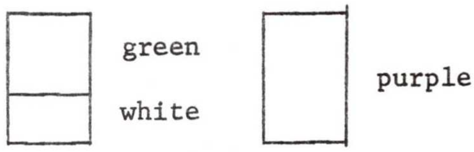
*Modified after W. J. Oosse, "Properties of Operations: a meaningful study," The Arithmetic Teacher, Vol. 16, No. 4 (Washington, D.C.: National Council of Teachers of Mathematics (NCTM), April, 1969), pp. 271-275.

Let $\ominus$ represent the operation on this set.

$\ominus$ shall mean to take any two elements (rods) of the set and stand them on their square end beside each other. The answer will be the rod which when placed upon the shorter of the two rods makes the heights as nearly equal as possible.

For example, white $\ominus$ purple = green, because green is the rod which, when placed on the shorter white rod, comes closest to making the two heights nearly equal.



Red $\ominus$ red = black (since black is the only rod which comes closest to making the heights nearly equal).

Try some other examples.

$\ominus$	black	white	red	green	purple
black					
white					green
red			black		
green		red			
purple					

Can you complete the table above?

Which of the following properties hold for the set and the operation $\ominus$?

(a) Closure.

(b) Commutativity.

(c) Associativity.

(d) Identity Element.

(e) Inverse for each Element.

Perhaps you might like to see what kind of an operation table you would get using more colored rods or maybe the first three only. See if the table obeys the properties listed above.

ACTIVITY C

NUMBER SYSTEMS

Look at Table D (on following page).

Check each square under N, Ra, and R where the particular number property holds. Where the property does not hold, state an example.

Now do the same for the real numbers D, but give an example for each property.

e.g. Addition

Closure property, yes, e.g. $17 + -15.2 \in D$

We will accept the real number properties without proof. You may try to prove some of them if you wish.

See how the different number systems compare now.

Which number system do you think is the least important? Why?

Which is the most important? Why?

ACTIVITY D

PROPERTY BEE

Examine the list of properties that follows. You will use these to play the property bee. Your teacher might suggest some abbreviations you can use to make the names simpler.

TABLE D

Operations	Addition				Multiplication			
Sets	N	Ra	R	D	N	Ra	R	D
Closure Properties								
Commutative Properties								
Associative Properties								
Distributive Properties								
Identity Element Properties								
Inverse Properties								

PROPERTY LIST

Name of Property*	Abbreviation Suggested
Commutative Property of Addition	
Commutative Property of Multiplication	
Associative Property of Addition	
Associative Property of Multiplication	
Distributive Property	
Additive Inverse Property	
Multiplicative Inverse Property (same as the Reciprocal Property)	
Identity Element Property of Addition	
Identity Element Property of Multiplication	
Well Defined Property of Addition	
Well Defined Property of Multiplication	
Definition of the Sum of Two Rational Numbers	
Definition of the Product of Two Rational Numbers	
Definition of Equivalent Ordered Pairs	
Difference Property	
Quotient Property	

The purpose of the property bee is to help you learn the mathematical properties which help you solve conditions and problems. Do you know what each property above means? Can you give an example for each property?

Can you name the properties used in solving the following condition of equality?

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*Van Engen _et_ _al_., _Seeing Through Mathematics_, special edition, Book 2 (Toronto: W. S. Gage Limited, 1964).

$w - 5 = -13.2$	Given
$w + -5 = -13.2$	______
$(w + -5) + 5 = -13.2 + 5$	______
$w + (-5 + 5) = -13.2 + 5$	______
$w + 0 = -13.2 + 5$	______
$w = -13.2 + 5$	______
$w = -8.2$	______

ACTIVITY E and ACTIVITY F

Your teacher will instruct you about these activities.

ACTIVITY G*

1. To solve a problem we must be able to translate from the English language to the mathematical or algebraic language.

   Translate the following English phrases into mathematical language:

e.g. The sum of a number and 2. $n + 2$

e.g. A number divided by 10. $n \div 10$ or $n/10$

a) A number decreased by 3.

b) The product of 3 and a number.

c) The cost in cents of n dolls costing 40¢ each.

d) The number 54 is 3 times a certain number.

e) One-half of a number.

f) The sum of six times a number and eight is equal to thirty-seven.

g) A number divided by seven is less than sixty-nine.

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*Modified after Topics in Mathematics, 29th Yearbook (Washington, D.C.: National Council of Teachers of Mathematics (NCTM), 1964), pp. 362-367.

h) Eleven decreased by a number is equal to negative six.

i) A number increased by nine is greater than three times the number.

j) A number divided by six and then increased by seventeen is greater than or equal to seventy-nine.

k) The sum of six times a number and five and a half is thirty-one.

l) The difference when seven is subtracted from four times a number is thirteen.

2. I thought of a number, multiplied it by five, then added 17, then subtracted the number I had first thought of and the result was 61. Translate this to math language. What was the number I thought of?

   Make up a problem similar to this one and see if one of your friends can solve it.

3. Now let's try translating mathematical language into English language:

e.g. $n + 8 = 39$ A number increased by 8 is equal to 39.

a) $1/4 \times n = 29$

b) $(2 \times n) + 9 < 26$

c) $29 \leq n \leq 45$

d) $(17 - n) \times 4.2 \neq 75$

e) $10n + 2$

f) $n - 19 = -7$

ACTIVITY H*

CONDITIONS WITH TWO UNKNOWNS (U = D × D)

1. Make a graph for the solution set of $y = 3x - 2$.

2. Irv says that the ordered pair (-1, -5) lies on the graph for the condition $y = 3x - 2$. Do you agree? How can you show that Irv is right?

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*Modified after Robert B. Davis, Explorations in Mathematics, A Text for Teachers (Palo Alto, Calif.: Addison Wesley Publishing Co., 1967), pp. 242-247.

3. Henry says that if you substitute the ordered pair (3.5, 8.5) into the condition for question 1, you will get a true statement. Do you agree with Henry? Does the ordered pair (3.5, 8.5) lie on the graph for $y = 3x - 2$?

4. Shelley says that she can write down all kinds of ordered pairs that will satisfy $y = 3x - 2$ without doing any calculating. What's her secret? See if you can write down some ordered pairs for $y = 3x - 2$.

5. Using the same graph paper from question 1, make a graph for the solution set of $y = -2x + 8$.

6. Without any hints find the solution set to this compound condition:

$$y = 3x - 2 \wedge y = -2x + 8$$

7. Mary-Jane says she found only one ordered pair in the solution set. Do you agree? What can you say about the point (2, 4)?

8. Find the solution set for:

$$a = 2b + 1 \wedge a = -2b + 5$$

9. Find the solution set for:

$$x = 3y + 1 \wedge x = -y + 9$$

10. Have you found a general method for solving these equations. Try to describe it.

11. Use your method to solve this condition:

$$y = 2x + 1 \wedge y = 4x + 4$$

Try and solve the following compound conditions:

12. $y = x \wedge y = 3x - 4$

13. $y = x \wedge y = 4x + 3$

14. $y = x + 1 \wedge y = 3x - 3$

15. Casey has discovered another method for solving compound conditions. How many methods have you found?

16. Casey used one of her methods to solve the following condition:

$$2r + s + 3t = 15 \wedge r = 4 - t \wedge t + 2 = 5\ 1/2$$

See if you can tabulate the solution set of this condition using one of your methods.

17. Have you found a method that works? Try it on this one:

$$-18 = 3s \wedge r + 2s = t \wedge r - 5 = 7$$

18. Look back at your solution for question 8. See if that will help you in solving this condition:

$$a > 2b + 1 \wedge a = -2b + 5$$

19. What's the best way of showing the solution set for $y > 3x + 3$? Show your solution.

20. How would you solve this condition:

$$x = 1/2\, y \vee y \leq 0$$

Show your solution.

TEACHER'S GUIDE

THE REAL NUMBER FIELD

ACTIVITY A

The main student objective of this activity is to introduce the students to some non-numerical examples of the field properties and to give them an opportunity to make up some examples of their own. Seeing these properties in a different context should make them more meaningful.

Essentially this activity is a review and you may choose to do only parts of it or make it optional for your brighter students. Try not to spend more than a day on it and make the most of some of the examples created by the students.

ACTIVITY B

The purpose of this activity is to introduce the students to some non-numerical mathematical systems that exhibit some of the same properties as the real numbers. By working with tangible objects and discovering some of the properties these ideas should become more real to the student. It should be an enjoyable and motivating activity.

Sometimes the best way to grasp an idea is to examine different manifestations of it in a different context. In this case the idea is mathematical properties in the system and the main emphasis should be on these rather than any computational skills, techniques, etc.

The experiments should be done by pupils working in small groups rather than individually. Introduce them to the properties - gradually perhaps the closure and identity properties only on the first day. After the first two experiments have been completed, many students or groups should be able to work on their own.

Another teaching method that might be useful is to ignore properties the first day and simply experiment with different finite mathematical systems and their operation tables. The period might be climaxed with a challenge to the students to make up their own set of elements and an operation. Suggestions might include rotating rectangles, triangles, or other geometric figures, binary numbers, etc. Subsequently the students can test these systems for the mathematical properties.

Caution - these systems should be closed or none of the other properties can ever be tested. Different groups of students might get involved testing another group's "creation" for the different properties.

Experiment 1

Since this is the first effort at completing an operations table, some of your weaker students might have difficulty in combining instructions. Remember it is always possible to find a single instruction which has the same effect as any pair of instructions.

Note: In all cases when testing an operation table for the different properties a quick way to determine commutativity is to check for symmetry about the main diagonal. Testing completely for associativity can be very tedious. The simplest and best approach here is to try a couple of examples and assume associativity holds unless you find a counter example.

The overhead projector might suit your purpose for demonstrating the moves and combinations possible via a transparency. You might prefer the students to cut out both Figures X and Y at the beginning of the experiment. Coloring in the shaded squares might be helpful. If time is short, you could duplicate Figure Y for the students.

Experiment 2

Students should be able to complete the operation table with very little difficulty. Have each group pick a student to execute the different commands to aid in completing the table.

As a challenge to aid students in making up their own systems, you might suggest such moves as 1/3 turn left, 1/3 turn right, etc. Students might make up systems with a different number of these moves, but encourage them to make up a system that is closed.

Experiment 3

As an alternative to making two cards, two pennies and the following instructions will give the same operation table:

M leave the pennies as they are

N turn both pennies over

O turn the left-hand penny

P turn the right-hand penny

Experiment 4

It is impossible to arrange all three pennies with heads up using the allowed moves. This should be obvious from the completed operation table which is closed for the four moves.

Be sure the students understand the distinction between the different moves. (Check the answer key.)

Experiment 5

The elements for this experiment could be cut out of colored paper and played in two dimensions if more convenient. In this case, the black unit squares (the identity) would be just a thin strip one unit long.

Note that in this case the associative property does not hold. This is important since this is their first encounter with a system that doesn't hold for all the properties. This may surprise some of your students who might think these properties are so obvious because the system is closed.

At this stage you may prefer to introduce some other examples of your own. Don't neglect to introduce closed systems that do not hold true for all the properties. Your better students might enjoy the challenge of creating their own mathematical systems. An excellent suggestion for this based on switches in various electric circuits is given in the NCTM, 27th Yearbook.*

ACTIVITY C

The purpose of this activity is to see how the real number system compares to the other main mathematical systems the students have studied. As suggested, let the students complete the table for N, Ra, and R on their own producing examples only where they think the properties don't hold. In making examples for the properties of D, suggest using some irrational numbers. This activity could be done in groups with students comparing their results.

Students should agree that N is the least important and D the most important if for no other reason than D encompasses all the other number systems.

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*"The Anatomy of a Mathematical System," Enrichment Mathematics for the Grades, 27th Yearbook (Washington, D.C.: National Council of Teachers of Mathematics (NCTM), 1963), pp. 273-281.

ACTIVITY D

PROPERTY BEE

The purpose of this activity is to make the learning of algebraic properties used in solving conditions a more enjoyable and motivating exercise. Emphasize that the purpose of learning these properties is to help them solve algebraic conditions.

The property bee or game is probably best played by two competing teams. You may divide the class into two teams or, say, eight teams and have a play-off similar to a curling or badminton draw (double knock-out) or any other competition you may prefer. Each member of each team must take his turn.

The game can be played using chains of conditions displayed on the overhead, with the two teams alternating in naming the property. To determine a winner, a point may be given for each correct answer and a point subtracted for each incorrect answer.

A review of some sample conditions and the properties used would be wise before beginning the competition. Give them the following list of properties to review.

To save time the following abbreviations for the properties might be useful. You may prefer some different abbreviations.

Name of Property	Abbreviation Suggested
Commutative Property of Addition	CA
" " Multiplication	CM
Associative Property of Addition	AA
" " Multiplication	AM
Distributive Property	DT
Additive Inverse Property	AIV
Multiplicative Inverse Property*	MIV
Identity Element Property of Addition	IEA
" " " Multiplication	IEM
Well Defined Property of Addition	WDA
" " " Multiplication	WDM
Definition of the Sum of Two Rational Numbers	D Sum
" Product of Two Rational Numbers	D Prod.
Definition of Equivalent Ordered Pairs	EOP
Difference Property	DP
Quotient Property	QP
Sum Property of "Less Than"	Sum <
Positive Multiplier Property of "Less Than"	PM <
Negative Multiplier Property of "Less Than"	NM <

*Same as the Reciprocal Property

It might be wise for the students to practice a few examples before playing the Property Bee. Perhaps a ten to fifteen minute trial game for two or three days might prepare the students and inform them where their weaknesses lie. Students should be allowed to keep the list of properties in front of them when they are playing the game as they should not be expected to memorize all these properties by name.

Please note that an extra condition appears to have been inserted in the chain in comparison to the textbook examples. This extra condition is to distinguish the definition of sum and definition of multiplication properties from the other properties in order that only one property can be named for each new condition in the chain. This saves possible confusion that might arise if two or more properties could be named to justify a condition in the chain.

If you wish to make any variations to the "bee," please feel free to do so.

Following are several chains of conditions of equality which should not present any major difficulty considering the students have dealt with similar types in previous units with the exception of the variation of the Distributive Property.*

I	a)	$9/16 = w + 5/8$	Given
	b)	$9/16 + -5/8 = (x + 5/8) + -5/8$	WDA
	c)	$1/16 = (w + 5/8) + -5/8$	D Sum
	d)	$1/16 = w + (5/8 + -5/8)$	AA
	e)	$1/16 = w + 0$	AIV
	f)	$1/16 = w$	IEA
II	a)	$-7 + t = 4\ 1/5$	Given
	b)	$7 + (-7 + t) = 7 + 4\ 1/5$	WDA
	c)	$7 + (-7 + t) = 11\ 1/5$	D Sum
	d)	$(7 + -7) + t = 11\ 1/5$	AA
	e)	$0 + t = 11\ 1/5$	AIV
	f)	$t = 11\ 1/5$	IEA

*van Engen et al., Seeing Through Mathematics, special edition, Book 2, Teacher's Guide (Toronto: W. J. Gage Limited, 1965).

III	a)	14.7 = 3.9 - 17	Given
	b)	14.7 = 3.9 + -17	DP
	c)	-3.9 + 14.7 = -3.9 + (3.9 + -n)	WDA
	d)	10.8 = -3.9 + (3.9 + -n)	D Sum
	e)	10.8 = (-3.9 + 3.9) + -n	AA
	f)	10.8 = 0 + -n	AIV
	g)	10.8 = -n	IEA
	h)	-10.8 = n	Def'n of Additive Inverse (Might leave this out of competition)
IV	a)	5/7 a = 20	Given
	b)	7/5 (5/7 a) = 7/5 × 20	WDM
	c)	7/5 (5/7 a) = 28	D Prod.
	d)	(7/5 × 5/7)a = 28	AM
	e)	1 a = 28	MIV
	f)	a = 28	IEM
V	a)	57.34 = -6.1 Z	Given
	b)	1/-6.1 × 57.34 = 1/-6.1 (-6.1 Z)	WDM
	c)	-9.4 = 1/-6.1 (-6.1 Z)	D Prod.
	d)	-9.4 = (1/-6.1 × -6.1)Z	AM
	e)	-9.4 = 1 Z	MIV
	f)	-9.4 = Z	IEM
VI	a)	5x + 7 = -13	Given
	b)	(5x + 7) + -7 = -13 + -7	WDA
	c)	(5x + 7) + -7 = -20	D Sum

	d)	$5x + (7 + -7) = -20$	AA
	e)	$5x + 0 = -20$	AIV
	f)	$5x = -20$	IEA
	g)	$1/5\ (5x) = 1/5 \times -20$	WDM
	h)	$1/5 \times (5x) = -4$	D Prod.
	i)	$(1/5 \times 5)x = -4$	AM
	j)	$1\ x = -4$	MIV
	k)	$x = -4$	IEM
VII	a)	$8/(x - .3) \sim 28/7$	Given
	b)	$8 \times 7 = (x - .3)28$	EOP
	c)	$56 = (x - .3)28$	D Prod.
	d)	$56 = 28x - 8.4$	DT
	e)	$56 = 28x + -8.4$	DP
	f)	$56 + 8.4 = (28x + -8.4) + 8.4$	WDA
	g)	$64.4 = (28x + -8.4) + 8.4$	D Sum
	h)	$64.4 = 28x + (-8.4 + 8.4)$	AA
	i)	$64.4 = 28x + 0$	AIV
	j)	$64.4 = 28x$	IEA
	k)	$1/28 \times 64.4 = 1/28(28x)$	WDM
	l)	$2.3 = 1/28(28x)$	D Prod.
	m)	$2.3 = (1/28 \times 28)x$	AM
	n)	$2.3 = 1\ x$	MIV
	o)	$2.3 = x$	IEM

VIII	a)	$4(x - 3) = 2x$	Given
	b)	$4(x + -3) = 2x$	DP
	c)	$4x + -12 = 2x$	DT
	d)	$-2x + (4x + -12) = -2x + 2x$	WDA
	d)	$(-2x + 4x) + -12 = -2x + 2x$	AA
	f)	$(-2 + 4)x + -12 = -2x + 2x$	DT
	g)	$2x + -12 = -2x + 2x$	D Sum
	h)	$2x + -12 = 0$	AIV
	i)	$(2x + -12) + 12 = 0 + 12$	WDA
	j)	$2x + (-12 + 12) = 0 + 12$	AA
	k)	$2x + 0 = 0 + 12$	AIV
	l)	$2x = 12$	IEA
	m)	$1/2(2x) = 1/2 \times 12$	WDM
	n)	$(1/2 \times 2)x = 1/2 \times 12$	AM
	o)	$1\ x = 1/2 \times 12$	MIV
	p)	$x = 1/2 \times 12$	IEA
	q)	$x = 6$	D Prod.

IX	a)	$9x + 47x = -224$	Given
	b)	$(9 + 47)x = -224$	DT
	c)	$56x = -224$	D Sum
	d)	$1/56(56x) = 1/56 \times -224$	WDM
	e)	$(1/56 \times 56)x = (1/56 \times -224)$	AM
	f)	$1\ x = (1/56 \times -224)$	MIV
	g)	$x = (1/56 \times -224)$	IEM
	h)	$x = -4$	D Prod.

X	a)	$4w = 3(w + 8)$	Given
	b)	$4w = 3w + 24$	DT
	c)	$-3w + 4w = -3w + (3w + 24)$	WDA
	d)	$-3w + 4w = (-3w + 3w) + 24$	AA
	e)	$-3w + 4w = 0 + 24$	AIV
	f)	$-3w + 4w = 24$	IEA
	g)	$(-3 + 4)w = 24$	DT
	h)	$1\ w = 24$	D Sum
	i)	$w = 24$	IEM

If the game is going well, and students are learning, then a few more of conditions of equality might be introduced at this stage. When you think the students are ready to try chains of inequalities and learn the three "less than" properties you might introduce a condition like the following in the middle of a property bee.

XI	a)	$8 + x < 3.8$	Given
	*b)	$-8 + (8 + x) < -8 + 3.8$	________ (Sum <)
	c)	$(-8 + 8) + x < -8 + 3.8$	AA
	d)	$0 + x < -8 + 3.8$	IVA
	e)	$x < -8 + 3.8$	IEA
	f)	$x < 3$	D Sum

*The property bee should become stalled here because students will be stymied by the name of this property which they have not yet learned (maybe?). <u>DO NOT TELL THEM YET</u>. Let them guess a few names and then halt the game. (Note the score if you wish to continue after the development of the following properties.)

At this stage where the property bee has been "torpedoed" by the insertion of an unknown property the students will want to learn about the property to continue the game. The following discovery activity should fill this gap in their background and permit them to continue the game. You may introduce all three "less than" properties at this point or develop a need for the last two in a similar manner to the first.

A. <u>The Sum Property of Less Than</u>:

1. Tabulate the solution set for the condition $x - 2 < 3$. $U = I$.

2. Write a standard description for the solution set of $x - 2 < 3$. $U = I$.

3. Mary Lou says that $x - 2 < 3$ is equivalent to $x < 5$. Do you agree?

4. Tabulate the solution set for $x + 3 < 8$. $U = I$.

5. Al says $x + 3 < 8$ is equivalent to $x - 2 < 3$. Do you agree?

6. Marlene figured out that $x + 13 < 18$ is equivalent to $x < 5$. How did she do it?

7. Jenny obtained the condition $x - 4 < 1$ from $x < 5$ and said they were equivalent. Make up three more conditions that are equivalent to $x < 5$. Compare your conditions with some of your classmates.

8. Bill said $x + 12 < 16$ is equivalent to $x < 5$. Henry says he's wrong. Who's right?

The purpose of this type of approach is to get the students to discover and use the sum property of less than in finding equivalent conditions. Hopefully the students will come up with a method equivalent to the sum property of less than. Call it, say, Marlene's rule, if you wish to introduce the formal name when everyone agrees on this method of finding equivalent conditions and add it to their property list for the property bee.

After this stage or before if you prefer you might ask the students to find which of the following conditions are equivalent to $x > -7$ ($U = D$) [Ans.: (a) (b) (f)]

(a) $x - 8 > -15$

(b) $x + 9 > 2$

(c) $x + 2 > 5$

(d) $x + 33 > 25$

(e) $x - 19 > -12$

(f) $x - 17 > -24$

Or you might ask them to make up equivalent conditions of their own.

e.g., Make up three conditions that are equivalent to $w + 9\ 1/3 > 15$ or $t - 1 < 49.3$.

A similar approach could be used to develop the remaining two less than properties.

B. The Positive Multiplier Property of Less Than:

1. Tabulate the solution set for the condition $3t > 18$. U = I.

2. Harry says that $3t > 18$ is equivalent to $5t > 6$. Henriette says he is wrong. Who is right?

3. Marcia says that $2t > 12$ is equivalent to $3t > 18$. Is she right? How did she do it?

4. Bob found out that $8t > 48$ is equivalent to $3t > 18$. How do you think he did this?

5. George found a method that showed him $1/2\ n > 3$ is equivalent to $3t > 18$. What's his method?

6. See if you can find a method to make three conditions equivalent to $w < 2\ 1/2$.

Following this question is probably an opportune time to ask the class for their methods and discover the property they are using. After agreement is reached on the positive multiplier property or whatever name you choose to call it for the present, the following exercises might prove beneficial.

Which of the following conditions are equivalent to $n > 4\ 1/3$? U = D. [Ans.: (b) (c) (e) (f)]

(a) $2n > 4\ 2/3$

(b) $3n > 13$

(c) $1/6\ n > 13/18$

(d) $5n < 21\ 2/3$

(e) $7n > 30\ 1/3$

(f) $9.23\ n > 40.0659$

Make up three conditions that are equivalent to 2/5 w < 6 or 5 1/2 x > 1/3.

C. The Negative Multiplier Property:

1. Tabulate the solution set of -2w < 16. U = I.

2. Marlene says that -2w < 16 is equivalent to w < -8. Do you agree?

3. Gary says he disagrees with Marlene and says that w > -8 is equivalent to -2w < 16. Who do you agree with?

4. Gary also comes up with the condition 4w > -32 and says this is equivalent to -2w < 16. How do you think he got from -2w < 16 to 4w > -32?

5. Janice got the condition 1/2 w > -4 from -2w < 16. How did she do it?

6. Can you find a method for showing 6w > -48 is equivalent to -2w < 16?

7. Henry says 10w > 80. Is he right?

8. Using your method (have you got one yet?) find three conditions equivalent to -1/2 c > 3.

9. What's your method? See how it compares with your classmates' methods. What do you call it?

10. Find out which of the following conditions are equivalent to -3t > 5. U = D. [Ans.: (c) (d) (e)]

    (a) -6t < 10

    (b) -9t < -15

    (c) t < -1 2/3

    (d) 3t < -5

    (e) 4 1/2 t < -7 1/5

    (f) 2t > -3 1/3

11. Make up three conditions equivalent to -2/5 r > 4 or -3y < 7 1/4.

After the three "less than" properties have been added to the students' lists, you may wish to resume the previous property bee game or start another one using conditions of the following type in addition to the previous kinds

e.g.	(a)	$-4s - 11.3 > 38.9$	Given
	(b)	$-4s + -11.3 > 38.9$	DP
	(c)	$(-4s + -11.3) = 11.3 > 38.9 + 11.3$	Sum <
	(d)	$-4s + (-11.3 + 11.3) > 38.9 + 11.3$	AA
	(e)	$-4s + 0 > 38.9 + 11.3$	AIV
	(f)	$-4s > 38.9 + 11.3$	IEA
	(g)	$-4s > 50.2$	D Sum
	(h)	$-1/4(-4s) < -1/4(50.2)$	NM
	(i)	$(-1/4 \cdot -4)s < -1/4 \cdot 50.2$	AM
	(j)	$1\ s < -1/4 \cdot 50.2$	MIV
	(k)	$s < -1/4 \cdot 50.2$	IEM
	(l)	$s < -12/55$	D Prod.

Another Property Bee?

(You might prefer this as a substitute or an additional version.)

Arrange your class in groups of 2 or 3, and draw up a single or double knock-out competition. When two teams begin a competition they may write up chains of conditions (involving a specified number, say 10 or 15) and exchange them. The winner is the team with the most correct properties. You might impose a time limit. No books allowed, just the list of properties. Add any other rules you think might help or ask the students for some suggestions on building the game.

ACTIVITY E

CHAIN PUZZLES

As a follow-up to the property bee the following activity should be excellent. The objective is to challenge the student to put in correct order a mixture of equivalent conditions. In essence, it is an intermediate step between identifying the properties used in solving conditions and solving conditions on their own.

For example, consider the following condition and its chain of equivalent conditions leading to the solution set.

$-1/3\ t - 11\ 2/5 > 36\ 3/5$ <u>Given</u>

$-1/3\ t + -11\ 2/5 > 36\ 3/5$

$(-1/3\ t + -11\ 2/5) + 11\ 2/5 > 36\ 3/5 + 11\ 2/5$

$-1/3\ t + (-11\ 2/5 + 11\ 2/5) > 36\ 3/5 + 11\ 2/5$

$-1/3\ t + 0 > 36\ 3/5 + 11\ 2/5$

$-1/3\ t > 36\ 3/5 + 11\ 2/5$

$-1/3\ t > 48$

$-3(-1/3\ t) < -3 \times 48$

$(-3 \cdot -1/3)t < -3 \times 48$

$1\ t < -3 \times 48$

$t < -3 \times 48$

$t < 144$

Cut out each equivalent condition separately, mix them up and place them in an envelope. The student's task is to put these conditions in the correct order starting with the given condition. In addition you might wish your students to name the properties used.

For classroom use, it would be wise to make up 20 to 40 conditions of equality and inequality with varying degrees of difficulty. Label the outside of the envelope with the given condition. You might include an answer key in the envelope to be used after completing the chain or post the answers in some convenient place.

Another use for the envelopes of conditions is to challenge the student to solve the given condition without opening the envelope. The previous activities should have provided a good background for the students solving conditions on their own and, if unsuccessful, the student could seek help from the solution in the envelope, from his classmates or the teacher. In this way, students are free to work and learn on their own and the teacher is involved only with students who need him.

ACTIVITY F

SOLVING CONDITIONS OF EQUALITY AND INEQUALITY IN ONE VARIABLE

The following are additional suggestions you might use as you see fit in developing your students' ability to solve conditions in one variable.

1. At this stage the student should be ready to try solving conditions on his own given the original condition only.

You might write a condition like the following on the board or overhead projector.

e.g. $3n + 11 = 134$

Then ask the students to solve this by developing a complete chain of conditions. You might prefer some easier conditions to start with:

e.g. $\chi - 11 = 13$ or $1/3\ y > -18$

Another approach might be to write the given condition on the board and in a space below the final solution set ask the students to complete the chain of conditions leading to the final solution set.

e.g. $.5\chi < .35$

) Students must
) complete
) the chain.

$\chi < .7$

The teacher must use his own discretion with respect to requiring the students to complete all of the chain. When convinced a student knows the properties for solving a condition satisfactorily or when doubtful as to the benefit of this procedure with a student, short-cuts in solving these conditions are quite in order.

e.g. A student might solve the following condition like so:

$-3y + 9 < 15$

$-3y < 6$

$y > -1/2$

To facilitate this type of condition-solving questions of the following type might be useful.*

(i) Is $-9y + 2y < 8$ equivalent to $-7y < 8$?

(ii) How do you obtain $y > -1\ 1/7$ from $-7y < 8$?

(iii) How can you obtain $2/7\ t = 9$ from $2/7\ t - 17\ 1/4 = -8\ 1/4$?

(iv) How do you get $t = 31\ 1/2$ from $2/7\ t = 9$?

The following are examples of conditions you could assign the students to use with any of the preceding methods or as

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*Modified after Robert B. Davis, Discovery in Mathematics, A Text for Teachers (Reading, Mass.: Addison Wesley Publishing Co., 1964), pp. 139-144.

you see fit. U = D

a) $x + 3\ 1/2 > 4\ 1/3$

b) $-5y < -14.2$

c) $13\ 1/2 - t < 14\ 1/3$

d) $13 = \sqrt{49} + s$

e) $w - 23 > 18\ 1/2$

f) $3/8\ r = -\sqrt{4}$

g) $2.2\ n = -11$

h) $5c - 17.3 = 14$

i) $5c - 17.3 > 14$

j) $-5c - 17.3 > 14$

k) $3r - 5r > -2$

l) $17\ 1/4 - 3/5\ z < 72$

m) $6(x - 3) = 3x$

n) $1/3(s - 2/5) = 7/10$

o) $(15r - .2) = 21r$

p) $14.7 = 3.8 - n$

q) $5/(n - .7) \sim 45/9$

r) $\sim (3k + 9) < 5$

You might wish the students to graph the solution set of some of these on the real number line.

2. You might prefer a more open-ended approach to solving conditions. For example you might write $2t + 5 < 15$ followed by the solution $t < 5$. Then ask the students to show as many different methods as they can of finding the solution. Some methods are trial and error, using chains of conditions, changing the inequality to an equality and solving and then substituting the inequality back again and using an algebraic formula. Compare the different methods and ask the student which he thinks would be the best method for solving all types of conditions accurately.

3. Compound conditions in one variable:*

The following discovery approach illustrates one method of

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*Modified after Robert B. Davis, Discovery in Mathematics, A Text for Teachers (Reading, Mass.: Addison Wesley Publishing Co., 1964), pp. 242-247.

introducing these types of conditions.

(i) Ask the students to make a graph of the solution set of $y + 7\ 1/2 > 4\ 1/4$ on the real number line.

(ii) Graph the solution set of $-\ 2/3\ y > -4$ on the same number line.

(iii) Now graph the solution set of $y + 7\ 1/2 > 4\ 1/4 \wedge -\ 2/3\ y > -4$.

Since U = D, the solution set of a simple condition of inequality is infinite and this rules out tabulating the solution set. The best pictorial way of displaying the solution set for conditions of inequality in one variable is the number line.

Ask the students to graph the solution sets of the following. If necessary, do one or more examples like the above.

a) $3(x + 4) = 12 \vee \sqrt{7} - x < \sqrt{7}$

b) $3r + 2\ 1/3\ r < -45 \wedge r - 49.2 > -61.8$

c) $r > -7 \vee 4/3 - r = 8\ 1/3$

d) $3t > -2 \wedge x - 14\ 1/4 < 14\ 7/8$

e) $-3y = 21.9 \vee 2y + 3y > -36.5$

f) $12\ 1/2 < x + 11\ 7/10 \wedge -6/5 + x < 1/2$

ACTIVITY G

The following exercises in this activity should serve to prepare your students for solving word problems in addition to preparing them for magic algebra. The essential feature of algebraic word problems is the use of a variable, letter or placeholder to represent the unknown. Therefore emphasis lies not on solving any condition but in writing the correct mathematical phrase or sentence. Opportunity is also provided in translating mathematical sentences into English.

This activity will be of particular value to your slower students. Some of your more creative students might enjoy the challenge of making their own problem from a condition.

ACTIVITY H

MAGIC ALGEBRA*

The primary purpose of playing magic algebra with your students is to prepare them for solving word problems. It should also give them some valuable practice in solving algebraic phrases and creating their own algebraic tricks.

Have you ever had a friend tell you what number you had calculated after a number of computations, without seeing your answer? How did he do it? Let's find out by trying one of these tricks.

Follow the steps given below:

Step 1. Take a number.

Step 2. Add 5.

Step 3. Multiply by 2.

Step 4. Subtract 8.

Step 5. Divide by 2.

Step 6. Subtract the number you started with.

Your answer should be 1. If not, check your work.

Try another number; your answer will still be 1.

Can you explain why using algebra?

Solution:		Algebraic Solution:	
Step 1.	Take a number.	n	n
Step 2.	Add 5.	$n + 5$	$n + 5$
Step 3.	Multiply by 2.	$2n + 10$	$2(n + 5)$
Step 4.	Subtract 8.	$2n + 2$	$2(n + 5) - 8$
Step 5.	Divide by 2.	$n + 1$	$\frac{2(n + 5) - 8}{2}$
Step 6.	Subtract the number you started with.	1	$\frac{2(n + 5) - 8}{2} - n$

Using a variable (n in this case), these tricks can always be proved. Make up one of your own that will give a different number as the answer.

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*Modified after W. H. Glenn, and D. A. Johnson, Fun With Mathematics (St. Louis, Mo.: Webster Publishing Co., 1961).

e.g.		Example:	Algebraic Solution:
Step 1.	Take a number.	12	n
Step 2.	Add the same number plus 2.	26	n + n + 2
Step 3.	Subtract 10	16	2n - 8
Step 4.	Divide by 2.	8	n - 4
Step 5.	Add 20.	28	n + 16
Step 6.	Subtract the original number.	16	16

You should always end up with 16.

Play some black magic on a friend by making him always end up with the number 13, no matter what number he chooses.

Secret:

Design a set of operations that will eliminate the original number chosen and put in the number you wish to end with.

This "magic" can be extended to many other interesting problems where the numbers people choose may be their age, the change in their pockets, the number in their family, the house number, when they were born, and many others you might like to try.

For example:

Your age is ______.

		Example:	Algebraic Solution:
Step 1.	Write down your age.	14	A
Step 2.	Multiply by 5.	70	5A
Step 3.	Add 20.	90	5A + 20
Step 4.	Multiply by 3.	270	15A + 60
Step 5.	Subtract 10 times your age.	130	5A + 60
Step 6.	Divide by 5.	26	A + 12
Step 7.	Subtract 12.	14	A

You might try combining this problem with the amount of change in the person's pocket (less than $1.00). Have the age in the hundreds place and the change in the ones and tens place.

e.g. 1749: Age 17 years, Change 49 cents.

With my crystal ball, I see you were born on ___________, 19__.

Name your friend's birthday.

Assign numbers to each month in order; e.g., January = 1, February = 2, March = 3, etc.

Imagine his birthday is August 23.

	Example:	Algebraic Solution:
	8	M
Multiply the number of the month by 5.	40	5M
Add 6.	46	5M + 6
Multiply by 10.	460	50M + 60
Add 9.	469	50M + 69
Multiply by 2.	934	100M + 138
Subtract 28.	910	100M + 110
Add the day of the month	933	100M + 133

Ask the answer. Then mentally subtract 110 from the answer (933) and you get 823. The 8 stands for August and 23 represents the day of the month.

"Hocus pocus ala kazam," your birthday is August 23.

Get the students to make up one of their own or a different set of directions for one of the examples.

Here is another one with two variables:

	Example:	Algebraic Solution:
Think of a number between 0 and 10.	7	
Multiply by 5.	35	5
Add 7.	42	$5\chi + 7$
Multiply by 2.	84	$10\chi + 14$
Add another number between 0 and 10 (picks 3).	87	$10\chi + 14 + y$
Subtract 3.	84	$10\chi + 11 + y$
Ask friend to tell you his number.		
Then subtract 11 mentally.	73	$10\chi + y$

$10x$ places the first number chosen in the tens place and y, the second number, is in the ones place.

Your friend's numbers are 7 and 3.

ACTIVITY I*

The purpose of this activity is to give the student an opportunity to discover solutions for conditions in two and three variables. Their background to date is based on Lesson III in UNIT 9, STM 2, in which they made tables and graphs of conditions in two variables using ordered pairs of natural numbers in the first quadrant only.

In the following suggested approach, students might work individually or in pairs of equal ability. Distribute graph paper to all and have more handy. Try to tell them as little as possible, using student discoveries to teach when necessary.

(1) Some students might plot ordered pairs of numbers only. Compare these graphs with straight-line graphs; remind them of the universe and ask which graph is correct.

Do not insist on a chart of values. Try and use a discovery approach. Let the students learn from each other. Slow learners might need to start out with a condition like $y = x + 4$.

(2), (3), (4) The purpose of these questions is to emphasize that a point lies on the graph of the solution set only if its co-ordinates produce a true statement when substituted into the condition.

A geometric approach is emphasized with the simple conditions since it is usually much easier to use the geometric approach to teach compound or simultaneous conditions.

The emphasis you need to use here will depend on your classes' background.

(5), (6), (7) By graphing the solution sets of questions 1 and 5 on the same graph paper, the answer to question 6 should be more easily discovered. Even if some of your students don't come up with a graphical method of solution don't insist on this method of solution. Let your students free to solve question 6 in any

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*Modified after Robert B. Davis, Explorations in Mathematics, A Text for Teachers (Palo Alto, Calif.: Addison Wesley Publishing Co., 1967), pp. 272-279.

way they can. Some might guess the answer at first, but encourage them to come up with a more systematic method. Other students might try making charts for each condition and looking for an ordered pair common to both charts.

(8), (9) More opportunity and chance to find and use a method for solving compound conditions.

(10) You may get all kinds of methods suggested here and hopefully one of them will be a combination of making charts and graphing each condition to find the point of intersection.

(11) Using Irv's or Harry's or whoever's method [see (10)], students should be able to solve this condition with no problem.

(12), (13), (14) Try and encourage students to look for a simpler non-graphical method if possible here. It is hoped that they might come up with the algebraic method of eliminating one unknown by substituting into a condition to get one unknown only.

(15) Let's hope one of your students comes up with the algebraic method here as this is the best way of solving conditions in three variables which come up in questions 16 and 17.

(16), (17) These are optional and you may leave it that way with your students also.

(18), (19), (20) Again, let your students devise their own methods here. They should be enlightened by a comparison of the graphical solution of conditions like $a = 2b + 1$ and $a > 2b + 1$.

You may have your own ideas on how important it is that all students learn one or more methods of solving compound conditions. There is very little doubt how important it is that this lesson should be approached with a spirit of originality, cleverness, and fun! For this reason you might find it necessary to give some of your slower students some simpler conditions that they can build confidence on.

At the end of the lesson you might ask the students how many methods they know for solving compound conditions. You might also ask how exciting it was?

ACTIVITY J

PIRATES' GOLD* AND THE RED BARON

The purpose of these activities is to provide practice, enrichment and add interest to the graphing of conditions. Pirates' Gold involves the graphing of simple conditions of equality and inequality in two variables. The correct "clue" to locate the hidden gold is the graph of $x + y = 8$. This is the only condition whose graph passes through the rock where the gold is hidden. Shoot the Red Baron involves the graphing of compound conditions of equality in two variables. The solution set of Fire Three is the only compound condition that "hits" the Red Baron.

Both graphs can be reproduced on paper for use as a class activity or as an individual assignment. You might encourage the student to add more "clues" to find another buried treasure or add more "shots" to hit the Red Baron twice. You might also encourage the students to construct one of their own graphs using whatever theme they prefer such as cars racing, throwing a spear, etc. It would be very easy to relate this activity to other subject areas such as literature.

ACTIVITY K

PROBLEMS:

The main student objective in teaching the solution of word problems is to get the students to use an algebraic method in solving these conditions. If the students can find the answer without using algebra then they are not developing the mathematical skills they will need later on.

Rather than tell the students to use algebraic methods of solution problems like the following which are solved so easily when algebra is used might have better results.

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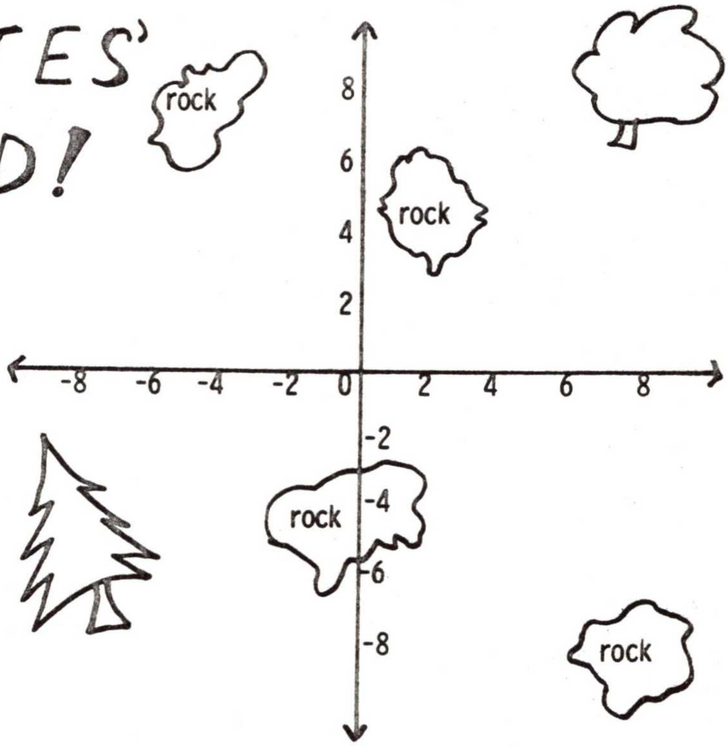
*Modified after E. M. Turner, <u>Teaching Aids for Elementary Mathematics</u> (New York, N.Y.: Holt, Rinehart and Winston, Inc., 1966), p. 139.

PIRATES' GOLD!



BEWARE!

"Clues"

g_1 $y = -2$

g_2 $x = 5$

g_3 $5x - y = 19$

g_4 $x \leq -7$

g_5 $x - 2y = 4$

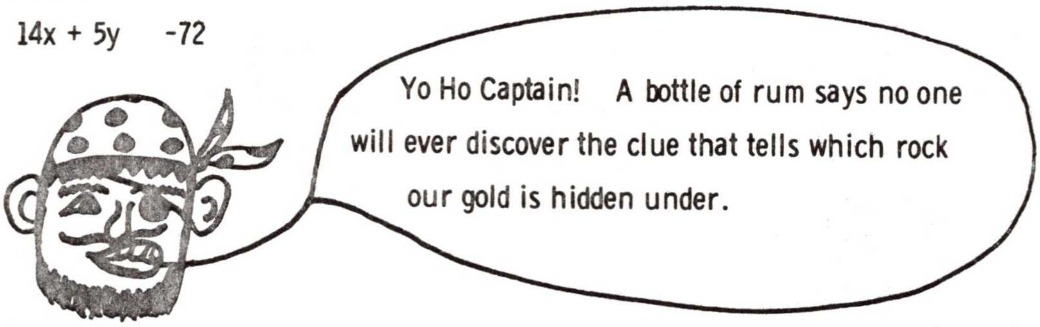
g_6 $x \geq 13$

g_7 $x + y = 8$

g_8 $x + 3y = 3$

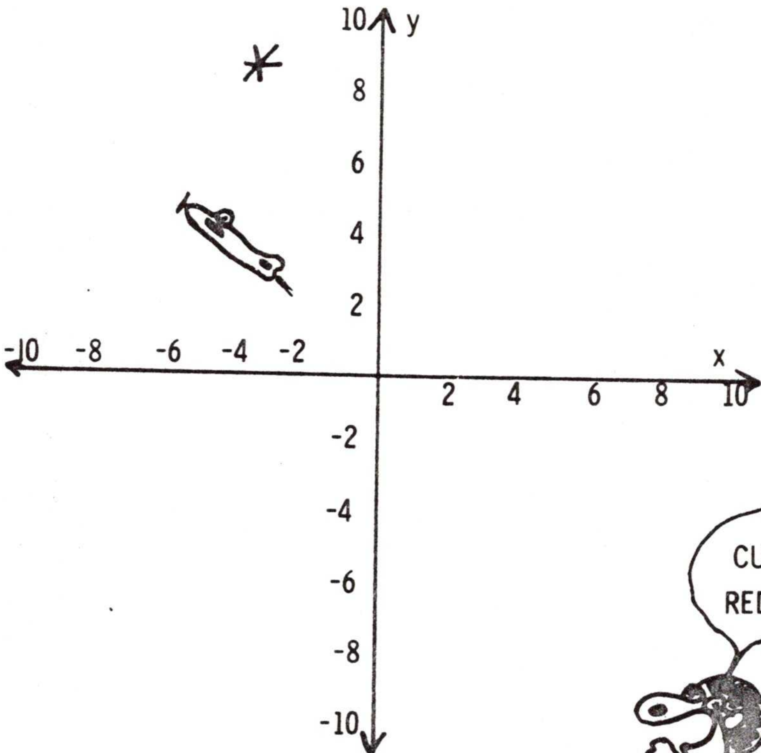
g_9 $y \geq 9\frac{1}{2}$

g_{10} $14x + 5y \quad -72$

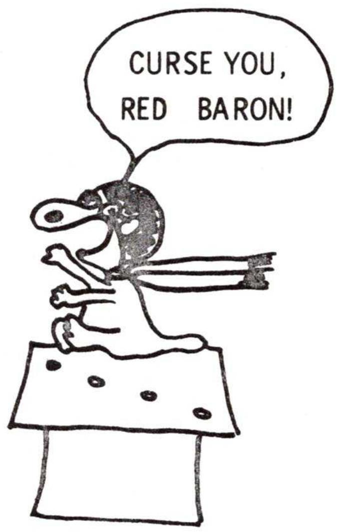


Modified after Ethel M. Turner, Teacher Aids for Elementary Mathematics (New York, N.Y.: Holt, Rinehart and Winston, Inc., 1966), p. 139.

SHOOT THE RED BARON!



Fire one: $y = x + 2 \wedge y = 4 - x$
Fire two: $y = 2x - 3 \wedge y = \frac{1}{2}x + 3$
Fire three: $y = -2x - 6 \wedge y = \frac{1}{4}x + 6$
Fire four: $y = -3x + 2 \wedge y = 1/3x - 4$
Fire five: $y = x + 3 \wedge y = -x - 7$



HELP SNOOPY BRING DOWN THE RED BARON BY FIRING COMPOUND CONDITIONS. THE SOLUTION SET OF ONLY ONE OF THESE WILL VANQUISH SNOOPY'S ARCH FOE.

You might suggest to the student that the use of a variable to represent the unknown would help him solve the problem much more easily or let him solve it completely on his own and see what kind of a method he uses.

Irving Glick gets on a bus at Lacombe. He is the only one to get on there. At Red Deer, seven more people get on and one man gets off. At Penhold, two people get off and four more get on. At Innisfail three more people get on. At Olds two people get off. At Carstairs, one half of the passengers get off. At Airdrie, again one half of the passengers get off. At the Calgary bus depot three people get off and Irving is the only passenger left on the bus.

How many passengers were on the bus when Irving got on?

SOLUTION:

The important algebraic technique here is the use of a variable to represent the unknown--in this case the number of passengers on the bus before Irving Glick got on.

The following table might make the solution more understandable:

	Number of Passengers
Before Irving got on:	P
After Irving got on:	P + 1
After the Red Deer stop:	P + 7
After the Penhold stop:	P + 9
After the Innisfail stop:	P + 12
After the Olds stop:	P + 10
After the Carstairs stop:	1/2(P + 10)

	Number of Passengers
After the Airdrie stop:	$1/4(P + 10)$
After the Calgary stop:	$1/4(P + 10) - 3$
At this point Irving is the only passenger left on the bus, so . . .	$1/4(P + 10) - 3 = 1$
Solving for P:	$1/4(P + 10) = 4$
	$(P + 10) = 16$
	$P = 6$

Therefore there must have been six passengers on the bus when Irving got on.*

A similar kind of problem follows:

On Wednesday, Dianne eats twice as many cherries as she did on Monday. On Thursday she ate twice as many as she did on Wednesday. On Friday she ate 50 cherries. On Saturday she ate twice as much as she did on Thursday. On Sunday and Tuesday she didn't eat any cherries. For the entire week she ate 230 cherries. How many cherries did she eat on Monday?

SOLUTION:

	Number of Cherries
Monday	C
Wednesday	2C
Thursday	2 (2C) = 4c
Friday	50
Saturday	2 (4C) = 8C
1 week	C + 2C + 4C + 50 + 8C = 230

*Modified after Robert B. Davis, Discovery in Mathematics, A Text for Teachers (Reading, Mass.: Addison Wesley Publishing Co., 1964), pp. 237-241.

Simplifying we obtain:

$$15C + 50 = 230$$

$$15C = 180$$

$$C = 12$$

Dianne ate 12 cherries on Monday.

Problems like the above might be used to introduce the students to an algebraic problem-solving method. When you feel a student is ready to work on his own then he or she should be directed to the activity cards.

ACTIVITY PROBLEM CARDS

These are a varied assortment of problems to provide mathematical experience ranging from easy to difficult and from simple pencil and paper activities to the gathering of scientific data based on classroom experiments. The problems have been printed on individual cards for easy use by individual students or groups of two and three. In most cases, the pupils should be able to read the cards, gather any information or materials necessary, and solve the problems with little or no help from the teacher. It is essential, however, that the teacher discuss the results of many of these problems with the group to ensure real understanding of the problem.

The purpose of the activity card approach is to allow children to:

1. discover for themselves the essential methods of solving problems.
2. work at their own ability level and rate of learning.
3. work on problems of particular interest to them.

4. learn from each other and work together co-operating.

Class teaching of mathematics is undesirable when it comes to solving problems since the abilities and interests of your students have such a wide range. But on occasion, class discussion might be valuable to draw attention to something well done or point out a common fault that needs to be corrected.

Organization of Cards

This can be done according to teacher preference. One suggestion is to arrange these cards in three groups:

a) problems with one variable

b) problems with two variables

c) enrichment problems (brain-teasers and brain-busters)

Preparing and Starting the Class

A good introduction is necessary to initiate a high quality standard of thinking, working, and discussion in addition to arousing interest. You might select a particular problem, discuss it thoroughly with the class and make the following points clear so that the students have no doubt about what is expected.*

1. The card is their problem, their challenge.

2. The card should be read carefully and discussed by both partners.

3. If the students are clear about what is to be done they should proceed on their own (including any collection of materials and information).

4. Ask the teacher for assistance only as a last resort.

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*Modified after R. A. J. Pethen, The Workshop Approach to Mathematics (Macmillan Co. Canada, Ltd., 1968).

5. Show and record your work on paper.

6. Make sure your card is completely answered.

7. Check with the teacher when finished. If the teacher is busy, then proceed with another card and check your previous problem later.*

Once you feel your students are ready to proceed on their own, let them loose. You might find it wise to post a set of instructions similar to the above for handy student reference.

The clue to a good activity card approach in the classroom is to use it at the right time and for proper lengths of time to sustain progress and interest.

The choice of content and materials for these cards is not intended to limit their use in any way. Please feel free to adapt, add to, or delete from the set of cards as best suits your classes and circumstances.

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*Modified after R. A. J. Pethen, The Workshop Approach to Mathematics (Macmillan Co. Canada, Ltd., 1968).