AN ACTIVE LEARNING UNIT ON REAL NUMBERS

by

DALE FISHER

Appendix A
from
"A Feasibility Study on Active Learning
With Real Numbers"

an M.Ed. thesis completed by
Dale Fisher
in the Department of Curriculum and Instruction
University of Calgary
July, 1970

Reprinted with permission of the author
by the Mathematics Council of
The Alberta Teachers' Association
March, 1971

AN ACTIVE LEARNING UNIT ON REAL NUMBERS

by

DALE FISHER

Appendix A
from
"A Feasibility Study on Active Learning With Real Numbers"

an M.Ed. thesis completed by
Dale Fisher
in the Department of Curriculum and Instruction
University of Calgary
July, 1970

Reprinted with permission of the author
by the Mathematics Council of
The Alberta Teachers' Association
March, 1971

TABLE OF CONTENTS*

*The following pages are reproduced from the Appendix of the writer's M. Ed. Thesis (A footnote reference is given in the Preface of this booklet.) and, consequently, the numbering of the pages starts with page 91.

PAGE

PREFACE

The active learning unit on real numbers detailed on the following pages was developed to cover all of the mathematical concepts listed in the unit entitled "Introduction to Real Numbers" in the Program of Studies for Junior High Schools of Alberta.* The topics listed for the unit in the Program of Studies are:

1. Extension of the number system to include irrational numbers.
2. Properties of the operations on the real numbers: closure, commutative, etc.
3. Additional properties of the number system: order, completeness, density.
4. An introduction to graphing on the real plane.
5. Solution of problems involving conditions or equations with real numbers.**

Activities that would foster an understanding of the concepts identified above and that would promote active learning on the part of students were collected from a wide variety of sources and modified to suit the needs of the unit or, where necessary, the author created original activities to fill a particular requirement. Many more activities were collected (forty-six in all) than would be needed by any particular student. The idea was to provide the student and teacher with a choice of activities based on individual interests and needs.

While the author attempted to provide students with many types of materials to provide a variety of experiences, the mathematical goals of the unit were kept in mind. It was easy to find new materials that were interesting, stimulating, and easily learned, but unless they served purposes other than just motivation and pleasure, they were not included.

---

*Alberta Department of Education, Program of Studies for Junior High Schools of Alberta (Edmonton: Queen's Printer, 1969).

**Ibid., p. 47.

Every effort was made to make the activities "ready-to-use" for both student and teacher, subject to the teacher's judgment as to whether or in what way any particular activity would be used.

The activity unit was tried out with eight classes of grade eight students from three Calgary Junior High Schools during April, May, and June, 1970. Evaluation of the feasibility of the unit was based on interviews with each of the four participating teachers as well from a checklist evaluation of specific activities and from student comment sheets. Using criteria that included differences in content, instructional demands, instructional effectiveness, and student attitudes, the real numbers active learning unit was judged by the teachers as preferable to the conventional textbook oriented approach. The four teachers indicated that they would use a majority of the activities in the real numbers unit in following years.* Data collected on student achievement and attitudes indicated achievement not significantly different from that of control classes following the conventional approach, and significantly more positive attitudes towards mathematics in the active learning classes as compared with the control classes.

Hopefully, the activities collected and described in detail on the following pages will be found useful by any junior high school teacher planning to pursue an active learning approach to mathematics with his students.

---

*For more information on the evaluation of the feasibility of the activity unit, see: Dale Fisher, "A Feasibility Study on Active Learning With Real Numbers" (Unpublished M. Ed. Thesis, Department of Curriculum and Instruction, The University of Calgary, Calgary, 1970).

STUDENT INTRODUCTION

THE REAL NUMBER SYSTEM

TO THE STUDENT:

The study of the real number system is the last number system to be developed in your junior high school mathematics program. This is a very important number system and therefore we are going to try and make it as interesting and enjoyable as possible in the hopes that you will understand it quite thoroughly.

The following activities will include experiments, games, individual and group competitions, puzzles, problems and other varied activities in addition to giving you a chance to make up some of your own.

There are far more activities presented here than you will have time for in class. Pick the ones, when you have a choice, that you think you can do and are the most interesting or fun.

If you think of a related activity that you would rather do instead of any of those suggested, check with your teacher and then try it out if the two of you think it will work.

Some of the activities have never been used in the classroom and others very little. If you get stuck, think it over twice again and then try and work it out with one of your classmates or your teacher.

There is an old Chinese proverb which is appropriate for this mathematics section:

I hear, I forget
I see, I remember
<u>I do</u>, <u>I understand</u>

We hope you will <u>understand</u> all about real numbers and the related activities when you are through.

TEACHERS' INTRODUCTION

THE REAL NUMBER SYSTEM

No attempt is made to present a rigorous development of the real number system, for this is the role of college mathematics. The main purpose of these activities is to simply help the student develop a feeling for the existence of irrational numbers, the real number system and some of their applications on the basis of the mathematics he has learned thus far.

The teacher should be aware that not all students are to be expected to complete all activities. These activities will vary in difficulty and interest depending on the individual student. When a choice of activities is to be made, it is hoped the student can make an intelligent decision. If not, he should seek teacher assistance and guidance.

STUDENT ACTIVITIES

SQUARES

ACTIVITY A*

Continue the pattern in this table:

SHAPE	.	. . . .	. . . .	. . . .		
NUMBER OF DOTS	1	4				

What shape have you drawn? ____________________

These numbers are called <u>square numbers</u>.

ACTIVITY B*

Here is another way to find square numbers -

Finish the chart:

---

*Ray W. Cleveland, unpublished manuscript, 1968.

$1 = 1$

$1 + 3 = 4$

$1 + 3 + 5 =$ _____

$1 + 3 + 5 + 7 =$ _____

$1 + 3 + 5 + 7 +$ _____ $=$ _____

$1 + 3 + 5 + 7 +$ _____ $+$ _____ $=$ _____

$1 + 3 + 5 + 7 +$ _____ $+$ _____ $+$ _____ $=$ _____ .

What kind of numbers did you add to get square numbers?

____

ACTIVITY C

Another way to find square numbers is to multiply each number by itself:

$1 \times 1 = 1$

$2 \times 2 = 4$

$3 \times 3 =$

$4 \times 4 =$

$5 \times 5 =$

The product 3×3 may be written as 3^2.

The number 2 is called an <u>exponent</u> and 3 is called the <u>base</u>.

Other examples are:

$$4 \times 4 = 4^2$$
$$7 \times 7 = 7^2$$
$$2/3 \times 2/3 = (2/3)^2$$
$$1\ 5/8 \times 1\ 5/8 = (1\ 5/8)^2$$

ACTIVITY D

Compute the value for each of the following:

(a) $8^2 = 8 \times 8 = 64$

(b) $5^2 = 5 \times 5 =$

(c) $0^2 =$

(d) $(1/3)^2 =$

(e) $12^2 =$

(f) $(-3/4)^2 =$

(g) $9^2 =$

(h) $(8/9)^2 =$

(i) $12^2 =$

(j) $(11/13)^2 =$

(k) $(-5/8)^2 =$

(l) $(.5)^2 =$

(m) $(2.2)^2 =$

Multiplying 4 by 4 and obtaining the result, 16, is called "squaring 4." In other words, 4^2 can be read "4 squared."

Can you think of another way of defining the "square" of a number?

From the examples above, you should agree that the square of a number is the second power of a number.

36 is the square of 6. $(6^2 = 36)$

9/25 is the square of 3/5. $((3/5)^2 = 9/25)$

As you can see, we may "square" any rational number.

But when we square a subset of the rational numbers, the positive integers, I_p, the squares of these numbers are called perfect squares or square numbers.

Look back at Activity D. Which of the numbers that you squared are perfect squares?

Are the square numbers that you worked with in Activities A, B, and C perfect squares?

ACTIVITY E

Squaring members of the set R_N.*

Square the following numbers:

---

*Modified after van Engen et al, Seeing Through Mathematics, special edition, Book 2, Toronto: W. S. Gage Limited, 1964.

$(-5)^2 = -5 \times -5 =$

$(-4/7)^2 =$

$(-10)^2 =$

$(-1\ 2/5)^2 =$

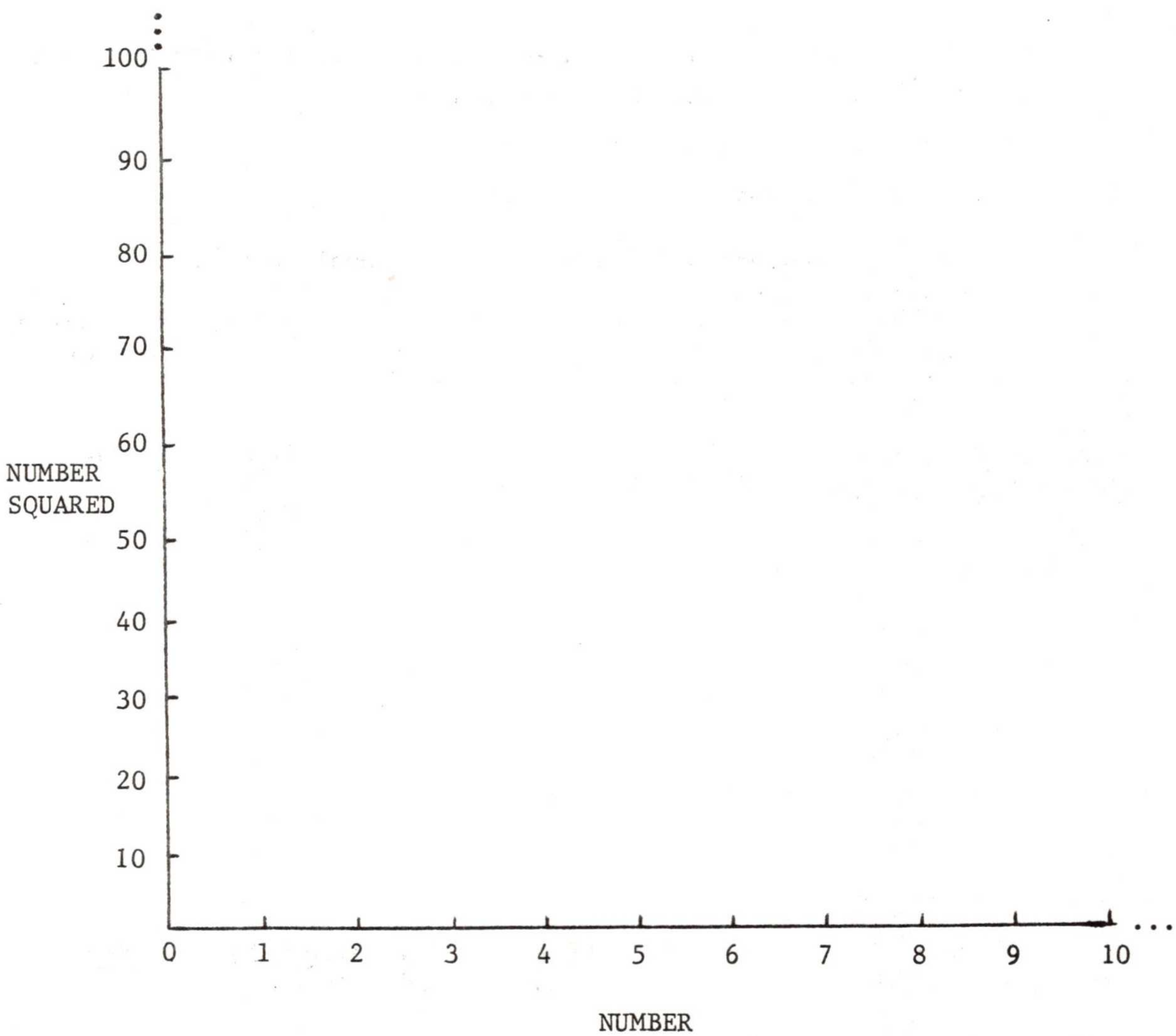
$(-3.3)^2 =$

$(-7)^2 =$

What do you notice about the square of a negative number?

ACTIVITY F

Tabulate the ordered pairs for the integers from 1 to 10 and the squares of these integers. Use the ordered pairs to complete the graph below:



Beginning at the origin, join these ordered pairs of numbers with one continuous line.

Examine your graph. Can you use this graph to find the approximate squares of other numbers?

Using your graph only, try and find the square of

(a) 2 1/2

(b) 7 1/2

(c) 4 3/4

Check your answers with your partner and see how closely you agree.

ACTIVITY G

Following is a list of rational numbers and their squares.

Try and pair them up.

Example: $(-6)^2 = 36$

$(-6)^2$, 4^2, 28 57/64, 4225, $(-5\ 3/8)^2$, $(-11)^2$, 16, 6.5^2, 65^2, 20^2, $3/8^2$, 121, $(7/8)^2$, 36, 49/64, 42.25, 400, 9/64

If you wish, make up your own mixture of numbers and their squares and see if your partner can match them.

ACTIVITY H

Try and express every integer from 1 to 25 as the sum of not more than four <u>square numbers</u>. Can it be done?

1 = 1	10 =	19 =
2 = 1 + 1	11 =	20 =
3 =	12 =	21 =
4 =	13 =	22 =
5 =	14 = 9 + 4 + 1	23 =
6 =	15 =	24 =
7 =	16 =	25 =
8 =	17 =	
9 =	18 =	

*If you have time, ask your teacher or check some mathematics books for information on how to find: triangular numbers; pentagonal numbers.

TEACHERS' GUIDE

SQUARES

LESSON OBJECTIVES:

1. Students learn how to find the square of a rational number.
2. Students learn that a perfect square or square number is the square of a positive integer.

Essential Activities

1. Activities D, E, and G - a bare minimum.
2. Remaining activities may be considered optional depending on student ability, interest and class time.
3. Strongly recommend activities A, B, C plus F and H, since they are not difficult and should prove interesting to students.

Additional Information

Activity F

Instructions are not too clear, I believe. Make it plain in tabulating the ordered pairs that:

(a) 1st component of ordered pairs are the numbers from 1 - 10.

(b) 2nd component of ordered pairs is the square of the 1st component, (e.g., (5, 25).

For the student graph, the use of graph paper should result in a much more accurate graph line as well as making the task of reading the graph to find squares easier and more accurate. Students should save this graph for a subsequent activity in the section on square roots.

Activity H

Triangular Numbers

SHAPE	.	.˙.		
NUMBER OF DOTS	1	3	?	?

Pentagonal Numbers

SHAPE	.	:˙:		
NUMBER OF DOTS	1	5	?	?

NOTE: Pentagonal numbers are the sum of triangular numbers and square numbers i.e. 5 = 4 + 1. Students might try adding some more numbers and shapes to each series.*

---

*Modified after M. A. Hervey, and B. H. Litwiller, "Polygonal numbers: a study of pattern," The Arithmetic Teacher, Vol. 17, No. 1 (January, 1970), pp. 33-38.

ANSWER KEY

SQUARES

ACTIVITY A

Number of Dots: 1 4 9 16 25 36

What shape have you drawn? Square

ACTIVITY B

$1 + 3 + 5 = 9$

$1 + 3 + 5 + 7 = 16$

$1 + 3 + 5 + 7 + 9 = 25$

$1 + 3 + 5 + 7 + 9 + 11 = 36$

$1 + 3 + 5 + 7 + 9 + 11 + 13 = 49$

To get square numbers we added consecutive ODD NUMBERS

ACTIVITY C

$3 \times 3 = 9$

$4 \times 4 = 16$

$5 \times 5 = 25$

ACTIVITY D

(a) 64

(b) 25

(c) 0

(d) 1/9

(e) 144

(f) 9/16

(g) 81

(h) 64/81

(i) 144

(j) 121/169

(k) 25/64

(l) .25

(m) 4.84

Which are perfect squares? a, b, e, g, i

The numbers 1 4 9 16 25 36 49.

ACTIVITY E

$(-5)^2 = 25$	$(-1\ 2/5)^2 = 1\ 24/25$
$(-4/7)^2 = 16/49$	$(-3.3)^2 = 10.89$
$(-10)^2 = 100$	$(-7)^2 = 49$

The square of a negative number is a positive number or:
A negative number times a negative number equals a positive number. (or any equivalent answer)

ACTIVITY F

Ordered Pairs:

(1,1)	(5,25)	(9,81)
(2,4)	(6,36)	(10,100)
(3,9)	(7,49)	
(4,16)	(8,64)	

Using graph to find squares of:

(a) 2 1/2 - 6 1/4
(b) 7 1/2 - 56 1/4 Approximate answers are satisfactory
(c) 4 3/4 - 22 9/16

ACTIVITY G

$(-6)^2 = 36$	$20^2 = 400$
$4^2 = 16$	$(3/8)^2 = 9/64$
$(-11)^2 = 121$	$(6.5)^2 = 42.25$
$(-5\ 3/8)^2 = 28\ 57/64$	$(7/8)^2 = 49/64$
$65^2 = 4{,}225$	

ACTIVITY H

$1 = 1$

$2 = 1 + 1$

$3 = 1 + 1 + 1$

$4 = 1 + 1 + 1 + 1$

$5 = 4 + 1$

$6 = 4 + 1 + 1$

$7 = 4 + 1 + 1 + 1$

$8 = 4 + 4$

$9 = 4 + 4 + 1$

$10 = 9 + 1$

$11 = 9 + 1 + 1$

$12 = 9 + 1 + 1 + 1$

$13 = 9 + 4$

$14 = 9 + 4 + 1$

$15 = 9 + 4 + 1 + 1$

$16 = 4 + 4 + 4 + 4$

$17 = 16 + 1$

$18 = 9 + 9$

$19 = 9 + 9 + 1$

$20 = 16 + 4$

$21 = 16 + 4 + 1$

$22 = 9 + 9 + 4$

$23 = 9 + 9 + 4 + 1$

$24 = 16 + 4 + 4$

$25 = 16 + 9$

NOTE: Answers are not unique - other sums possible

e.g. $17 = 16 + 1$

$17 = 9 + 4 + 4$

SQUARE ROOTS

ACTIVITY A

Do you remember the mathematics unit in which you learned how to find the area of a square figure? For example, what would be the area of a square tile with sides of length 12 inches?

e.g. χ represents length of a side of the square tile.

$$\frac{12}{1} \sim \frac{\chi}{12}$$

$$\chi = 12 \times 12$$

$$\chi = 144$$

The area of the square tile is 144 square inches.

Suppose we changed the problem so that you had to reverse the above operation to obtain the solution. That is, if given the area of a square, could you find the length of its sides?

For example, if you had a square with an area of 4 square inches, what would the length of its sides be?

____________ inches

Area =
4 sq. inches
or
$\chi^2 = 4$ sq. in.
$\chi = ?$

HINT: Find a number that, multiplied by itself, equals the area.

e.g. $\chi^2 = 4$

or $\chi \cdot \chi = 4$

Find χ and the problem is solved.

Ask your teacher for some square figures. You will notice that each has the area printed on it. Your job is to find the length of the side by <u>computation</u> - <u>rulers are a no-no!</u>

Use the hint given above.

Complete the table below for at least six of these squares:

(See next page)

	Area of Square	Length of Side (χ)	Check: Does Area of Square = χ^2
e.g.	4	2	$4 = 2^2$
a)			
b)			
c)			
d)			
e)			
f)			

Why don't you make up your own square, even if it's not a cut-out, and exchange your problem with a friend's and see if each of you can solve the length of the square's side.

ACTIVITY B

Congratulations!

By completing Activity A, in particular the second column, you have discovered a new operation called Finding the square root of a number.

This is the inverse operation of finding the square of a number, just as subtraction is the inverse of addition and division is the inverse of multiplication.

The symbol for the new operation is $\sqrt{\ }$

For example $5^2 = 25$

$\sqrt{25} = 5$

That is, the square root of 25 is 5.

How do you know? - $(5 \times 5 = 25)$

Compare the two operations again:

Find the square of 7: $7^2 = 7 \times 7 = 49$

Find the square root of 49: $\sqrt{49} = \sqrt{7 \times 7} = 7$

Find the square root of the following numbers by completing the table:

	Number	Number Factored into EQUAL Factors	Square Root of Number ($\sqrt{\ }$)
e.g.	6^2	6 x 6	6
a)	64	8 x 8	
b)	9	-3 x -3	
c)	169	13 x 13	
d)	4/9	-2/3 x -2/3	
e)	625	25 x 25	
f)	225/121	15/11 x 15/11	
g)	64	-8 x -8	

Notice that a positive rational number may have both a:

1) positive square root, e.g., $\sqrt{64} = 8$.
2) negative square root, e.g., $\sqrt{64} = -8$.

How do we know which is correct?

Most mathematicians agree to use the square root symbol ($\sqrt{\ }$) to mean the positive square root only, or as it is also called, the principal square root. (See your text, p. 218.)

If the negative square root is wanted, then a negative sign is placed in front of the square root symbol.

Complete these: e.g. $-\sqrt{9} = -3$

$\sqrt{9} = 3$

$\sqrt{100} =$

$-\sqrt{100} =$

$-\sqrt{81} =$

$-\sqrt{1/4} =$

$\sqrt{121} =$

$-\sqrt{121} =$

Do you think it possible to find the square root of a negative number?

____

Why or why not? ____

____

Before going any further, be sure you understand what the following terms mean:

SQUARE: a) the number obtained by multiplying two equal numbers or factors:

e.g., $2 \times 2 = 4$. 4 is the square of 2.

SQUARE: b) a plane geometric figure having four sides of equal length and each angle has a measure of 90 degrees.

SQUARING: a) the multiplication of a number by itself, or

b) the second power of a number.

SQUARE ROOT: one of the two equal factors of a number:

e.g., 3 is the square root of 9 because $3 \times 3 = 9$.

ACTIVITY C

USING A TABLE OF SQUARES AND SQUARE ROOTS

To save time, a table of squares and square roots is very helpful in finding these kinds of numbers. Turn to page 461 in STM 2 and study the table.

If you need help in answering the following exercises, turn to page 256 of your text.

Using the table on page 461, find the square and the square root of each of the following numbers:

(See next page)

	Number (χ)	Square (χ^2)	Square Root ($\sqrt{\chi}$)
e.g.	13	169	3.606
a)	38		
b)	49		
c)	4		
d)	212		
e)	68		
f)	5		
g)	137		

ACTIVITY D

The exact square root of 2?

The table on page 461 lists $\sqrt{2}$ as 1.414.

This should mean $(1.414)^2 = 2$. Does it? ____________________

1.414×1.414 equals 1.999396

but $1.999396 \neq 2$

Obviously the book has made a mistake.

Try $(1.415)^2$. Does this equal 2? __________

What does it equal? __________

Try 1.4141 - __________

Try 1.4142 - __________

See if you can find the exact square root of 2.

ACTIVITY E

Calculating Square Root

Perhaps if we had a particular method for calculating the square root of a number, finding the square root of 2 might be easier.

Usually, mathematicians use a table of squares and square roots, or a slide rule, or a computer, but occasionally they might be stuck without any of these time-saving aids. Then they probably use one of the following methods.

1. GUESS METHOD*

Let's calculate the square root of 72. ($\sqrt{72}$)

First, find which integers surround $\sqrt{72}$

You know that $\sqrt{72}$ is greater/less than $\sqrt{64}$. $\sqrt{64} = 8$, therefore $\sqrt{72}$ is a number greater than 8. Is $\sqrt{72}$ a number greater/less than $\sqrt{81}$? $\sqrt{81} = 9$, therefore $\sqrt{72}$ is a number less than 9. This tells us that $\sqrt{72}$ is some number between 8 and 9.

Before we proceed with finding $\sqrt{72}$, you might find it helpful to practice the above procedure. Without using the table of squares and square roots, find the integers that bound the following numbers:

a) $\sqrt{72}$ lies between ______ and ______.

b) $\sqrt{75}$ lies between ______ and ______.

c) $\sqrt{26}$ lies between ______ and ______.

d) $\sqrt{13}$ lies between ______ and ______.

e) $\sqrt{95}$ lies between ______ and ______.

f) $\sqrt{135}$ lies between ______ and ______.

Returning to the calculation of $\sqrt{72}$, using the "guess method," we simply make successive guesses to continue our calculation.

The following table might help you make your calculation easier. Try and find $\sqrt{72}$ to the nearest thousandth.

(See next page)

---

*Modified after D. C. Attridge, et al., ASTC Mathematics, (Toronto: GINN and Company, 1968), pp. 81-82.

	Guess	Guess "Squared"	Is Value Less than # (n), i.e., $\sqrt{72}$	Is Value Greater than #(n), i.e. $\sqrt{72}$	Range of Values for n, i.e. $\sqrt{72}$
i)	8	64	X		$8 < \sqrt{72}$
ii)	9	81		X	$8 < \sqrt{72} < 9$
iii)	8.5	72.25		X	$8 < \sqrt{72} < 8.5$
iv)	8.46	70.56	X		$8.46 < \sqrt{72} < 8.5$
v)					
vi)					
vii)					
viii)					

This method requires a lot of patience, but is easy to understand.

2. "NEWTON'S METHOD"

This is usually a shorter method and involves taking the average of two guesses. You will find the procedure detailed on pages 259-260 of your text.

3. ALGORITHMIC METHOD

This is also a good way of finding square root, although the method will likely make very little sense to you. Ask your teacher about this one if you don't like either of the first two methods.

Choose at least two of the following numbers and calculate the square root by whichever method you prefer (to the nearest thousandth).

a) $\sqrt{39}$

b) $\sqrt{88}$

c) $\sqrt{7}$

d) $\sqrt{155}$

e) $\sqrt{19}$

f) $\sqrt{140}$

Check your answers on page 461 of your text.

ACTIVITY F

One more square root calculation--maybe! If you have not already done so, from Activity D, calculate the $\sqrt{2}$ to five decimal places, using the method you prefer. Check your answer with your partner. Do you agree?

____

Does the number you have "squared" equal the numeral 2 exactly?

If not, how close are you? ______________________________

Do you think you will ever find the exact square root of 2?

Why or why not? ______________________________

____

If you did find it, or think you might find it, show your answer to the teacher, some mathematicians, your friends, parents, everyone, because you will be famous. You will have done what mathematicians have believed impossible for thousands of years.

BRAIN TEASER

What number multiplied by itself is equal to itself?

____

How many such numbers do you know? ______________________________

Can you make any general multiplication statements about these numbers, i.e., statements that are true for any rational number:

____

____

TEACHERS' GUIDE

SQUARE ROOTS

LESSON OBJECTIVES:

1. Students learn the reverse operation of squaring a number--finding the square root of a number.

2. Students learn how to find square root by using a Table of Squares and Square Roots and/or one of several methods of calculation.

Background Information

ACTIVITY A

This is meant to be a discovery introduction to a simple calculation of square root. For classroom needs you will have to cut out a number of "perfect squares" e.g. squares of area 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, etc. To challenge some of your better students you might give them squares of mixed numerals or decimals e.g. - 6.25 - $(2\ 1/2)^2$, 19.36 - $(4.4)^2$, 11 1/9 - $(3\ 1/3)^2$, etc.

ACTIVITY B

The concept of the square root of a number being an equal factor is important (one of the two equal factors that form the number (product)).

e.g., $64 = 4 \times 16$ $\quad$ $64 = 2 \times 4 \times 8$

$64 = 2 \times 32$ $\quad$ $64 = 8 \times 8$

8 is the square root since it is one of an equal pair of factors that form the product. 4 is not a square of 64 since $4 \neq 16$.

In regard to finding the square root of a negative number, the answer is NO with respect to the numbers studied to date. The square root of a negative number is an imaginary number, but the decision is yours as to whether or not you want to discuss these at this time.

ACTIVITY C

Before or after completing this activity you might wish to have the students refer back to Activity F in the section on Squares and ask them to find the square roots of numbers given the square and using their graph. Perhaps you might introduce this exercise after Activity A in this section or at some other point, if at all, depending on your judgement.

ACTIVITY D

Unless you think the student is completely wasting his time, let him work on finding $\sqrt{2}$ as long as he is interested. There is a meaningful follow-up to this activity in the section on irrationals.

ACTIVITY E

1. Guess Method - basically an intuitive approach. Easy to understand but method can be long and laborious, particularly with larger numbers.

2. Newton's Method - while not labelled as such in STM 2 (probably because it is an adaptation of the original) this is a meaningful method.

3. Algorithmic Method - see page 169 STM 2 Teachers' Guide - probably the quickest way to calculate square root, but the most difficult technique to understand.

You may find the majority of students prefer the third method, after they have an understanding of the operation of square root, simply because it's quicker. If you think the student is spending too much time learning the different methods or making a choice, you might help him make a decision based on his ability and interest. In any case, no great stress need be laid on calculating square root, since 99.9 per cent of the students will probably use a Table of Squares and Square Roots if they encounter any future need for this operation.

ACTIVITY F

This is an open-ended activity intended as an introduction to the next section on irrational numbers.

Essential Activities

A, B, C, and D - these are the activities I think are most meaningful and beneficial to the majority of students. Activities E and F, dealing with calculation of square root simply involve a technique which we seldom use and quickly forget, but may be useful to the student in understanding square root and the existence of irrational numbers.

ANSWER KEY

SQUARE ROOTS

ACTIVITY A

$x^2 = 4$ sq. in. $x = 2$ length of side = 2 in.

Your answers will depend on the squares your teacher gave you.

ACTIVITY B (Square Root)

a) 8
b) -3
c) 13
d) -2/3
e) 25
f) 5/11
g) -8

Complete these:

$\sqrt{100} = 10$

$-\sqrt{100} = -10$

$-\sqrt{81} = -9$

$-\sqrt{1/4} = -1/2$

$\sqrt{121} = 11$

$-\sqrt{121} = -11$

The square root of a negative number is impossible to find considering your study of numbers to date.

The simplest explanation is based on the definition of square root, and from this we must conclude that there is no negative product formed by multiplying two equal numbers.

e.g., a positive number times a positive number = a positive number

a negative number times a negative number = a positive number

ACTIVITY C

	x^2	$\sqrt{x}$
a)	1444	6.164
b)	2401	7
c)	16	2
d)	44944	14.56
e)	4624	8.246
f)	25	2.236
g)	18769	11.705

ACTIVITY D

$(1.415)^2 = 2.002225$

$(1.4141)^2 = 1.99967881$

$(1.4142)^2 = 1.99996164$

$(1.4143)^2 = 2.00024449$

Good Luck!

ACTIVITY E

1. Guess Method

a) 8 and 9

b) 8 and 9

c) 5 and 6

d) 3 and 4

e) 9 and 10

f) 11 and 12

$\sqrt{72} = 8.485$ (to nearer thousandth)

3. Algorithmic Method

a) 6.245

b) 9.381

c) 2.646

d) 12.450

e) 4.359

f) 11.832

ACTIVITY F

$\sqrt{2} = 1.41421$

BRAIN TEASER

Two numbers:	0 and 1
For any a,	$a \times 0 = 0$
For any a,	$a \times 1 = a$

THE STORY OF JOE SINE

One day Joe Sine went to call on his new neighbors who lived in the adjacent house. He was a handsome tangent, a confirmed bachelor.

Joe was met at the door by two sisters who had anything but congruent figures. The first, Deca Gon, had real construction problems. Her discontinuous curves were intersected at various angles by parallel lines. The second sister, Polly Gon, was dressed in a pretty co-ordinate set and it was obvious that her natural curves ran into imaginary numbers. Just looking from the first to the second, Joe found his interest compounding rapidly. What poor Joe did not know was that Polly knew all the angles and was an expert at taking squares.

Joe was invited to come in and sit down. Deca proved to be as square as she looked and just sat there like a log. Polly, at a given sine, sent Deca out to find some roots to make tea. While she was gone, Polly served Joe pi. Then she used the complementary angle and Joe was soon reduced to zero power. Next she introduced an "If . . . then proposition." That is, if Joe would marry her, then . . . Joe said yes, then began to consider the possibility of spending the rest of his life adjacent to her side. He suddenly became very nervous and began crossing and uncrossing his legs for joint variation. What would be a Hero's formula for getting out of this mess? Joe was so scared he nearly had a corollary coronary.

But the tension was suddenly broken when Deca came bursting through the door with a tremendous discovery. Against all probability she had dug up a freak of nature, a square root![*]

---

[*]Mary Rector, "The Story of Joe Sine," In CHIPS from the Mathematical Log, ed. Josephine P. Andree (Norman, Oklahoma: The University of Oklahoma, 1966), p. 58.

IRRATIONAL NUMBERS

ACTIVITY A

Are you still puzzled about $\sqrt{2}$? Let's see if we can solve the problem by examining the following decimal numbers:

2.236...	$.\overline{14257}$
$.625\overline{0}$	$38.\overline{12}$
$.3\overline{3}$	.01001000100001...
$9.\overline{0}$	.3149765389...
4.1232332333...	17.634979779777...

If you were to place the 10 decimal numbers above into two distinct groups, how would you group them? Try placing each of the above numbers into Group A or B according to your own definition.

Group A	Group B

What's the difference between your two groups of numbers?

Do your two groups agree with your partner's?

Check the way your teacher grouped them. Do you agree with him/her?

Whether or not you agreed with your teacher, you should notice that only 5 of the 10 numbers are rational numbers. Remember, any rational number can <u>always</u> be expressed by a repeating decimal, if you include 0 as a repetend. (If necessary, review pages 217-221 of your text.) Remember also that <u>any</u> rational number can be expressed as a fraction.

BUT, what about numbers like 4.1232332333... that don't have a repetend. Are they rational numbers? _______ No, and since they are not rational, we call them ir rational numbers. Not rational - irrational, get it?

We can also conclude that irrational numbers can't be expressed as fractions. Right? Try changing an irrational number to a fraction--when you have lots of time!

ACTIVITY B

Classify the following numbers as rational or irrational:

(a) $0.75\overline{0}$

(b) $2.\overline{31}$

(c) $0.\overline{7}$

(d) 0.721374

(e) -3.1121112...

(f) $-2.\overline{1423}$

(g) 3.9763674...

(h) .7565565556...

A rational number can be defined as a repeating infinite decimal. The symbol R denotes these numbers.

An irrational number can be defined as a non-repeating infinite decimal. The symbol $\bar{R}$ denotes these numbers.

What about the $\sqrt{2}$? Which set do you think it belongs to: R or $\bar{R}$?

If you picked $\bar{R}$, you are correct. No matter how hard you try, you will never find a repeating pattern in the decimals. Using the computer, some mathematicians have calculated $\sqrt{2}$ to over a thousand decimal places--and still no repeating pattern.

The number π is also an irrational number. Wrong, you say! What about 3.14 or 3 1/7? Those are just approximations. If you don't believe this, ask your teacher to show you the computer's calculation of π to two thousand decimal places. See if you can find a repeating pattern.

There are some very interesting stories about the number π. Check your library to see if it has any information about this irrational number.

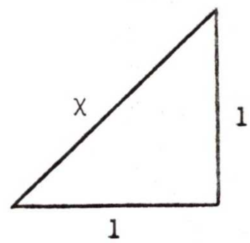
Those of you who like to prove statements might be interested in a proof that an irrational number like $\sqrt{2}$ is not rational. Ask your teacher for a copy of this.

ACTIVITY C

Read the following story:

THE PYTHAGOREAN MAFIA!*

Over two thousand years ago, there was a group of extremely dedicated and brilliant mathematicians called the Pythagorean Brotherhood. Their founder, as you may have guessed, was the famous Greek mathematician, Pythagoras. One of their favorite activities was geometry and one day while working with the pictured triangle, they were trying to express the side χ in the form of a fraction. They knew that $\chi^2 = 2$, so all they had to do was find a number times itself that equalled 2. They tried, and tried, and tried. They tried 1 2/5 with no success. They tried 1 41/100, but it didn't work. They tried 1 207/500, but again no luck.



They kept on trying until finally Pythagoras discovered a proof that showed side χ could not be expressed as a fraction. This meant that χ was not a rational number.

Pythagoras and the rest of the members of the brotherhood were so deeply shocked about this discovery that they swore all the members to the utmost secrecy. Apparently they believed that knowledge about a number that was not rational would cause a scandal in the rest of the community and possibly the world. Alas, one of the members, Hippasus, couldn't keep the secret. Legend says that the other Pythagoreans were so angry that they took Hippasus for a walk along a steep cliff by the sea and he never returned.

ACTIVITY D

Which of the following are rational numbers?

(a) $\sqrt{1}$	(e) $\sqrt{30}$	(i) $\sqrt{76}$
(b) $\sqrt{5}$	(f) $\sqrt{9/26}$	(j) $-\sqrt{16/49}$
(c) $\sqrt{16}$	(g) $-\sqrt{9}$	(k) $\sqrt{60}$
(d) $\sqrt{4/9}$	(h) $-\sqrt{29}$	(l) $-\sqrt{81}$

---

*Modified after Howard Eves, An Introduction to the History of Mathematics (New York: Holt, Rinehart and Co., Inc. 1953); and Isaac Asimov, Chap. 7 "Digging for Roots," Realm of Numbers (Boston: Houghton Mifflin Company, 1959), pp. 114-132.

Perhaps you have observed that the square root of any number is irrational unless:

1. the number is a perfect square.
2. the number is a fraction composed of perfect squares.

ACTIVITY E

Make up 4 irrational numbers of your own. Are they non-repeating and non-terminating?

(a)

(b)

(c)

(d)

Just for curiosity, see what kind of irrational numbers your partner made up.

ACTIVITY F

Generate your own irrational number:

1. Select one blue and one green cube from the game of <u>TUF</u>.*
2. Write down a decimal point. (Ignore the decimal point, $\odot$, on the green cube.)
3. Shake and roll the two cubes or dice and record after the decimal point the two numbers on the top face, in any order, i.e., .37.
4. Roll the cubes again and record the next two numbers after the preceding two: i.e., .3749.
5. Repeat step #4, about six more times.

Is your number an irrational number?

Could you continue this process indefinitely?

Make up another one if you have time.

---

*TUF, (Baltimore: Avalon Hill Co., 1969).

Compare your number(s) with your partner's.

ACTIVITY G

Do (a) or (b) - not both:

(a) An ancient approximation for Pi (π) was 335/113. Is this value correct to seven decimal places? (Check your answer with the computer's and if they differ, state by how much.)

____

(b) We often use $\pi = 22/7$ as an approximation today. Calculate this fraction to 6 decimal places and compare your answer with the computer's. By how much do you differ?

____

ACTIVITY H

REAL NUMBERS:

A REAL NUMBER is any number that can be named by an infinite decimal.

Are all the numbers we have been studying to date, real numbers?

____

The set of real numbers is the union of the rational numbers and the irrational numbers, or the union of the repeating infinite decimals and the non-repeating infinite decimals.

Your text uses the symbol D to name the set of real numbers. (See page 263.)

Does $R \cup \bar{R} = D$?

Examine the Venn diagram on page 262 of your text. Read over exercises R to V (page 263) if you have difficulty understanding it.

Do questions 1-12 on page 265 of your text.

ACTIVITY I

A GAME OF REAL NUMBERS*

This numbers game can be played by 2 or more players. To win the game, you must be able to detect and write down on a piece of paper more numbers (of a particular kind) than any other player. The numbers to be detected and written down are those that can be made from what appears on the top faces of the cubes or dice shaken out.

To play the game, select one blue, one green, one red, one yellow, and one blank cube from the game of TUF (five cubes in all). The blank cube represents the square root symbol ($\sqrt{}$).

A game is finished after every player has had one shake. Play proceeds according to the following rules.

1. TIME RULE - players should agree on a time limit for each shake. e.g., 1, 2, 3, 5, 10 minutes or any other time the players agree on.

2. SCORING - when time is up, each player receives one point for each correct appropriate number. Players lose one point for each number that shouldn't be on the list.

3. TYPE OF NUMBER - Before each player shakes the cubes, he must specify which kind or type of number is to be written down. This could be any one of the following:

   (a) The Integers (I) or the subsets I_p or I_N

   (b) The Rational numbers (R) or the subsets R_p or R_N

   (c) The Irrational numbers ($\bar{R}$)

   (d) The Real Numbers (D)

   Use the Venn Diagram on page 262 of STM2 to help you imagine the kinds of numbers and their relationships.

   IMPORTANT: Since this section is based on the study of irrational numbers, one player must call irrational numbers for one shake of each game.

4. ORDER OF SHAKING - each player rolls a blue and a green cube. The player with the highest total begins, next highest total

---

*Modified after Layman E. Allen, The Real Numbers Game (Downsview, Ont.: WFF'n PROOF Publishers).

goes second and so on.

5. Ignore the $\odot$ symbol on the green cube.

6. Each symbol can be used once only in making up numbers.

OPTIONAL RULES:

7. The $\odot$ symbol on the green cube may be used to denote exponentiation. e.g.,

$$2 \odot 3 = 2^3 = 8$$

8. BIG BOASTER'S RULE

   If a player thinks he has written down all of the specified numbers possible on the shake before the time limit has expired, he can stop the game immediately by declaring "BIG BOAST." For his bravery, he will get a bonus if right, a penalty if wrong.

   If a "Boaster" is correct and has all possible numbers written down for that shake and no incorrect ones, he gains one point for each correct number listed and the other players receive a score of 0.

   If a "Boaster" has missed one or more numbers, he loses one point for each number he missed or for each incorrect one and scores 0 for the shake. The other players score points as stated in rule 2.

HERE IS AN EXAMPLE:

Suppose there are four players:

Suppose the following symbols appeared on the top faces of the cubes rolled on the table -

$$4 \quad 7 \quad 1/2 \quad - \quad \sqrt{}$$

If the set of Integers (I) had been called, each of the following numbers could be listed since they can be formed from the symbols above:

$4,\ 7,\ 3(7-4),\ 47,\ 74,\ 2(\sqrt{4}),\ 5(7-\sqrt{4}),\ -4,\ -7,$

$-3(4-7),\ -47,\ -74,\ -2(-\sqrt{4}),\ -5(\sqrt{4}-7)$

Total = 14 numbers

If the set of positive Rational Numbers (R_p) had been called, each of the following numbers could be listed in addition to the positive integers above:

1/2, 3 1/2(4-1/2), 6 1/2(7-1/2), 7 1/2, 4 1/2,

47 1/2, 74 1/2, 1 1/2($\sqrt{4-1/2}$), 46 1/2(47-1/2),

73 1/2(74-1/2)

Total = 17 numbers

If the set of Rational Numbers (R) had been called, each of the following numbers could be listed in addition to the numbers listed for the sets I and R_p:

-1/2, -47 1/2, -74 1/2, -3 1/2(1/2-4), -6 1/2

(1/2-7), -1 1/2(1/2-$\sqrt{4}$), -46 1/2(1/2-47),

-73 1/2(1/2-74)

This would make a possible total of 35 numbers.

If the set of Irrational Numbers ($\bar{R}$) had been called, each of the following numbers could be listed:

$\sqrt{7}$, $\sqrt{1/2}$, $\sqrt{3}$($\sqrt{7-4}$), $\sqrt{47}$, $\sqrt{74}$, $\sqrt{46\ 1/2}$($\sqrt{47-1/2}$),

$\sqrt{6\ 1/2}$($\sqrt{7-1/2}$), $\sqrt{3\ 1/2}$($\sqrt{4-1/2}$), $\sqrt{73\ 1/2}$($\sqrt{74-1/2}$),

$\sqrt{7\ 1/2}$, $\sqrt{4\ 1/2}$, $\sqrt{47\ 1/2}$, $\sqrt{74\ 1/2}$, $-\sqrt{7}$, $-\sqrt{1/2}$, $-\sqrt{3}$($-\sqrt{7-4}$),

$-\sqrt{47}$, $-\sqrt{74}$, $-\sqrt{7\ 1/2}$, $-\sqrt{4\ 1/2}$, $-\sqrt{47\ 1/2}$, $-\sqrt{74\ 1/2}$

Total = 22 numbers.

If the set of Real Numbers (D) had been called all of the above numbers could have been listed for a total of 57 numbers.

The winner would be the one with the highest point total.

SPECIAL CHALLENGE:

After you have had a few games, maybe you can think of some new rules to make a different game. Or maybe you can make up a completely new one. If you do, let your teacher know and try it out with your friends. Lots of luck.

ACTIVITY J*

Do you remember the property that stated that there is at least one rational number between any two given rational numbers? This property gave the impression that the number line was continuous and was called the Density Property.

For example take two rational numbers like 2.4 and 2.5. Between these we have 2.45. Or take two rational numbers like 2.45 and 2.46. Between these we have 2.455--and so on. But is $\sqrt{2}$ a rational number? Is the number 2.45373773777. . . a rational number? The answer is no and you must be thinking that there are a bunch of holes or empty spaces in the number line. You are right. And the only way to plug the holes is to fill in with irrational numbers. Then our number line (the real number line) is continuous and we won't "fall through" anywhere.

Let us try fitting an irrational number like 2.45373773777. . . into the number line. Try it on your own first.

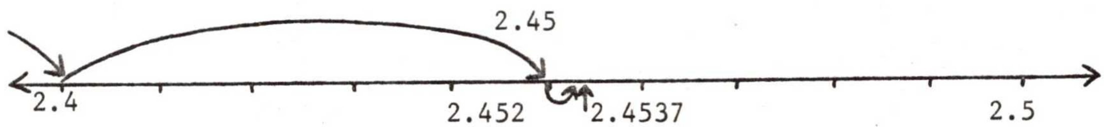
You should see that it would lie

to the right of 2.4

to the right of 2.45

to the right of 2.453

to the right of 2.4537 and so on.



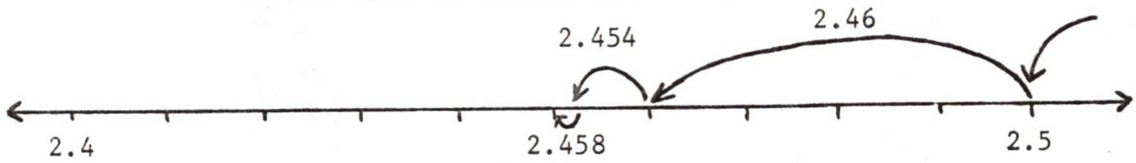
From the other side you should see that it would lie

to the left of 2.5

to the left of 2.46

to the left of 2.454

to the left of 2.4538 and so on.



---

*Modified after M. P. Dolciani, et al., Structure and Method 8, Modern School Mathematics Series, (Boston: Houghton Mifflin Company, 1967), pp. 235-236.

Of course you could continue this process for the rest of your life. Interested? The important idea here is that there is room (lots of room) on the number line for irrational numbers. When we combine the irrational numbers with the rational numbers to get the real numbers we can say that there is exactly one point on the number line corresponding to any given real number, (completeness property).

Another way of stating this property is that no matter where you locate a point in a line, there is a real number that can be associated with that point. That is, there are no "holes" in the real number line.

A. If you are interested, try fitting an irrational number of your own on the number line or try one of these:

(1) 0.237927779237779. . .

(2) 3.141601637947. . .

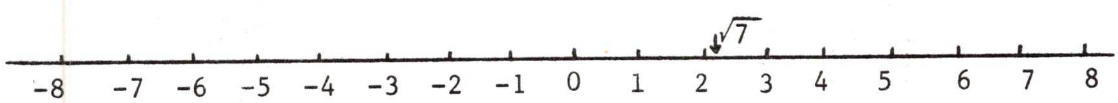
B. Copy the number line segment below:

On your copy, indicate where the graph of π lies in relation to the graphs of 3.14, 3.141, 3.1415 and 3.15, 3.142, 3.1416.

ACTIVITY K

To give you an idea of how the values of some irrational numbers compare to those of integers, try placing the following irrational numbers on the real number line as best as you can. (Use arrows--e.g. $\sqrt{7}$.)

(a) $\sqrt{12}$	(e) 1.673733. . .	(i) -6.4987342
(b) $\sqrt{1}$	(f) $-\sqrt{2}$	(j) $-\sqrt{31}$
(c) $-\sqrt{7}$	(g) $\sqrt{40}$	(k) $\sqrt{5}$
(d) $\sqrt{50}$	(h) π	(l) $-\sqrt{13}$



ACTIVITY L

$\sqrt{2}$ on the Number Line*

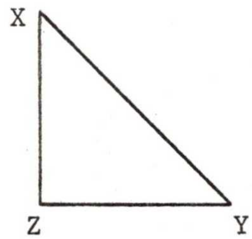
The problem in this activity is to measure a line equal to $\sqrt{2}$ inches in length. The method used will be based on the formulas for the area of a right triangle and the area of a square. Do you remember the formula?

Area of a right triangle =

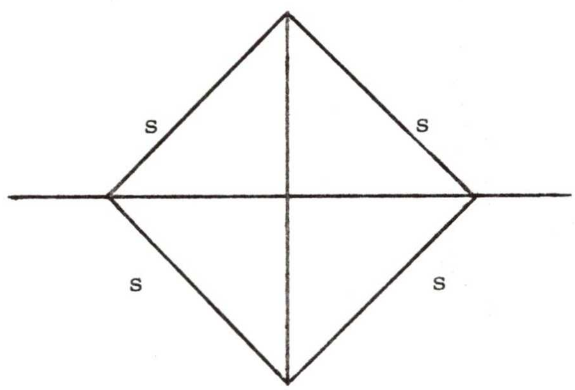
Area of a square =

Suppose we have a right triangle like Δ XYZ

Area of Δ XYZ = 1/2 ab
= 1/2 (1) (1)
= 1/2 sq. in.



Now if we take four of these triangles and fit them together, like the pieces of a jigsaw puzzle, we notice that they form a square. Since each triangle had an area of 1/2, and there are four such triangles in the square, the area of the square is 4(1/2), or 2.

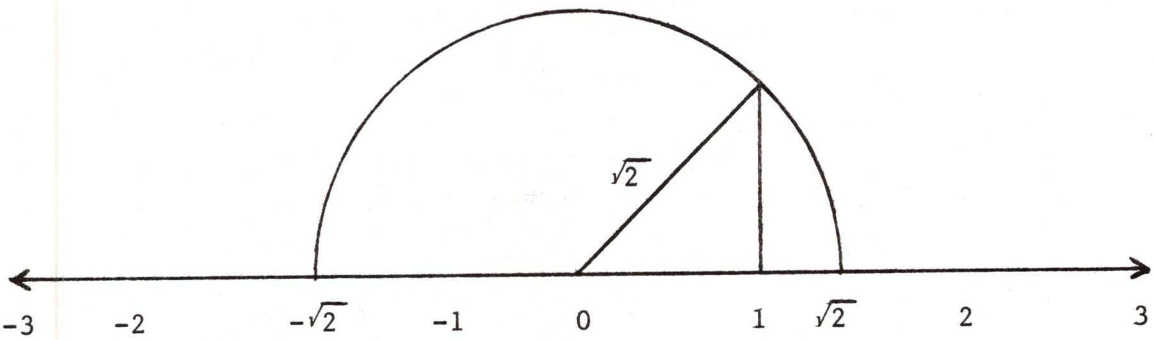


But the area of a square is equal to the length of its side multiplied by itself, is equal to 2(s^2=2). From our previous work we know s = $\sqrt{2}$.

---

*School Mathematics Study Group (SMSG), "Part III: An Experimental Approach to Functions," rev. ed. Mathematics Through Science (Stanford, California: Leland Stanford Junior University, 1964).

Now let's find $\sqrt{2}$ on the number line using this method. All we have to do is construct a right triangle with the two sides of length one inch and transfer the length of the third side to the number line.



This we can do by drawing a circle whose centre is at point 0 on the number line and whose radius is the same length as the third side of the triangle (use your compass). This circle cuts the number line in two points: $\sqrt{2}$ and $-\sqrt{2}$.

NOTE: This is still an approximate value and the accuracy depends on your construction. The idea is to find a point on the number line that corresponds to an irrational number and not a rational number.

Using the same number line if you wish, try and find points on the number line for the following coordinates.

(a) $2\sqrt{2}$ (d) $\sqrt{2} + 1$

(b) $-2\sqrt{2}$ (e) $\sqrt{2} - 1$

(c) $3\sqrt{2}$ (f) $-(\sqrt{2} + 1)$

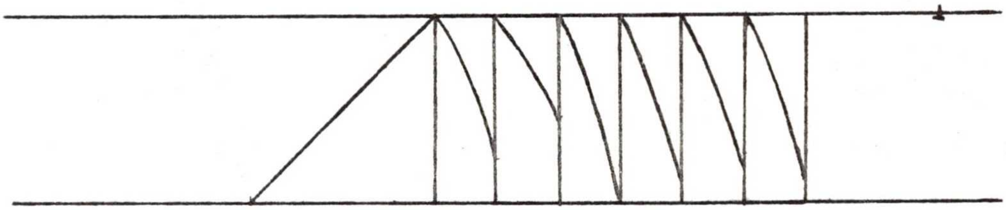
ACTIVITY M*

Irrationals on the Number Line

This activity is a special challenge and while you may be able to follow the construction below you should know the Pythagorean Theorem to understand this method.

---

*J. F. Pearcy and K. Lewis, Experiments in Mathematics, stages 1, 2 and 3 (London, Ont.: Longmans, Green and Co., Ltd., 1966).

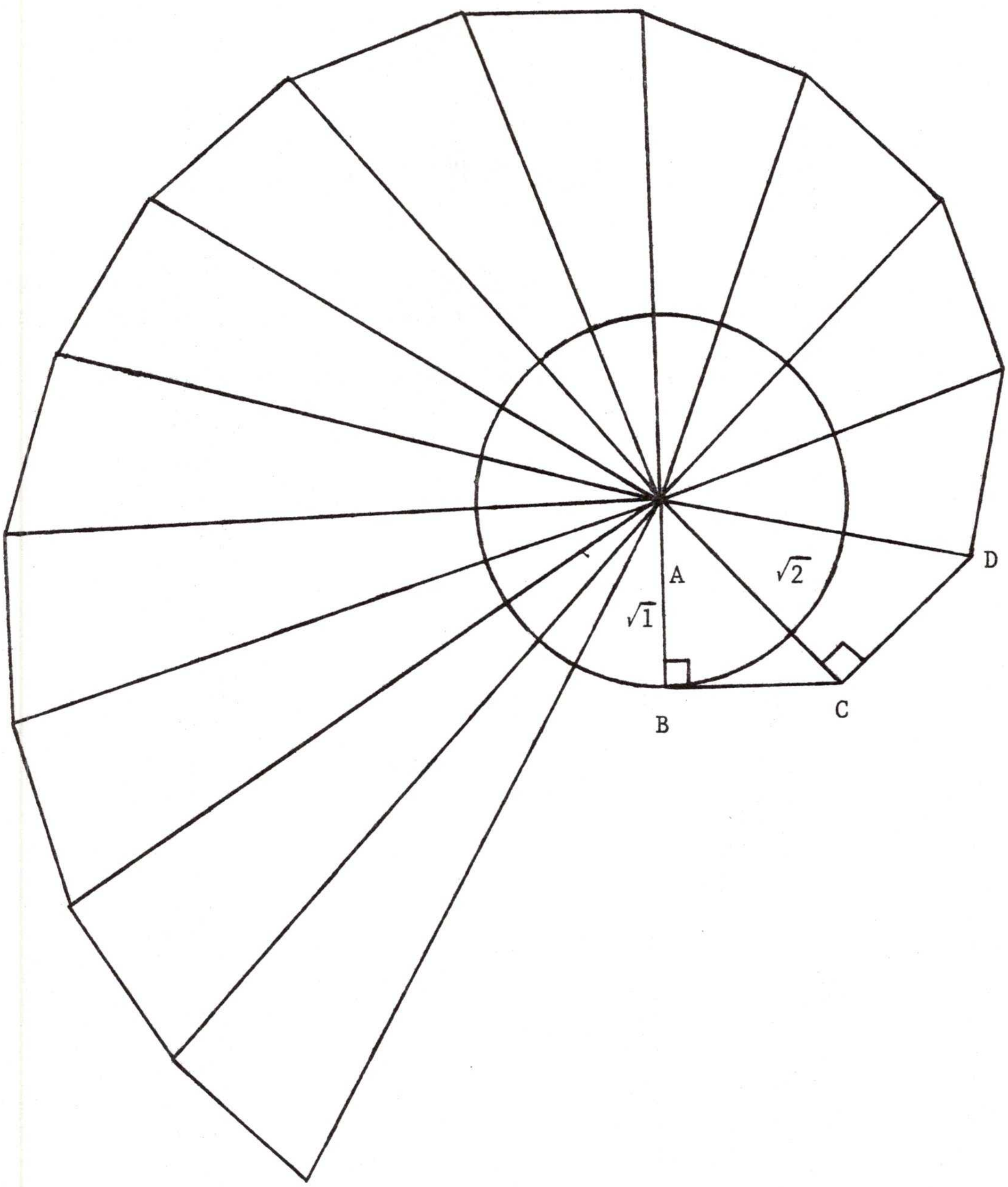


The method you used for graphing $\sqrt{2}$ (ACTIVITY L) can be extended to give a method for determining successively $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, etc. In the above construction let L be a line parallel to the number line and one inch away from it. Construct $\sqrt{2}$ as in ACTIVITY L. Draw or construct a perpendicular to $\sqrt{2}$ which meets L at point M. Using a compass with radius OA, and centre 0, draw an arc that meets the number line at $\sqrt{3}$. Applying the same technique to $\sqrt{3}$, you can locate $\sqrt{4}$. You can continue this technique for $\sqrt{3}$, $\sqrt{6}$, etc.

How would you graph $-\sqrt{3}$? $-\sqrt{7}$?

ACTIVITY N

Archimedes Spiral*



*E.D. Ripley and G. E. Tait, Mathematics Enrichment (Toronto: Copp Clark Publishing Co., 1966).

CONSTRUCTION

Draw $\triangle$ ABC

$\angle$ ABC = 90°

$\overline{AB} = \overline{BC}$ = 1 inch

$\overline{AC} = \sqrt{2}$ inches

At C draw a 1 inch segment CD perpendicular to AC.

$\overline{AD} = \sqrt{3}$. Do you know why?

You can continue constructing right-angled triangles in this fashion to obtain line segments equal to $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, etc.

The pattern forms the figure called Archimedes' Spiral.

TEACHERS' GUIDE

IRRATIONAL NUMBERS

LESSON OBJECTIVES:

1. Students learn about irrational numbers by trying to find the square root of certain rational numbers and discovering they cannot be expressed by repeating decimals and/or by observing and discovering differences in decimal patterns between a mixture of rational and irrational numbers.

   NOTE: Understanding and appreciating the existence of irrational numbers is fundamental to the development of the Real Number System and, I believe, the most important new concept that is developed in this unit.

2. Students learn that the set of real numbers (D) is the union of the set of rational numbers (R) and the set of irrational numbers ($\bar{R}$).

3. Students learn that there is a one-to-one correspondence between the set of real numbers and the set of all points in the number line. (The completeness property.)

4. Students learn a technique(s) of locating certain irrational numbers on the number line.

ACTIVITY A

This is essentially a simple discovery approach to irrational or non-repeating decimal numbers. Note that I have not distinguished between terminating and non-terminating repeating decimals. Unless some students become confused here, I don't believe this distinction is important as long as they realize that both types are repeating decimals.

ACTIVITY B

See the answer key for the computer calculation of π.

A proof that $\sqrt{2}$ is an irrational number follows on page 140. Show this to your top students or <u>any</u> others interested. As a special challenge they might try proving $\sqrt{3}$, is an irrational using a proof similar to the one shown.

ACTIVITY D

This activity should help students understand the real number game.

ACTIVITY E

This is an opportunity to let the students create their own irrationals and show their knowledge and understanding.

ACTIVITY F

This activity should provide some amusing and active reinforcement and understanding. Don't let students spend too much time on this one; e.g., 5 - 15 minutes - ?

ACTIVITY H

A good student may not need to do this activity. With most students it should help prepare them for the real numbers game.

ACTIVITY I

Note that I have stressed each game must include some player specifying irrational numbers on one shake. To save class time, you could display the cubes (dice) the students will use in the game and have them study the rules and example at home. Perhaps some might wish to make up their own game from sugar cubes or wooden blocks.

I don't think it is particularly important that players succeed in finding <u>all</u> possible numbers for a particular set as it is that they specify the <u>correct</u> ones for each set. Your better students might be challenged to find all possibilities.

This game will vary in motivation and activity with each student of course. I think all students should play it at least once since it not only provides understanding and reinforcement for the study of irrational numbers, but all number systems the students have studied to date and therefore is an excellent summary of their study of number systems.

ACTIVITY J

Good students might skip this activity in favour of activities M and N.

ACTIVITY K

Stress that all these locations on number line are approximate.

ACTIVITY L

There are many different constructions for locating an irrational number on the number line, but most depend on an understanding of the Pythagorean Theorem. This activity is the simplest I could find and should not prove too difficult to most of your students, I hope. You might be wise to go over the construction with some, using more accurate measurements and a more accurate number line.

Exercises(a) to (f) can be assigned at your discretion depending on the students' capabilities. Note these can be located without any further constructions.

ACTIVITY M and N

For your top and very interested students.

Students should try the activity enclosed on the Pythagorean Theorem to really understand the theory and construction for these activities, even though they may be capable of repeating the construction.

Suggested Time Limit: 3 - 5 days.

Optional Activities

H and J might not be necessary for some of your better mathematicians.

M and N are not recommended for your less talented mathematicians.

Proof for an irrational number ($\sqrt{2}$)--see Activity B

$\sqrt{2}$ - NOT RATIONAL?*

How can we prove $\sqrt{2}$ is irrational? A computing machine can quickly give us thousands of decimal digits of its numeral, and we can see that no repeating pattern has emerged. But perhaps $\sqrt{2}$ is a rational number whose simplest fraction has a very great numerator and a very great denominator. It might easily require millions of digits, instead of thousands, before a repeat begins. We can never prove the number is irrational by looking at digits, no matter how many of them we produce.

Let us try and prove $\sqrt{2}$ is rational--That is, write $\sqrt{2}$ as a fraction. But first we should review the following facts:

"A. All whole numbers are either even or odd. No number is both even and odd. (The even whole numbers are 0, 2, 4, 6, . . .: the odd ones are 1, 3, 5, 7, . . .)

B. Any even whole number can be written as two times another whole number. For example,

$$10 = 2 \times 5,$$

$$12 = 2 \times 6.$$

Any number that is twice a whole number is even.

C. The square of an even number is even. The square of an odd number is odd. (Try some examples to convince yourself.)

D. Any fraction is equivalent to the fraction changed to its simplest form--that is, the form whose numerator and denominator have 1 as their greatest common divisor. For example, 30/42 is equivalent to 5/7, which is in simplest form.

E. If a fraction is in simplest form, then its numerator and denominator are not both even. (For then, according to statement B, we could divide numerator and denominator by 2.) At least one of them is odd.

Now let us try to write $\sqrt{2}$ as a fraction. This fraction (if there is one) can be in simplest form, by statement D. The denominator is some counting number, q, and the numerator is a whole number, p.

---

*National Council of Teachers of Mathematics, More Topics in Mathematics (30th yearbook, Washington, D.C.: NCTM, 1968).

Then we have

$$p/q = \sqrt{2}, \quad \text{or} \quad p/q \times p/q = \sqrt{2} \times \sqrt{2},$$

$$\text{or} \quad p^2/q^2 = 2, \text{ by the definition of } \sqrt{2}.$$

Now, multiplying both sides of the equation by q^2, we get

$$p^2/q^2 \times q^2 = 2 \times q^2,$$

$$\text{or} \qquad p^2 = 2 \times q^2.$$

Thus p^2 is even, by statement B; and statement C implies that p is even.

Then q is odd, by statement E. Further, using the result that p is even and applying fact B again, we see that there is some whole number r such that $p = 2 \times r$. Therefore, we can substitute $2 \times r$ for p, in the equation $p^2 = 2 \times q^2$, so that

$$(2 \times r)^2 = 2 \times q^2, \quad \text{or} \quad (2 \times r) \times (2 \times r) = 2 \times q^2.$$

Now the associative and commutative properties of multiplication permit us to regroup the expression on the left as follows:

$$(2 \times 2) \times (r \times r) = 2 \times q^2$$

$$\text{or} \qquad 4 \times r^2 = 2 \times q^2,$$

or, when we divide both sides by 2,

$$2 \times r^2 = q^2$$

Now statements B and C imply that q^2 and q are even.

Hence, q is both even and odd. But, by fact A, this is impossible; q does not exist and there is no fraction for $\sqrt{2}$. This completes the proof--but if you have found it difficult, then perhaps you should start again to appreciate its subtleties."

TWO THOUSAND DECIMALS OF PI CALCULATED ON IBM 704

3.14159	26535	89793	23846	26433	83279	50288	41971	69399	37510
58209	74944	59230	78164	06286	20899	86280	34825	34211	70679
82148	08651	32823	06647	09384	46095	50582	23172	53594	08128
48111	74502	84102	70193	85211	05559	64462	29489	54930	38196
44288	10975	66593	34461	28475	64823	37867	83165	27102	19091
45648	56692	34603	48610	45432	66482	13393	60726	02491	41273
72458	70066	06315	58817	48815	20920	96282	92540	91715	36436
78925	90360	01133	05305	48820	46652	13841	46951	94151	16094
33057	27036	57595	91953	09218	61173	81932	61179	31051	18548
07446	23799	62749	56735	18857	52724	89122	79381	83011	94921
98336	73362	44065	66430	86021	39494	63952	24737	19070	21798
60943	70277	05392	17176	29317	67523	84674	81846	76694	05132
00056	81271	45263	56082	77857	71342	75778	96091	73637	17872
14684	40901	22495	34301	46549	58537	10507	92279	68925	89235
42019	95611	21290	21960	86403	44181	59813	62977	47713	09960
51870	72113	49999	99837	29780	49951	05973	17328	16096	31859
50244	59455	34690	83026	42522	30825	33446	85035	26193	11881
71010	00313	78387	52880	58753	32083	81420	61717	76691	47303
59825	34904	28755	46873	11595	62863	88235	37875	93751	95778
18577	80532	17122	68066	13001	92787	66111	95909	21642	01989
38095	25720	10654	85863	27886	59361	53381	82796	82303	01952
03530	18529	68995	77362	25994	13891	24972	17752	83479	13151
55748	57242	45415	06959	50829	53311	68617	27855	88907	50983
81754	63746	49393	19255	06040	09277	01671	13900	98488	24012
85836	16035	63707	66010	47101	81942	95559	61989	46767	83744
94482	55379	77472	68471	04047	53464	62080	46684	25906	94912
93313	67702	89891	52104	75216	20569	66024	05803	81501	93511
25338	24300	35587	64024	74964	73263	91419	92726	04269	92279
67823	54781	63600	93417	21641	21992	45863	15030	28618	29745
55706	74983	85054	94588	58692	69956	90927	21079	75093	02955
32116	53449	87202	75596	02364	80665	49911	98818	34797	75356
63698	07426	54252	78625	51818	41757	46728	90977	77279	38000
81647	06001	61452	49192	17321	72147	72350	14144	19735	68548
16136	11573	52552	13347	57418	49468	43852	33239	07394	14333
45477	62416	86251	89835	69485	56209	92192	22184	27255	02542
56887	67179	04946	01653	46680	49886	27232	79178	60857	84383
82796	79766	81454	10095	38837	86360	95068	00642	25125	20511
73929	84896	08412	84886	26945	60424	19652	85022	21066	11863
06744	27862	20391	94945	04712	37137	86960	95636	43719	17287
46776	46575	73962	41389	08658	32645	99581	33904	78027	59009

*D. G. Seymour, and R. Gidley, EUREKA (Palo Alto, California: Creative Publications, 1968).

You might be interested to know that it took the computer about 70 hours to perform this calculation. If you were to sit down with paper and pencil to work this out, you would have to live longer than Methuselah before you would be finished.

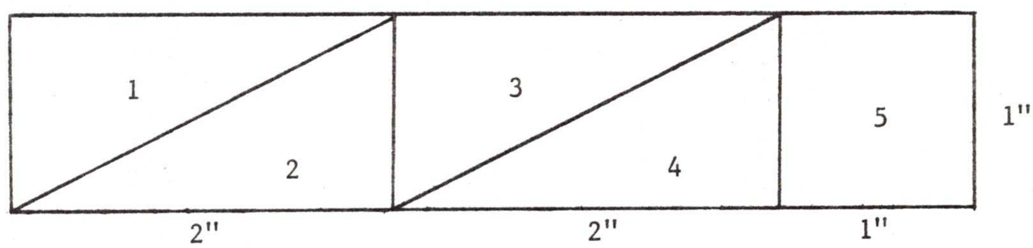
As mentioned in your lesson notes, this discovery by the Pythagoreans was very shocking. Another legend states that Pythagoras swore his associates to secrecy and slaughtered one hundred oxen as a sacrifice.

PYTHAGORAS' THEOREM*

PART 1

Equipment: Card, scissors, and glue.

Take a rectangle of card 5 inches × 1 inch and mark it out:

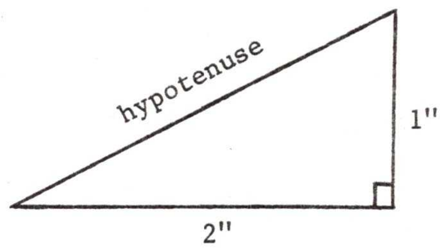


Cut out the five pieces then:

(a) Assemble pieces 1, 2, 3 and 4 to make a square.

(b) Assemble all five pieces to make a square.

(c) Draw the following triangle in the centre of a page.



Place the square formed in (b) on the hypotenuse of this triangle.

Now cut up another strip in exactly the same way as before, place piece number 5 on the shortest side of the above triangle and the square

---

*J. F. Pearcy and K. Lewis, Experiments in Mathematics, stages 1, 2, 3. (London, Ont.: Longmans, Green and Co., Ltd., 1966).

formed in (a) on the 2 inch side. Stick these three squares down around the triangle.

The square on the hypotenuse is made up of the same pieces which go to make the other two squares on the other sides of the right angled triangle.

This shows that the area of the square on the hypotenuse is equal to the area of the squares on the other two sides of this triangle.

Pythagoras' Theorem proves that this result is true for every right angled triangle.

This is but one dissection to demonstrate Pythagoras' Theorem. Many others exist and it is left as an extra exercise for you to find some of them.

ANSWER KEY

ACTIVITY A

Group $.625\overline{0}$, $.3\overline{3}$, $9.\overline{0}$, $.\overline{14257}$, $38.\overline{12}$

Group 2.236 ... , 4.123233 ... , .010010001... ,

.3149765389... , 17.634979779...

It does not matter whether each group of five numbers is in group A or B as long as you have the same five numbers in each group.

The basic difference between the two groups is that one group has a repetend or repeating decimal while the other group does not have a repetend or repeating decimal.

ACTIVITY B

Rational numbers: (a), (b), (c), (f)

Irrational numbers: (d), (e), (g), (h)

ACTIVITY D

Rational numbers: (a), (c), (d), (g), (j), (l)

ACTIVITY F

You could go on rolling the cubes until you are old and grey. After the first 2 or 3 rolls it would be next to impossible to roll a repeating decimal or rational number.

ACTIVITY G

(a) π (335/113) = 2.9646017

π (computer) = 3.1415926

Difference = 0.1769909

(b) π (22/7) = 3.142857

π (computer) = 3.141592

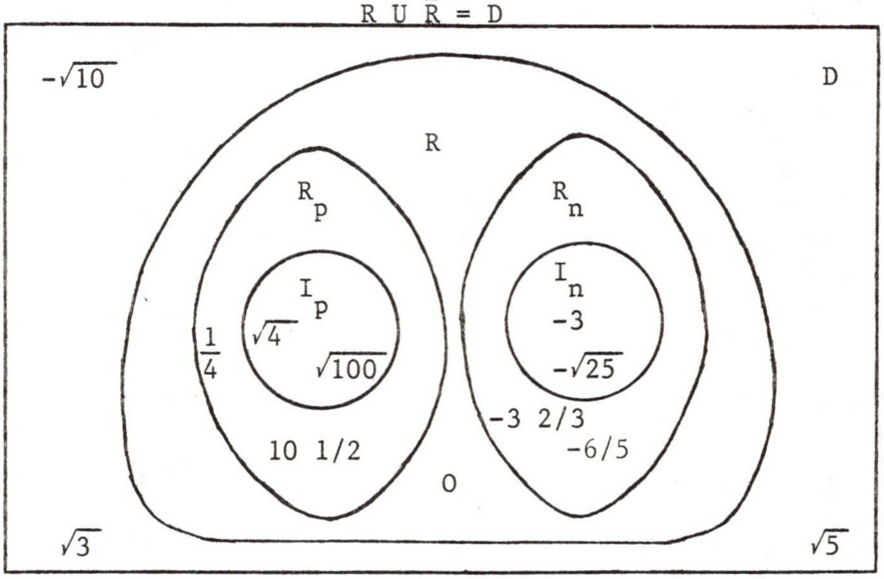
Difference = .001265

22/7 > π by 1/2500 or .04%

ACTIVITY H

Yes, all the numbers you have studied to date (both R and $\bar{R}$) are real numbers since both sets can be named by infinite decimals.

$R \cup \bar{R} = D$



ACTIVITY I

I'm glad I didn't holler "Big Boast."

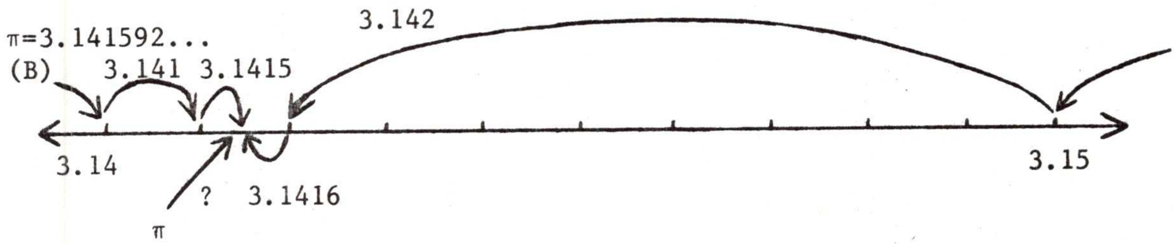
Here are some more irrational numbers:

$(4-\sqrt{7})$, $(\sqrt{7}-4)$, $(\sqrt{1/2}-7)$,

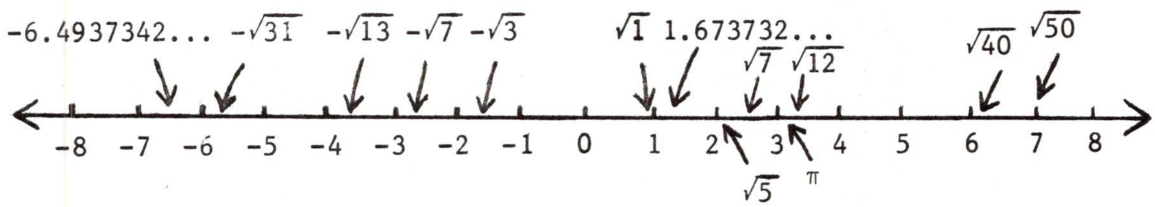
$(4-\sqrt{1/2})$, $(\sqrt{1/2}-4)$, $(\sqrt{7}-1/2)$

Did you find any others not listed in the example?

ACTIVITY J

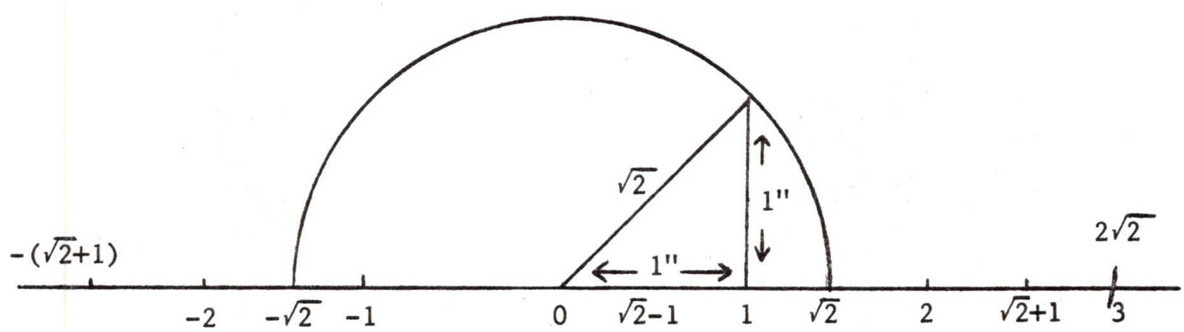


ACTIVITY K



ACTIVITY L

All these numbers can be located on the number line using your original construction for locating $\sqrt{2}$ and a compass and/or ruler.



ACTIVITY M

To graph the negative irrationals - e.g. $-\sqrt{3}$, $-\sqrt{7}$, etc.

Simply rotate the compass, with centre at 0, to the other side (negative side) of the number line. See how $-\sqrt{2}$ was located on the number line in Activity L.

ACTIVITY N

Note: For all of activities L, M, and N the unit of measure need not be restricted to one inch. It could be 1 foot, 1 centimeter or any unit segment.

Labelling the hypotenuses (hypoteni?) of Archimedes Spiral from $\sqrt{1}$ to $\sqrt{16}$ might help some students understand this construction. If they use 1 inch as their unit measure, their construction can be checked at $\sqrt{4}$, $\sqrt{9}$, or $\sqrt{16}$.

STUDENT GUIDE

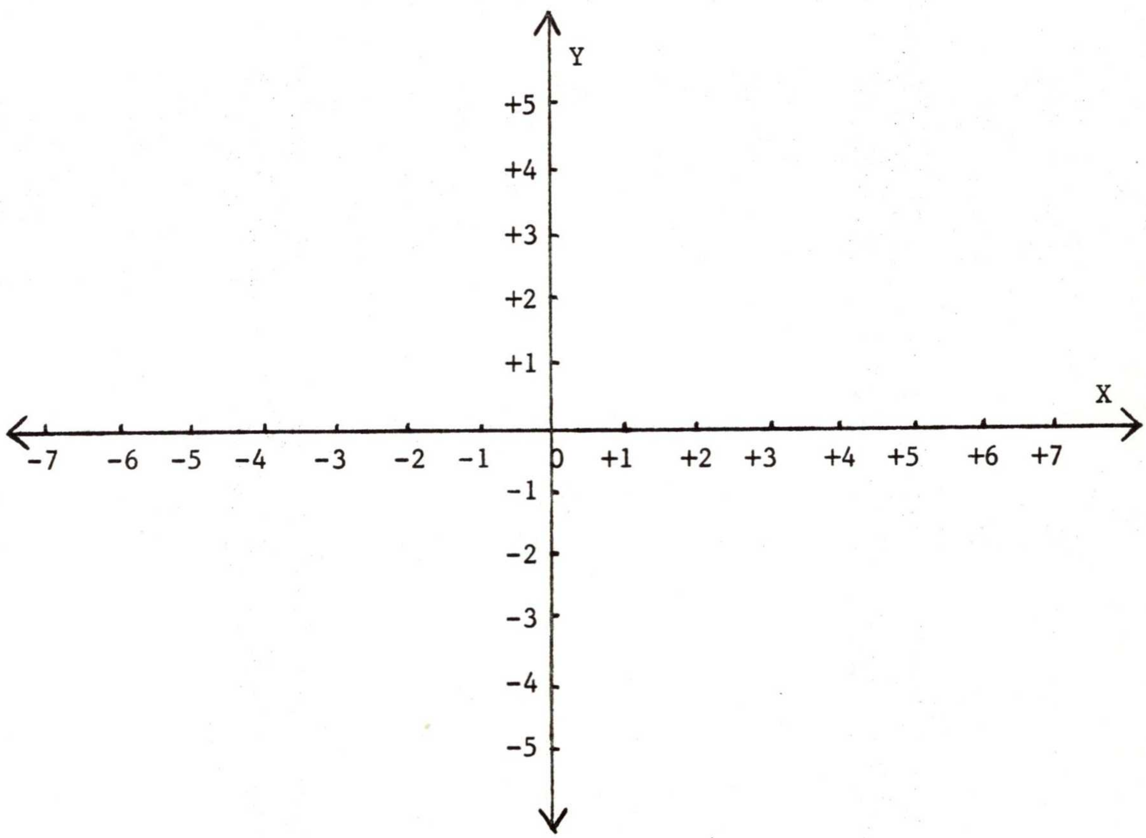
GRAPHING IN THE REAL NUMBER PLANE

In this particular section, you are going to see how certain mathematical procedures can be used to study the growth patterns of living organisms. An amazing discovery about living organisms (micro-organisms, plants, or animals--including humans) is that all have quite similar growth patterns. We shall investigate these growth patterns with a population of mold, but first we need to know some additional mathematical procedures.

ACTIVITY A

The Rectangular Coordinate System

There are a number of ways you can learn about associating ordered pairs of real numbers with the rectangular coordinate system (see the diagram below)



*1. You might ask your teacher to play the class game called "Living Coordinates." This is an activity in which every student in the class is identified by an ordered pair of numbers according to his position in the class seating plan. To be able to play you must learn how to "stay alive" by standing in response to certain questions the teacher will ask.

*2. You could play the game of "Tic-Tac-Toe," "Battleship," or "Go" on a grid similar to the one above and try and "discover" how to identify points on the grid with ordered pairs of real numbers. Since you might already know how to play one of these games, you will probably find this more fun and interesting. It will certainly be a challenge for you to solve this problem without the teacher telling you.

*3. Another method would be to study an explanation from a textbook or notes from your teacher about the rectangular coordinate system and complete any exercises that you might be assigned.

ACTIVITY B

Connect the Coordinates (dots)**

Graph each of the following points in the real coordinate plane

---

*For more specific details on the activities named in suggestions 1 and 2, the reader is referred to the following teacher's guide on graphing in the real number plane. An excellent reference for suggestion 3 is: School Mathematics Study Group (SMSG), Mathematics and Living Things, Student Text, (Stanford, Calif.: Leland Stanford Junior University, 1965), pp. 129-137. The material in this text on these pages was actually duplicated and distributed with the original experimental unit. While the interested reader may wish to consult this reference to view the original context, he might be interested to know that the teachers thought this material presented too much of a conventional approach. Two of the teachers omitted the majority of this material and presented the students with the opportunity to discover these mathematical concepts on their own. They did this by trying to extend the game of "Tic-Tac-Toe" and "Battleship" to the complete rectangular coordinate grid from their previous experience with just the first quadrant. This method was very successful and well received by the students in the opinion of these two teachers. For this reason the investigator would suggest this approach in preference to the original material and suggestions for use.

**D. G. Seymour and R. Gidley, EUREKA (Palo Alto, Calif.: Creative Publications, 1968).

and connect the points in order with line segments. What do you see?

(-1, -2), (-6, -7), (-5, -7), (-4, -6), (-3, -7), (-2, -6),

(-1, -7), (0, -6), (1, -7), (2, -6), (3, -7), (4, -6), (5, -7),

(6, -7), (1, -2), (3, -2), (4, -1), (5, -2), (9, -2), (10, -1),

(11, -1), (12, 0), (13, 0), (14, 1), (15, 1), (16, 2),

(5, 2), (4, 1), (3, 2), (3, 4), (4, 5), (5, 5 1/2), (4, 6), (3, 6),

(2, 7), (0, 7), (-3, 4), (-3, 2), (-4, 1), (-5, 2), (-16, 2),

(-15, 1), (-14, 1), (-13, 0), (-12, 0), (-11, -1), (-10, -1),

(-9, -2), (-5, -2), (-4, -1), (-3, -2), (-1, -2).

ACTIVITY C

The World's Most Famous Beagle!

I. Plot the following points and join them in order with straight lines:

(-8, 2), (-6, -2), (-6, -6), (-8, -8), (-12, -6), (-12, 0),

(-8, 8), (-4, 9), (-2, 8), (0, 4), (4, 4), (7, 1), (7, -1),

(2, -4), (-2, -4), (-2, -8), (0, -10), (-6, -10), (-4, -8),

(-4, -4), (-6, -4). Join these points (-3, -3), (-2, -2),

(-2, 4).

Make a big dot ● at the points (-4, 4) and (-2, 4).

Make an oblong circle around the X axis between 2 and 4.

Shade the circle in.

Do you recognize this World War I flying ace?

Make a big dot at the points (-4, 4) and (-2, 4).

Make an oblong circle around the X axis between 2 and 4.

Shade the circle in.

Do you recognize this World War I flying ace?

What's his name?

What is the name of the comic strip he appears in?

To make him look more authentic, shade in his ear.

II. Take the above ordered pairs and divide each value by 2. This will give you the points:

(-4, -1), (-3, -1), (-3, -3), (-4, -4), (-6, -3) etc.

Complete the rest of these and join them up as before. What has happened to Snoopy? ______________________________

____

III. Using the original ordered pairs, keep the X-value the same and double the Y-value. This will give you the points:

(-8, 4), (-6, -4), (-6, -12) etc.

Complete these, plot them and join them up. Describe your dog now. ______________________________

____

IV. Using the original ordered pairs, keep the Y-value the same and double the X-value. Plot these, join them up and describe Snoopy now. ______________________________

____

V. This time take the original values again, leave the Y-value alone and add 1 to every value of X. What's happened to Snoopy? ______________________________

____

Experiment with some other techniques of changing the X and Y values and see what happens to Snoopy. Compare them or try them out on your partner.

ACTIVITY D

The following activity is a more challenging one for those of you who enjoy plotting animal figures. See what kind of animal you end up with this time. Any guesses!

NOTE that the instructions are very similar except that the ordered pairs of numbers are in table form.

What about plotting Charlie Brown, Woody Woodpecker, Yogi Bear, Robin Hood? Get a coloring book or some picture of your favorite cartoon character, or any other character or figure you prefer (not too detailed). Trace their picture on graph paper and then map out a set of ordered pairs that when joined up will show your hero. Try it out on your partner. Have fun.

ACTIVITY E

Tic-Tac-Toe

Perhaps you have played this game using coordinates in the first quadrant. Have you ever played X's and O's? This game can be played with many variations of the rules. To play you simply need a graph of the real plane and two players or two teams. Each team takes turns calling ordered pairs to be marked on the graph. The object of the game is to get 4 of your points on the graph that can be joined in a straight line. The winning player or team is the one with the most straight lines (or points if you award a point for each line).

After playing this one or more times, can you think of some new rules or changes in the old rules to make the game different and/or more interesting. Call it your version of Tic-Tac-Toe.

If you like these kinds of games ask your teacher about:

1. "Battleship."[*] Each team tries to sink the other team's "Navy" by firing shots of ordered pairs onto the enemy battlefield (graph).

2. "The Point-Set Game."[**] A version of the ancient Japanese war game called "GO." See who will be the first to holler "atari." A very interesting and challenging game.

---

[*] Edith E. Biggs, and James Maclean, Freedom to Learn (Don Mills, Ont.: Addison-Wesley (Canada) Ltd., 1969), p. 51.

[**] Robert, B. Davis, Discovery in Mathematics. A Text for Teachers (Reading, Mass.: Addison Wesley Publishing Co., 1964), pp. 55-66.

GRAPHING PICTURE*

	X	Y
	-2	3
*	-3	4
	-4	3
	-2	3
	-2	-3
	2	-3
	2	3
*	3	4
	4	3
	2	3
	lift pencil	
	-2	-3
	-1½	-4
*	1½	-4
	2	-3
	-2	-3
	lift pencil	
	-5½	-2½
	-6	-3
	-6	-1½
	-8	-2
	-6	½
	-8	1
	-6	2
	-9	5
	-6	4
	-6	5
	-4	3½
	-5	7
	-2	5
	-2	9
	0	5
	2	9
	2	5
	5	7
	4	3½
	6	5
	6	4
	9	5
	6	2
	8	1
	6	½

X	Y
8	-2
6	-1½
6	-3
5½	-2½
lift pencil	
-4	-2
-5	-3
-5	-5
-9	-7
-7	-4
-9	-5
-9	-3½
-12	-3
-11	½
-12	1
-11	3½
-12	5
-11	5
-11½	7½
-11	8
-12	11
-9	10
-10½	14
-7	12
-7	13
-6	12
-6	14
-5	12
lift pencil	
-6	9½
-5½	12
-4½	11
-4	14
-2½	12
-2	13
-½	12
0	12½
½	12
2	13
2½	12
4	14
4½	11

X	Y
5½	12
6	9½
lift pencil	
5	12
6	14
6	12
7	13
7	12
10½	14
9	10
12	11
11	8
11½	7½
11	5
12	5
11	3½
12	1
11	½
12	-3
9	-3½
9	-5
7	-4
9	-7
5	-5
lift pencil	
4	-2
5	-3
5	-6
3	-8
2	-8
1	-9
-1	-9
-2	-8
-3	-8
-5	-6
-5	-5
lift pencil	
-4	-5
-2½	-6
2½	-6
4	-5

*Robert C. Madison, Graphing Pictures (Des Moines, Iowa: Central Iowa Mathematics Project (CILAMP), 1969).

ACTIVITY F

Growing Mold

The experience and knowledge you have gained from the previous activities will be used in this biological experiment. The experiment is based on a study of the growth pattern of a mold culture and will involve the ability to read a rectangular coordinate graph to identify the position of the mold.

Materials and Supplies

You will need an aluminum pie or cake tin, 10 × 10 to the inch graph paper, gelatin, a bouillon cube, Saran wrap, a rubber band or Scotch tape, scissors and a ruler for each group.

Procedure

1. To prepare the tin, cut the graph paper to fit into the bottom of the tin. The X and Y axes must be constructed on the graph paper and then rubber cement or glue should be used to hold the graph paper to the bottom of the tin.
2. To prepare the gelatin mix, combine a cold mix of gelatin with a hot mix of a bouillon cube and be ready to pour before the mix sets.
3. Pour a thin layer of the mix onto the graph paper in the bottom of the tin. Put the tin aside to cool and set for approximately 5 minutes. While it is "setting," it will be contaminated with mold from the air.
4. Cover the top of the tin with an excess of Saran wrap or a similar transparent wrap and fasten with a rubber band or Scotch tape.
5. Store in a dark place where the temperature is fairly uniform.
6. Observe and record your observations every day.

Recording

The mold will probably become visible on the 2nd, 3rd, or 4th day and as soon as it does, the following method should be used.

1. Identify the position of each dot of the mold on the coordinate grid at the bottom of the tin.
2. Transfer the position of each dot of mold to a separate piece of graph paper according to both its position and the number of squares covered. This procedure is to be completed every day.
3. The data should be used to complete the following table (see Table M).
4. From the total growth data in the table a graph should be plotted. Use the day number for the x-axis and the y-axis for area of squares covered. (See Figure A on the second page following.)

TABLE M*

RECORD OF MOLD GROWTH

A	B	C	
DAYS	TOTAL AREA TO DATE (SQUARE UNITS)	INC. AREA OVER PREVIOUS DAY (SQUARE UNITS)	PERCENT OF INCREASE- INC. AREA / TOTAL AREA
Start (Friday)			
2nd. Saturday 3rd. Sunday			
4th. Monday 5th. Tuesday 6th. Wednesday 7th. Thursday 8th. Friday			
9th. Saturday 10th. Sunday			
11th. Monday 12th. Tuesday You may go on . . .			

*Modified after Mathematics for Living Things, Student Text, rev. ed. (Stanford, Calif.: Leland Stanford Junior University 1965), p. 141.

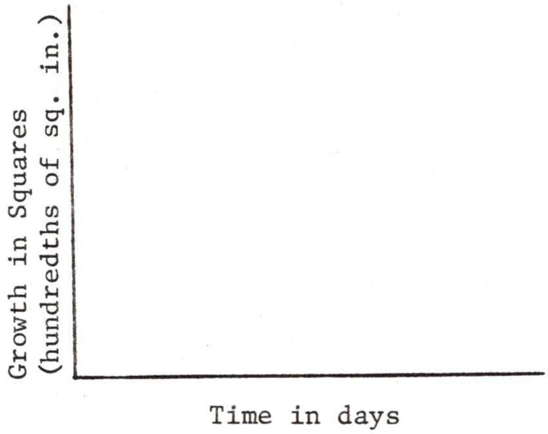


FIGURE A

5. Upon completing the experimental procedures, analyse your results by comparing them with a "typical" growth curve and the growth curves obtained from other living organisms. (See your teacher for this information.)

NOTE: The detailed instructions necessary for this student experiment are found on pages 138-143 of Mathematics for Living Things, Student Text, SMSG (Stanford, Calif.: Leland Stanford Junior University, 1965). These pages were duplicated for the students in the original unit and the reader will find complete information for the student experiment from this source. For additional information see Activity F in the Teacher's Guide of this section of the Appendix.

TEACHER'S GUIDE

GRAPHING IN THE REAL PLANE

LESSON OBJECTIVES:

Students should learn how to associate ordered pairs of real numbers with points in the real plane.

ACTIVITY A

The purpose of the activities mentioned in the student guide is to teach the student the mathematical procedures involved in identifying points in the real plane. In using these activities, teachers should select the most appropriate one based on the abilities and interests of the students. (See the explanatory note at the end of Activity A, student guide.)

1. "Living Coordinates"
   This is an excellent introductory activity although the students should have some preliminary introduction to the rectangular coordinate system. For full details the reader is referred to pages 109-112 of Mathematics for Living Things, Teachers' Commentary, rev. ed., School Mathematics Study Group (Stanford, Calif.: Leland Stanford Junior University, 1965). A summary description is also included in Chapter 4 of this study.
2. "Battleship"
   This game is described on page 51 of Freedom to Learn, E. E. Biggs and J. R. Maclean (Don Mills, Ont.: Addison-Wesley (Canada) Ltd., 1969). Please note that this game can easily be modified to correspond with the identification of points on the grid according to the intersection of two lines.
   "Go"
   A version of this game is described under the heading "The Point Set Game" by Robert B. Davis in the text Discovery in Mathematics, A Text for Teachers (Reading, Mass.: Addison Wesley Publishing Co., 1964).
3. As mentioned previously in the student guide the reader is referred to a detailed and thorough development of the rectangular coordinate system along with various exercises in pages 129-137 of School Mathematics Study Group of Mathematics for Living Things, Student Text, rev. ed. (Stanford, Calif.: Leland Stanford Junior University, 1965). There is also a published teacher's guide and answer key for these activities found on pages 102-109 of School Mathematics Study Group of

Mathematics for Living Things, Teachers' Commentary, rev. ed. (Stanford, Calif.: Leland Stanford Junior University, School Mathematics Study Group, 1965). The teacher's guide in the original experimental unit included duplicated copies of these pages.

ACTIVITIES B, C, and D

These are puzzle and enrichment activities that students can work on according to their interests. They should prove highly motivating to most students and if additional activities of this nature are desired, the manual Graphing Pictures by Robert C. Madison provides a rich source of materials. However, before supplying the students with many more plotting activities of this type, you might encourage them to make up their own with the aid of pictures or a coloring book. An example of the caricature of "Leo the Lion" which would result from the instructions given in Activity D is included for the teacher's use.

ACTIVITY F

Growing Mold

This is *the* major activity of this section and is an excellent example of active learning. Not only does it provide the student with "concrete" experience and an opportunity to use his mathematical skills gained from the previous unit, but it offers the teacher some excellent opportunities to coordinate mathematics with the subject areas of science and social studies.

Background Information

Growth is a major biological concept and if it can be measured early enough and late enough in the life of the organ (such as heart, liver, brain, etc.), organism (total living thing), or population, then three phases should be evident in the final graph:

*1. The lag phase: illustrating a slow start, a period of cell or organism adjustment.

*2. The grand phase or exponential phase, where the cells are multiplying exponentially (1 cell divides into 2, the 2 divide into 4-8-16-32-64- etc., becoming astronomical in number before limiting factors begin to be felt), and

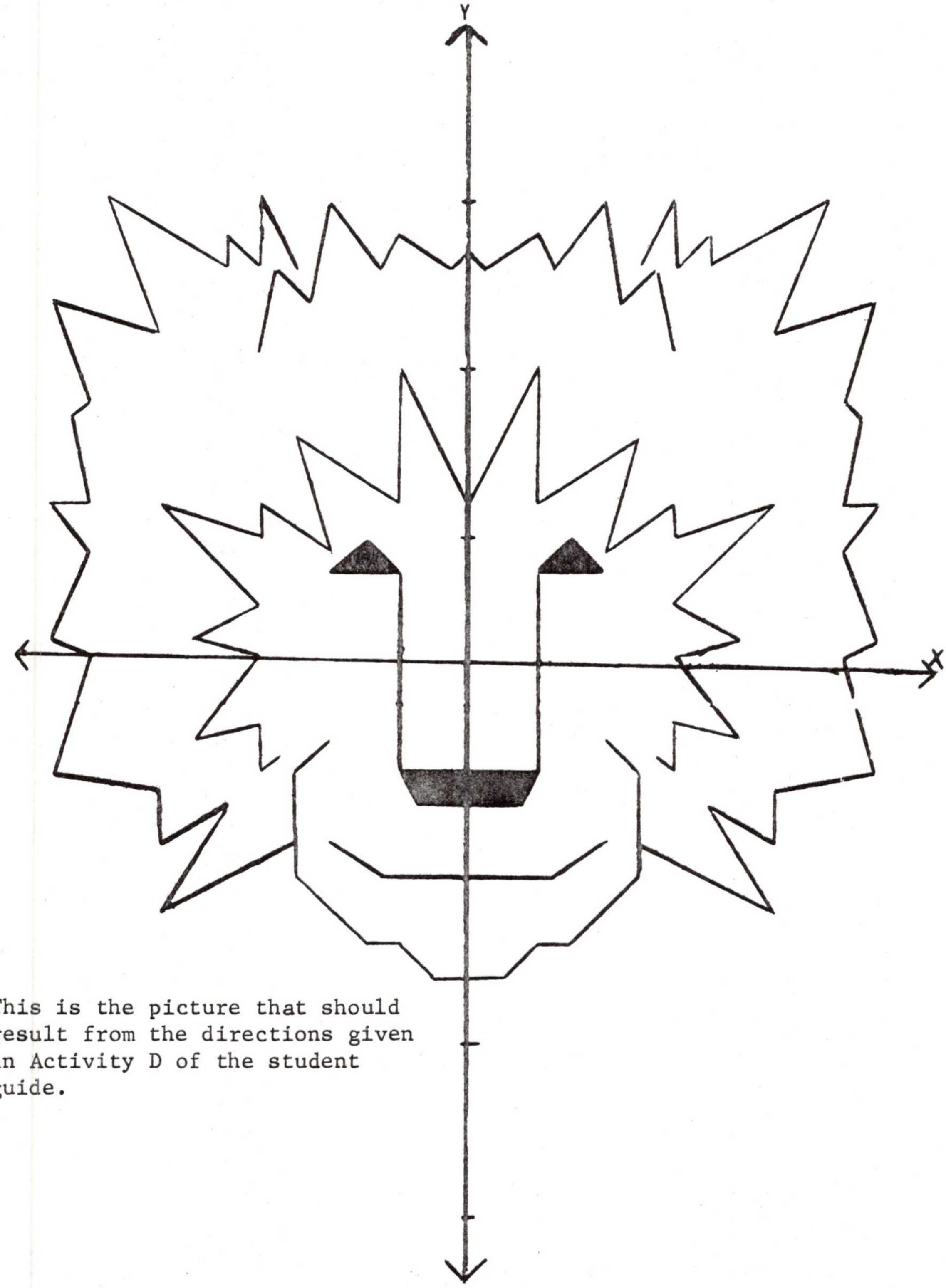
*3. The stationary phase or senescence, a levelling off as a result of a completed set of limiting factors, such as

(continued on third page following)

---

*School Mathematics Study Group, Mathematics for Living Things, Teachers' Commentary, rev. ed. (Stanford, Calif.: Leland Stanford Junior University, 1965), p. 100.

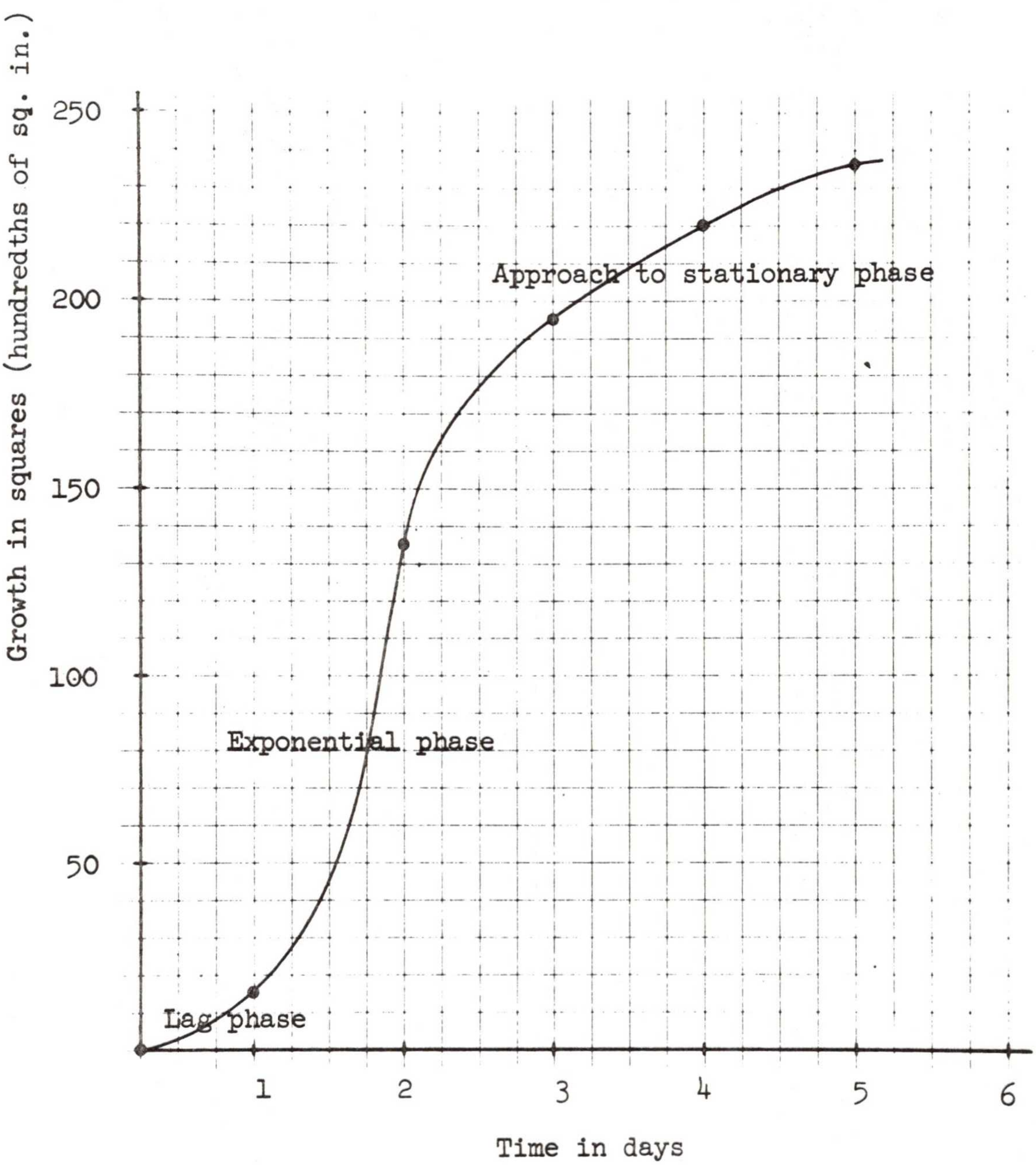
THE LION



This is the picture that should result from the directions given in Activity D of the student guide.

Source: Robert C. Madison, Graphing Pictures (Des Moines, Iowa: Central Iowa Mathematics Project (CILAMP), 1969).

The following graph shows the results obtained during five days of growth in an activity carried out according to the instructions in the student text.



School Mathematics Study Group (SMSG), Mathematics for Living Things, Teachers' Commentary (Stanford, Calif.: Leland Stanford Junior University, 1965), p. 113.

metabolism, regulation, food supply, and relationships of the organisms with each other and their environment.

Materials, Recording and Procedure

In addition to the details and reference made under Activity F of the Student Guide of this section the following pages should be of assistance to the teacher: Pages 100-101 and pages 112-120 of Mathematics for Living Things, Teacher's Commentary, rev. ed. School Mathematics Study Group (Stanford, Calif.: Leland Stanford Junior University, 1965). The majority of these pages were duplicated for teacher reference and use in the original unit.

Upon completion of the experiment on growing mold, a further study of the biological process of growth is encouraged by plotting and studying the growth curves of other living organisms such as gourd fruit, bacteria, chickens, corn plants, the liver and brain of human boys and girls as well as the population growth curves of the U.S. and Canada. The study and analyses of these growth curves contributes not only to an understanding of the biological phenomena of growth but also provides an opportunity to discuss and study the problems of population explosion, birth control and pollution.

CANADA'S GROWTH CURVE

The final activity on plotting growth curves is not only an excellent summary but a superb opportunity to introduce such terms as "population explosion," birth control, pollution, population control, etc. Population statistics for Canada and the world are listed. Population figures for Canada were not officially taken until 1851 and 1969 is the most recent. The growth curves that result from the data are excellent for comparing with the standard growth curve on page 162.

OFFICIAL POPULATION STATISTICS FOR:

A. Canada

Year	Population
1851	2,436,000
1861	3,230,000
1870	3,625,000
1880	4,225,000
1890	4,779,000
1900	5,301,000
1910	6,998,000
1920	8,556,000
1930	10,208,000
1940	11,381,000
1950	13,712,000
1960	17,870,000
1969	21,066,000
1980	?
1990	?
2000	?

B. World

Year	Population (Millions)
1650	545
1750	728
1800	906
1850	1,171
1900	1,608
1930	2,070
1940	2,295
1950	2,517
1960	3,005
1967	3,402
1980	?
1990	?
2000	?

THE REAL NUMBER FIELD

Hello, Sports Fans!

You have all heard of the soccer field and the football field. Have you heard of the real number field? It might be called a mathematical game--so let's compare it to a game of, say, soccer!

GAME	SOCCER	REAL NUMBER FIELD
EQUIPMENT	boots, uniform, ball, goal posts, etc.	2, -4, 1/3, 3 5/8, $\sqrt{11}$, etc. (All the real numbers - D)
RULES	Point is scored by kicking ball into net. Ball cannot be passed with hands during play, etc.	commutative, associative, closure, distributive, inverse, identity, etc.
PLAYS	Centre passes to left forward. Right half back passes to centre, etc.	adding, multiplying, subtracting, dividing, finding square root, etc.

Do you recognize the "equipment" for the real number field? You should from many of the previous activities.

You should be quite familiar with most of the "plays" also.

What about the rules? These are very important. Sometimes it is easier to learn these by applying them to different situations. Let's try.

ACTIVITY A

SOME MATHEMATICAL "RULES" OR PROPERTIES*

I. If two things can be done in any order with the same results, then the operation is commutative. e.g., $a + b = b + a$

For example, if Irving puts on his left sock first and then puts on his right sock, the result is that both socks are on. If Irving puts on his right sock first and then his left sock, the result is the same. (In other words, it does not matter which order he puts his socks on.) Therefore, the operation of putting on the socks illustrates which rule or property?

____

Which of the following show this property?

(a) to go through a doorway and then open the door.

(b) to put on a coat and hat.

(c) to put on your shoes and socks.

(d) to put on a swimming suit and to jump in the pool.

(e) to add 7 to 19.

(f) to cook your dinner and to eat it.

(g) to multiply 4 by 13.

(h) to wash your face and to put on make-up.

Make up a couple of examples of your own and try them out on your friends.

II. If three things are done in a particular order, the result is the same no matter which two are done first. e.g. $(a + b) + c = a + (b + c)$

Which "rule" or property is this? ______________________

Does this rule hold for:

(a) the operation of putting on your boots and your hat and then your coat?

(b) the operation of mixing three different colors of paint to produce a new color?

---

*Modified after Properties of Operations with Numbers UNIT 2, Experiences in Mathematical Discovery Series (Washington, D.C.: (NCTM) National Council of Teachers of Mathematics, 1966).

(c) the operation of making a right turn followed by a left turn and then a right turn?

(d) the operation of adding 11, 17, and 31?

(e) the operation of multiplying 2, 4, and 6?

(f) the problem of $(16 \div 8 \div 4)$?

(g) the problem of $16 - 8 - 4$?

Can you make up a couple of examples of your own?

III. Which of the "rules" or properties combines two different kinds of plays in a particular order?

Maybe this example will help:

3 boys and 2 girls are going to the zoo.
The bus fare is 10 cents each.
How much will it cost for the five children to go to the zoo on the bus?

Two ways to solve:

1. It will cost 3×10, or 30 cents for the boys.
   It will cost 2×10, or 20 cents for the girls.
   It will cost $30 + 20$, or 50 cents for the girls and the boys.

$$(3 \times 10) + (2 \times 10) = 50$$

2. The number of children is $3 + 2$, or 5.
   It will cost 5×10, or 50 cents to take the boys and the girls.

$$(3 + 2) \times 10 = 50$$

   Therefore $(3 \times 10) + (2 \times 10) = (3 + 2) \times 10$

Which method is the easiest? ________________

Which "rule" or property does this illustrate? ____________

Use this property to determine which of the following is true.

(a) $2 \times (5 + 6) = (2 \times 5) + (2 \times 6)$

(b) $4 + (5 \times 3) = (4 + 5) \times (4 + 3)$

(c) $a \cdot (b + c) = (a + b) + (b + c)$

(d) $12 \times (10 - 9) = (12 \times 10) - (12 \times 9)$

(e) $4a \times 3a = (4 + 3)a$

Try out a couple of examples or a problem of your own on your partner.

IV. Which property states that an operation on any two members of a set results in a member of the same set?

Suppose we have a set A of integers from 0 to 500. 300 and 410 are members of this set. 300 + 410 = 710. Is this a member of the set A?

Which property does not hold for set A for the operation of addition?

Does this property hold for the following operations?

(a) the set of even numbers and addition.

(b) the set of even numbers and multiplication.

(c) the set of odd numbers and addition.

(d) the set of paints, red, yellow, and blue, and the operation of mixing the paints.

(e) the numbers 0 and 1 and addition.

(f) the numbers 0 and 1 and multiplication.

V. Doing something and then "undoing" it is an example of the

________________________ "rule" or property.

For example, the <u>inverse</u> of putting your hat on is taking it off.

1. Complete the following:

ACTIVITY	INVERSE
e.g. sitting down	standing up
e.g. going to sleep	waking up
(a) putting on shoes	
(b) making a pencil mark	

ACTIVITY	INVERSE
(c)	subtracting 3
(d) walking 3 blocks south	
(e)	dividing a number
(f)	walking down 5 flights of stairs
(g) squaring a number	
(h) subtracting 17	
(i) multiplying by 8	
(j) a right turn	

2. Which of the following have no inverses?
   (a) reading a book.
   (b) turning on a light.
   (c) putting on a coat.
   (d) scrambling eggs.
   (e) cutting the grass.
   (f) driving from Calgary to Edmonton.

VI. The ________________________ is a certain element of the set which remains unchanged when combined with any other member of the set.

1. For ordinary addition what is the element? Answer this by completing the statements below.
   (a) $5 + \_\_\_\_\_\_ = 5$
   (b) $\sqrt{6} + \_\_\_\_\_\_ = \sqrt{6}$
   (c) $\_\_\_\_\_\_ + 100 = 100$
   (d) $x + \_\_\_\_\_\_ = x$

2. What is the element that fits the property for ordinary multiplication? _______________ Make up a couple of examples.

(a)

(b)

(c)

ACTIVITY B

FINITE MATHEMATICAL SYSTEMS

Experiment 1*

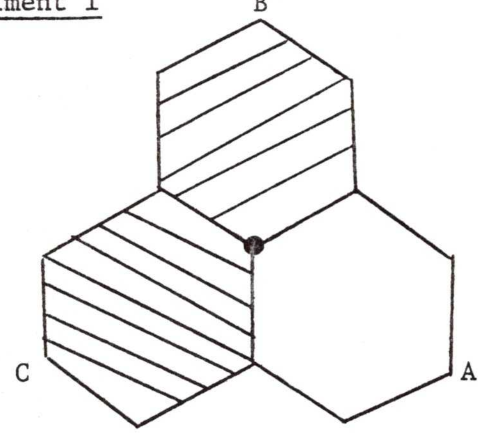


Figure X

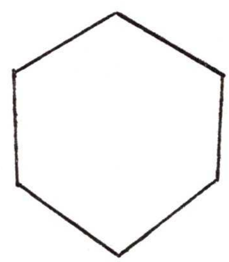


Figure Y

Trace Figure Y on a piece of paper and cut it out. This is your hexagonal counter.

Instruction A means: leave your counter where it is.

Instruction B means: rotate your counter about the black dot through 1/3 turn anti-clockwise.

Instruction C means: rotate your counter about the black dot through 1/3 turn clockwise.

Starting each time with your counter on the white hexagon in Figure X, find out where your counter would land after each of the following pairs of instruction:

---

*Modified after The School Mathematics Project (SMP), Book D (Cambridge, Mass.: Cambridge University Press, 1970), pp. 1-2.

(a) B followed by C.

(b) C followed by B.

(c) A followed by C.

After B followed by C your counter lands on the white hexagon. This result could be obtained more simply by using instruction A.

Then we could say instead:

B followed by C gives A

Can you replace the other two pairs of instructions by a single instruction? (Always start from the white hexagon.)

See if you can complete the table:

"Followed by"		Second Instruction A	B	C
First	A			C
Instruc-	B			A
tion	C		A	

Now, let's see if the set {A,B,C} and the operation "followed by" have the properties of some of the mathematical systems you have studied.

CLOSURE

In Experiment 1, the members of the set {A,B,C} can always be combined by the operation "followed by" to give a result which belongs to this set. There are no results in the table which do not belong to the set {A,B,C}. Therefore the set {A,B,C} is "closed" under the operation "followed by."

COMMUTATIVE

(i) A followed by B =

(ii) B followed by A =

(iii) Does A followed by B = B followed by A?

(iv) Does A followed by C = C followed by A?

(v) Does B followed by C = C followed by B?

Therefore the set {A,B,C} is commutative under the operation "followed by."

ASSOCIATIVE

Step (a) C followed by B = A

Step (b) A followed by C = C

Does C followed by B followed by C = C?

Using parentheses we can write this expression as:

(C followed by B) followed by C = C

This means we did step (a) first.

What would happen if we did step (b) first?

e.g. C followed by (B followed by C) = ____________

B followed by C = A

C followed by A = C

Note we get the same result no matter which way we group them?

What is the result of:

(i) A followed by (B followed by C)

(ii) (A followed by B) followed by C

If we always get the same result no matter which way we group any three members in the same order of set {A,B,C} then the set {A,B,C} is <u>associative</u> when "followed by."

IDENTITY

Look again at the operation table from Experiment 1:

"Followed by"		Second Instruction A	B	C
First	A	A	B	C
Instruc-	B	B	C	A
tion	C	C	A	B

You can see that:

A followed by A gives A.
A followed by B gives B.
A followed by C gives C.
B followed by A gives B.
C followed by A gives C.

Notes that any member of the set remains unchanged when combined with A.

Therefore we say that A is the identity of the set.

What is the identity when rational numbers are:

(i) added? ____________________

(ii) multiplied? ____________________

INVERSES

Look again at the operation table.

The identity is A and three pairs of instructions combine to give the identity.

A followed by A gives A.
B followed by C gives A.
C followed by B gives A.

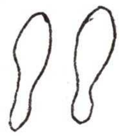
When two members of a set can be combined in either order to give the identity, they are called inverses of each other.

Note that every member of the set has an inverse.

Each of the five properties we have discussed holds for the set {A,B,C} and the operation "followed by."

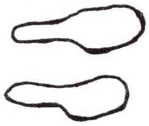
Now let's see if you can determine whether these five properties hold for the following system.

Experiment 2: "Command Arithmetic"*



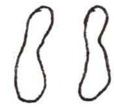
FRONT FACE

(F)



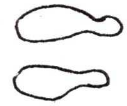
RIGHT FACE

(R)



ABOUT FACE

(A)



LEFT FACE

(L)

To obey each of these commands, start with your feet pointed toward the front of the room.

FRONT FACE (F): Do not move.

RIGHT FACE (R): Make a quarter (90°) turn to the right so that your feet face in the direction shown.

ABOUT FACE (A): Make a half (180°) turn so that you point in the opposite direction.

LEFT FACE (L): Make a quarter (90°) turn to the left.

Take turns giving and obeying some of these commands with another student.

The operation in the system of drill commands is "followed by."
Use $\oplus$ to represent "followed by."
Example: What single command is equivalent to $R \oplus A$?

Solution: The command $R \oplus A$ means Right Face followed by About Face. Starting with the feet pointed to the front, you should end up in the same position as the single command Left Face (L).

Therefore: $R \oplus A = L$.

What single command is the same as:

$A \oplus L =$

$L \oplus R =$

$R \oplus F =$

$A \oplus A =$

*Modified after Ray W. Cleveland, unpublished manuscript, 1969.

Complete the table:

	⊕	F	R	A	L
		Second Command			
First	F				
Com-	R	R		L	
mand	A			F	R
	L		F		

(Check your table with the correct answer before proceeding.)

TESTING FOR MATHEMATICAL PROPERTIES:

CLOSURE

Is every element in the table a member of the set {F,R,A,L}?

Does the closure property hold for Experiment 2?

COMMUTATIVE

Test the operation ⊕ for commutivity by checking to see if the expressions below result in the same position.

(a) F ⊕ L and L ⊕ F.

(b) A ⊕ R and R ⊕ A.

(c) R ⊕ L and L ⊕ R.

(d) F ⊕ A and A ⊕ F.

Do you think ⊕ for the set of commands is commutative?

ASSOCIATIVE

Test the operation ⊕ for associativity by checking to see if the expressions below result in the same position.

(a) (A ⊕ R) ⊕ R and A ⊕ (R ⊕ R).

(b) (A ⊕ L) ⊕ F and A ⊕ (L ⊕ F).

(c) (R $\oplus$ L) $\oplus$ R and R $\oplus$ (L $\oplus$ R).

(d) (F $\oplus$ R) $\oplus$ L and F $\oplus$ (R $\oplus$ L).

Does the operation $\oplus$ for set of commands appear to be associative?

IDENTITY

Examine the operation table.

Is there a command which does not change the position occupied?

Which command is it?

INVERSE

Look again at the operation table.

The identity is F.

Is there a command which will make each of the following statements true?

(a) A $\oplus$ ______ = F.

(b) F $\oplus$ ______ = F.

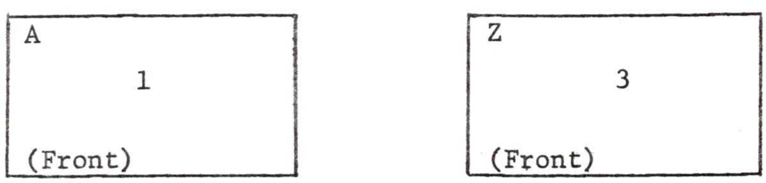
(c) L $\oplus$ ______ = F.

(d) R $\oplus$ ______ = F.

Is there an inverse for each element of the set {F,L,A,R} under $\oplus$?

Experiment 3*

Cut out two rectangles and mark one A and the other Z. Mark them with numbers front and back as shown below.



---

*Modified after The School Mathematics Project (SMP), Book D (Cambridge, Mass.: Cambridge University Press, 1970), pp. 12-13.

A	
	2
(Back)	

Z	
	4
(Back)	

Place them in the position below to start each move.

A	
	1
(Front)	

Z	
	3
(Front)	

Move M: leave the cards as they are.

Move N: turn both cards over.

Move O: turn card A over only.

Move P: turn card Z over only.

Move M gives the number 13.

Move N gives the number 24.

What numbers do you get with moves O and P?

* means the operation "followed by."

O * P gives the number 24.
The number 24 is equal to Move N.
Therefore O * P = N.

What number is given by M * N? O * M?

Try some other combinations:

*	M	N	O	P
M		N		
N				
O				N
P		O		

Can you complete the table above?

Using the above table, test the set {M,N,O,P} and the operation *, for the following properties:

(a) Closure.

(b) Commutativity.

(c) Associativity.

(d) Identity Element.

(e) Inverse for each Element.

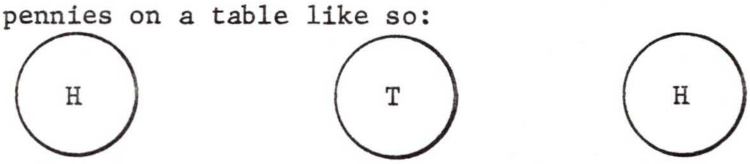
Can you think of some different moves and operations which you can make an operation table for and test for the above properties?

What about one card only and some different moves?

What about different shaped cards?

<u>Experiment 4</u>*

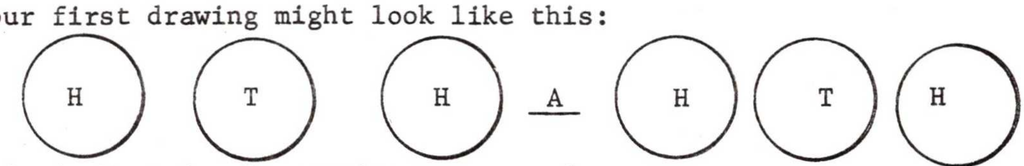
Place three pennies on a table like so:



You are allowed to turn any two pennies over at one time. See if you can do this so that all three pennies are heads up.

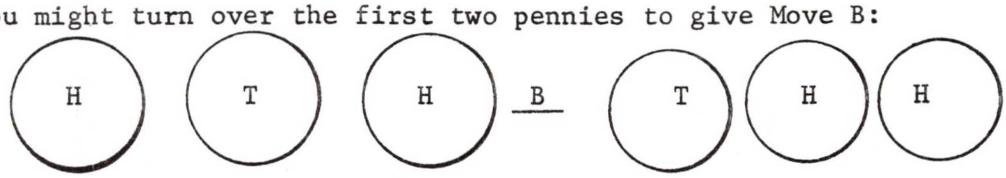
Make a drawing of each of the possible situations. Remember-- turn over only two pennies at a time.

Your first drawing might look like this:



This is Move A. No pennies were moved.

You might turn over the first two pennies to give Move B:



---

*Modified after The School Mathematics Project (SMP), Book D (Cambridge, Mass.: Cambridge University Press, 1970), pp. 15-16.

You should find two more moves. Call them C and D. Remember to start from A each time. Make diagrams of moves C and D.

Complete the table below for your moves under the operation followed by, ⓧ .

	ⓧ	Second Move A	B	C	D
First Move	A		B		
	B				
	C				
	D				

Determine whether:

(a) your table is closed.

(b) your set of moves has an identity.

(c) your table is commutative.

(d) your table is associative.

(e) each member of {A,B,C,D} has an inverse.

Can you think of some different moves you might make? What would happen if you used only two coins? four coins?

Make an operation table and see if the properties hold for this new set and operation.

Experiment 5*

Pick one each of the white, red, green, and purple colored rectangular rods plus a black rod (a one-unit square piece of black paper) out of the sack of colored rods.

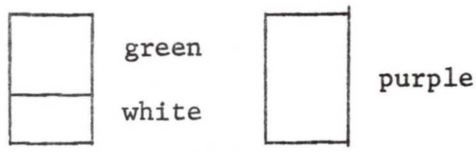
---

*Modified after W. J. Oosse, "Properties of Operations: a meaningful study," The Arithmetic Teacher, Vol. 16, No. 4 (Washington, D.C.: National Council of Teachers of Mathematics (NCTM), April, 1969), pp. 271-275.

Let ⊖ represent the operation on this set.

⊖ shall mean to take any two elements (rods) of the set and stand them on their square end beside each other. The answer will be the rod which when placed upon the shorter of the two rods makes the heights as nearly equal as possible.

For example, white ⊖ purple = green, because green is the rod which, when placed on the shorter white rod, comes closest to making the two heights nearly equal.



Red ⊖ red = black (since black is the only rod which comes closest to making the heights nearly equal).

Try some other examples.

⊖	black	white	red	green	purple
black					
white					green
red			black		
green		red			
purple					

Can you complete the table above?

Which of the following properties hold for the set and the operation ⊖ ?

(a) Closure.

(b) Commutativity.

(c) Associativity.

(d) Identity Element.

(e) Inverse for each Element.

Perhaps you might like to see what kind of an operation table you would get using more colored rods or maybe the first three only. See if the table obeys the properties listed above.

ACTIVITY C

NUMBER SYSTEMS

Look at Table D (on following page).

Check each square under N, Ra, and R where the particular number property holds. Where the property does not hold, state an example.

Now do the same for the real numbers D, but give an example for each property.

e.g. Addition

Closure property, yes, e.g. $17 + -15.2 \in D$

We will accept the real number properties without proof. You may try to prove some of them if you wish.

See how the different number systems compare now.

Which number system do you think is the least important? Why?

Which is the most important? Why?

ACTIVITY D

PROPERTY BEE

Examine the list of properties that follows. You will use these to play the property bee. Your teacher might suggest some abbreviations you can use to make the names simpler.

TABLE D

Operations	Addition				Multiplication			
Sets	N	Ra	R	D	N	Ra	R	D
Closure Properties								
Commutative Properties								
Associative Properties								
Distributive Properties								
Identity Element Properties								
Inverse Properties								

PROPERTY LIST

Name of Property*	Abbreviation Suggested
Commutative Property of Addition	
Commutative Property of Multiplication	
Associative Property of Addition	
Associative Property of Multiplication	
Distributive Property	
Additive Inverse Property	
Multiplicative Inverse Property (same as the Reciprocal Property)	
Identity Element Property of Addition	
Identity Element Property of Multiplication	
Well Defined Property of Addition	
Well Defined Property of Multiplication	
Definition of the Sum of Two Rational Numbers	
Definition of the Product of Two Rational Numbers	
Definition of Equivalent Ordered Pairs	
Difference Property	
Quotient Property	

The purpose of the property bee is to help you learn the mathematical properties which help you solve conditions and problems. Do you know what each property above means? Can you give an example for each property?

Can you name the properties used in solving the following condition of equality?

---

*Van Engen _et_ _al_., _Seeing Through Mathematics_, special edition, Book 2 (Toronto: W. S. Gage Limited, 1964).

$w - 5 = -13.2$	Given
$w + -5 = -13.2$	______
$(w + -5) + 5 = -13.2 + 5$	______
$w + (-5 + 5) = -13.2 + 5$	______
$w + 0 = -13.2 + 5$	______
$w = -13.2 + 5$	______
$w = -8.2$	______

ACTIVITY E and ACTIVITY F

Your teacher will instruct you about these activities.

ACTIVITY G*

1. To solve a problem we must be able to translate from the English language to the mathematical or algebraic language.

   Translate the following English phrases into mathematical language:

e.g. The sum of a number and 2. $n + 2$

e.g. A number divided by 10. $n \div 10$ or $n/10$

a) A number decreased by 3.

b) The product of 3 and a number.

c) The cost in cents of n dolls costing 40¢ each.

d) The number 54 is 3 times a certain number.

e) One-half of a number.

f) The sum of six times a number and eight is equal to thirty-seven.

g) A number divided by seven is less than sixty-nine.

---

*Modified after Topics in Mathematics, 29th Yearbook (Washington, D.C.: National Council of Teachers of Mathematics (NCTM), 1964), pp. 362-367.

h) Eleven decreased by a number is equal to negative six.

i) A number increased by nine is greater than three times the number.

j) A number divided by six and then increased by seventeen is greater than or equal to seventy-nine.

k) The sum of six times a number and five and a half is thirty-one.

l) The difference when seven is subtracted from four times a number is thirteen.

2. I thought of a number, multiplied it by five, then added 17, then subtracted the number I had first thought of and the result was 61. Translate this to math language. What was the number I thought of?

   Make up a problem similar to this one and see if one of your friends can solve it.

3. Now let's try translating mathematical language into English language:

e.g. $n + 8 = 39$ A number increased by 8 is equal to 39.

a) $1/4 \times n = 29$

b) $(2 \times n) + 9 < 26$

c) $29 \leq n \leq 45$

d) $(17 - n) \times 4.2 \neq 75$

e) $10n + 2$

f) $n - 19 = -7$

ACTIVITY H*

CONDITIONS WITH TWO UNKNOWNS ($U = D \times D$)

1. Make a graph for the solution set of $y = 3x - 2$.

2. Irv says that the ordered pair (-1, -5) lies on the graph for the condition $y = 3x - 2$. Do you agree? How can you show that Irv is right?

---

*Modified after Robert B. Davis, Explorations in Mathematics, A Text for Teachers (Palo Alto, Calif.: Addison Wesley Publishing Co., 1967), pp. 242-247.

3. Henry says that if you substitute the ordered pair (3.5, 8.5) into the condition for question 1, you will get a true statement. Do you agree with Henry? Does the ordered pair (3.5, 8.5) lie on the graph for $y = 3x - 2$?

4. Shelley says that she can write down all kinds of ordered pairs that will satisfy $y = 3x - 2$ without doing any calculating. What's her secret? See if you can write down some ordered pairs for $y = 3x - 2$.

5. Using the same graph paper from question 1, make a graph for the solution set of $y = -2x + 8$.

6. Without any hints find the solution set to this compound condition:

$$y = 3x - 2 \wedge y = -2x + 8$$

7. Mary-Jane says she found only one ordered pair in the solution set. Do you agree? What can you say about the point (2, 4)?

8. Find the solution set for:

$$a = 2b + 1 \wedge a = -2b + 5$$

9. Find the solution set for:

$$x = 3y + 1 \wedge x = -y + 9$$

10. Have you found a general method for solving these equations. Try to describe it.

11. Use your method to solve this condition:

$$y = 2x + 1 \wedge y = 4x + 4$$

Try and solve the following compound conditions:

12. $y = x \wedge y = 3x - 4$

13. $y = x \wedge y = 4x + 3$

14. $y = x + 1 \wedge y = 3x - 3$

15. Casey has discovered another method for solving compound conditions. How many methods have you found?

16. Casey used one of her methods to solve the following condition:

$$2r + s + 3t = 15 \wedge r = 4 - t \wedge t + 2 = 5\ 1/2$$

See if you can tabulate the solution set of this condition using one of your methods.

17. Have you found a method that works? Try it on this one:

$$-18 = 3s \wedge r + 2s = t \wedge r - 5 = 7$$

18. Look back at your solution for question 8. See if that will help you in solving this condition:

$$a > 2b + 1 \wedge a = -2b + 5$$

19. What's the best way of showing the solution set for $y > 3x + 3$? Show your solution.

20. How would you solve this condition:

$$x = 1/2\ y \vee y \leqq 0$$

Show your solution.

TEACHER'S GUIDE

THE REAL NUMBER FIELD

ACTIVITY A

The main student objective of this activity is to introduce the students to some non-numerical examples of the field properties and to give them an opportunity to make up some examples of their own. Seeing these properties in a different context should make them more meaningful.

Essentially this activity is a review and you may choose to do only parts of it or make it optional for your brighter students. Try not to spend more than a day on it and make the most of some of the examples created by the students.

ACTIVITY B

The purpose of this activity is to introduce the students to some non-numerical mathematical systems that exhibit some of the same properties as the real numbers. By working with tangible objects and discovering some of the properties these ideas should become more real to the student. It should be an enjoyable and motivating activity.

Sometimes the best way to grasp an idea is to examine different manifestations of it in a different context. In this case the idea is mathematical properties in the system and the main emphasis should be on these rather than any computational skills, techniques, etc.

The experiments should be done by pupils working in small groups rather than individually. Introduce them to the properties - gradually perhaps the closure and identity properties only on the first day. After the first two experiments have been completed, many students or groups should be able to work on their own.

Another teaching method that might be useful is to ignore properties the first day and simply experiment with different finite mathematical systems and their operation tables. The period might be climaxed with a challenge to the students to make up their own set of elements and an operation. Suggestions might include rotating rectangles, triangles, or other geometric figures, binary numbers, etc. Subsequently the students can test these systems for the mathematical properties.

Caution - these systems should be closed or none of the other properties can ever be tested. Different groups of students might get involved testing another group's "creation" for the different properties.

Experiment 1

Since this is the first effort at completing an operations table, some of your weaker students might have difficultyin combining instructions. Remember it is always possible to find a single instruction which has the same effect as any pair of instructions.

Note: In all cases when testing an operation table for the different properties a quick way to determine commutativity is to check for symmetry about the main diagonal. Testing completely for associativity can be very tedious. The simplest and best approach here is to try a couple of examples and assume associativity holds unless you find a counter example.

The overhead projector might suit your purpose for demonstrating the moves and combinations possible via a transparency. You might prefer the students to cut out both Figures X and Y at the beginning of the experiment. Coloring in the shaded squares might be helpful. If time is short, you could duplicate Figure Y for the students.

Experiment 2

Students should be able to complete the operation table with very little difficulty. Have each group pick a student to execute the different commands to aid in completing the table.

As a challenge to aid students in making up their own systems, you might suggest such moves as 1/3 turn left, 1/3 turn right, etc. Students might make up systems with a different number of these moves, but encourage them to make up a system that is closed.

Experiment 3

As an alternative to making two cards, two pennies and the following instructions will give the same operation table:

M leave the pennies as they are

N turn both pennies over

O turn the left-hand penny

P turn the right-hand penny

Experiment 4

It is impossible to arrange all three pennies with heads up using the allowed moves. This should be obvious from the completed operation table which is closed for the four moves.

Be sure the students understand the distinction between the different moves. (Check the answer key.)

Experiment 5

The elements for this experiment could be cut out of colored paper and played in two dimensions if more convenient. In this case, the black unit squares (the identity) would be just a thin strip one unit long.

Note that in this case the associative property does not hold. This is important since this is their first encounter with a system that doesn't hold for all the properties. This may surprise some of your students who might think these properties are so obvious because the system is closed.

At this stage you may prefer to introduce some other examples of your own. Don't neglect to introduce closed systems that do not hold true for all the properties. Your better students might enjoy the challenge of creating their own mathematical systems. An excellent suggestion for this based on switches in various electric circuits is given in the NCTM, 27th Yearbook.*

ACTIVITY C

The purpose of this activity is to see how the real number system compares to the other main mathematical systems the students have studied. As suggested, let the students complete the table for N, Ra, and R on their own producing examples only where they think the properties don't hold. In making examples for the properties of D, suggest using some irrational numbers. This activity could be done in groups with students comparing their results.

Students should agree that N is the least important and D the most important if for no other reason than D encompasses all the other number systems.

---

*"The Anatomy of a Mathematical System," Enrichment Mathematics for the Grades, 27th Yearbook (Washington, D.C.: National Council of Teachers of Mathematics (NCTM), 1963), pp. 273-281.

ACTIVITY D

PROPERTY BEE

The purpose of this activity is to make the learning of algebraic properties used in solving conditions a more enjoyable and motivating exercise. Emphasize that the purpose of learning these properties is to help them solve algebraic conditions.

The property bee or game is probably best played by two competing teams. You may divide the class into two teams or, say, eight teams and have a play-off similar to a curling or badminton draw (double knock-out) or any other competition you may prefer. Each member of each team must take his turn.

The game can be played using chains of conditions displayed on the overhead, with the two teams alternating in naming the property. To determine a winner, a point may be given for each correct answer and a point subtracted for each incorrect answer.

A review of some sample conditions and the properties used would be wise before beginning the competition. Give them the following list of properties to review.

To save time the following abbreviations for the properties might be useful. You may prefer some different abbreviations.

Name of Property	Abbreviation Suggested
Commutative Property of Addition	CA
" " Multiplication	CM
Associative Property of Addition	AA
" " Multiplication	AM
Distributive Property	DT
Additive Inverse Property	AIV
Multiplicative Inverse Property*	MIV
Identity Element Property of Addition	IEA
" " " Multiplication	IEM
Well Defined Property of Addition	WDA
" " " Multiplication	WDM
Definition of the Sum of Two Rational Numbers	D Sum
" Product of Two Rational Numbers	D Prod.
Definition of Equivalent Ordered Pairs	EOP
Difference Property	DP
Quotient Property	QP
Sum Property of "Less Than"	Sum <
Positive Multiplier Property of "Less Than"	PM <
Negative Multiplier Property of "Less Than"	NM <

*Same as the Reciprocal Property

It might be wise for the students to practice a few examples before playing the Property Bee. Perhaps a ten to fifteen minute trial game for two or three days might prepare the students and inform them where their weaknesses lie. Students should be allowed to keep the list of properties in front of them when they are playing the game as they should not be expected to memorize all these properties by name.

Please note that an extra condition appears to have been inserted in the chain in comparison to the textbook examples. This extra condition is to distinguish the definition of sum and definition of multiplication properties from the other properties in order that only one property can be named for each new condition in the chain. This saves possible confusion that might arise if two or more properties could be named to justify a condition in the chain.

If you wish to make any variations to the "bee," please feel free to do so.

Following are several chains of conditions of equality which should not present any major difficulty considering the students have dealt with similar types in previous units with the exception of the variation of the Distributive Property.*

I	a)	$9/16 = w + 5/8$	Given
	b)	$9/16 + -5/8 = (x + 5/8) + -5/8$	WDA
	c)	$1/16 = (w + 5/8) + -5/8$	D Sum
	d)	$1/16 = w + (5/8 + -5/8)$	AA
	e)	$1/16 = w + 0$	AIV
	f)	$1/16 = w$	IEA
II	a)	$-7 + t = 4\ 1/5$	Given
	b)	$7 + (-7 + t) = 7 + 4\ 1/5$	WDA
	c)	$7 + (-7 + t) = 11\ 1/5$	D Sum
	d)	$(7 + -7) + t = 11\ 1/5$	AA
	e)	$0 + t = 11\ 1/5$	AIV
	f)	$t = 11\ 1/5$	IEA

---

*van Engen et al., Seeing Through Mathematics, special edition, Book 2, Teacher's Guide (Toronto: W. J. Gage Limited, 1965).

III	a)	14.7 = 3.9 - 17	Given
	b)	14.7 = 3.9 + -17	DP
	c)	-3.9 + 14.7 = -3.9 + (3.9 + -n)	WDA
	d)	10.8 = -3.9 + (3.9 + -n)	D Sum
	e)	10.8 = (-3.9 + 3.9) + -n	AA
	f)	10.8 = 0 + -n	AIV
	g)	10.8 = -n	IEA
	h)	-10.8 = n	Def'n of Additive Inverse (Might leave this out of competition)
IV	a)	5/7 a = 20	Given
	b)	7/5 (5/7 a) = 7/5 × 20	WDM
	c)	7/5 (5/7 a) = 28	D Prod.
	d)	(7/5 × 5/7)a = 28	AM
	e)	1 a = 28	MIV
	f)	a = 28	IEM
V	a)	57.34 = -6.1 Z	Given
	b)	1/-6.1 × 57.34 = 1/-6.1 (-6.1 Z)	WDM
	c)	-9.4 = 1/-6.1 (-6.1 Z)	D Prod.
	d)	-9.4 = (1/-6.1 × -6.1)Z	AM
	e)	-9.4 = 1 Z	MIV
	f)	-9.4 = Z	IEM
VI	a)	5x + 7 = -13	Given
	b)	(5x + 7) + -7 = -13 + -7	WDA
	c)	(5x + 7) + -7 = -20	D Sum

	d)	$5x + (7 + -7) = -20$	AA
	e)	$5x + 0 = -20$	AIV
	f)	$5x = -20$	IEA
	g)	$1/5\ (5x) = 1/5 \times -20$	WDM
	h)	$1/5 \times (5x) = -4$	D Prod.
	i)	$(1/5 \times 5)x = -4$	AM
	j)	$1\ x = -4$	MIV
	k)	$x = -4$	IEM
VII	a)	$8/(x - .3) \sim 28/7$	Given
	b)	$8 \times 7 = (x - .3)28$	EOP
	c)	$56 = (x - .3)28$	D Prod.
	d)	$56 = 28x - 8.4$	DT
	e)	$56 = 28x + -8.4$	DP
	f)	$56 + 8.4 = (28x + -8.4) + 8.4$	WDA
	g)	$64.4 = (28x + -8.4) + 8.4$	D Sum
	h)	$64.4 = 28x + (-8.4 + 8.4)$	AA
	i)	$64.4 = 28x + 0$	AIV
	j)	$64.4 = 28x$	IEA
	k)	$1/28 \times 64.4 = 1/28(28x)$	WDM
	l)	$2.3 = 1/28(28x)$	D Prod.
	m)	$2.3 = (1/28 \times 28)x$	AM
	n)	$2.3 = 1\ x$	MIV
	o)	$2.3 = x$	IEM

VIII	a)	$4(x - 3) = 2x$	Given
	b)	$4(x + -3) = 2x$	DP
	c)	$4x + -12 = 2x$	DT
	d)	$-2x + (4x + -12) = -2x + 2x$	WDA
	d)	$(-2x + 4x) + -12 = -2x + 2x$	AA
	f)	$(-2 + 4)x + -12 = -2x + 2x$	DT
	g)	$2x + -12 = -2x + 2x$	D Sum
	h)	$2x + -12 = 0$	AIV
	i)	$(2x + -12) + 12 = 0 + 12$	WDA
	j)	$2x + (-12 + 12) = 0 + 12$	AA
	k)	$2x + 0 = 0 + 12$	AIV
	l)	$2x = 12$	IEA
	m)	$1/2(2x) = 1/2 \times 12$	WDM
	n)	$(1/2 \times 2)x = 1/2 \times 12$	AM
	o)	$1 x = 1/2 \times 12$	MIV
	p)	$x = 1/2 \times 12$	IEA
	q)	$x = 6$	D Prod.
IX	a)	$9x + 47x = -224$	Given
	b)	$(9 + 47)x = -224$	DT
	c)	$56x = -224$	D Sum
	d)	$1/56(56x) = 1/56 \times -224$	WDM
	e)	$(1/56 \times 56)x = (1/56 \times -224)$	AM
	f)	$1 x = (1/56 \times -224)$	MIV
	g)	$x = (1/56 \times -224)$	IEM
	h)	$x = -4$	D Prod.

X	a)	$4w = 3(w + 8)$	Given
	b)	$4w = 3w + 24$	DT
	c)	$-3w + 4w = -3w + (3w + 24)$	WDA
	d)	$-3w + 4w = (-3w + 3w) + 24$	AA
	e)	$-3w + 4w = 0 + 24$	AIV
	f)	$-3w + 4w = 24$	IEA
	g)	$(-3 + 4)w = 24$	DT
	h)	$1\ w = 24$	D Sum
	i)	$w = 24$	IEM

If the game is going well, and students are learning, then a few more of conditions of equality might be introduced at this stage. When you think the students are ready to try chains of inequalities and learn the three "less than" properties you might introduce a condition like the following in the middle of a property bee.

XI	a)	$8 + x < 3.8$	Given
	*b)	$-8 + (8 + x) < -8 + 3.8$	______ (Sum <)
	c)	$(-8 + 8) + x < -8 + 3.8$	AA
	d)	$0 + x < -8 + 3.8$	IVA
	e)	$x < -8 + 3.8$	IEA
	f)	$x < 3$	D Sum

*The property bee should become stalled here because students will be stymied by the name of this property which they have not yet learned (maybe?). <u>DO NOT TELL THEM YET</u>. Let them guess a few names and then halt the game. (Note the score if you wish to continue after the development of the following properties.)

At this stage where the property bee has been "torpedoed" by the insertion of an unknown property the students will want to learn about the property to continue the game. The following discovery activity should fill this gap in their background and permit them to continue the game. You may introduce all three "less than" properties at this point or develop a need for the last two in a similar manner to the first.

A. The Sum Property of Less Than:

1. Tabulate the solution set for the condition $x - 2 < 3$. $U = I$.

2. Write a standard description for the solution set of $x - 2 < 3$. $U = I$.

3. Mary Lou says that $x - 2 < 3$ is equivalent to $x < 5$. Do you agree?

4. Tabulate the solution set for $x + 3 < 8$. $U = I$.

5. Al says $x + 3 < 8$ is equivalent to $x - 2 < 3$. Do you agree?

6. Marlene figured out that $x + 13 < 18$ is equivalent to $x < 5$. How did she do it?

7. Jenny obtained the condition $x - 4 < 1$ from $x < 5$ and said they were equivalent. Make up three more conditions that are equivalent to $x < 5$. Compare your conditions with some of your classmates.

8. Bill said $x + 12 < 16$ is equivalent to $x < 5$. Henry says he's wrong. Who's right?

The purpose of this type of approach is to get the students to discover and use the sum property of less than in finding equivalent conditions. Hopefully the students will come up with a method equivalent to the sum property of less than. Call it, say, Marlene's rule, if you wish to introduce the formal name when everyone agrees on this method of finding equivalent conditions and add it to their property list for the property bee.

After this stage or before if you prefer you might ask the students to find which of the following conditions are equivalent to $x > -7$ ($U = D$) [Ans.: (a) (b) (f)]

(a) $x - 8 > -15$

(b) $x + 9 > 2$

(c) $x + 2 > 5$

(d) $x + 33 > 25$

(e) $x - 19 > -12$

(f) $x - 17 > -24$

Or you might ask them to make up equivalent conditions of their own.

e.g., Make up three conditions that are equivalent to $w + 9\ 1/3 > 15$ or $t - 1 < 49.3$.

A similar approach could be used to develop the remaining two less than properties.

B. <u>The Positive Multiplier Property of Less Than</u>:

1. Tabulate the solution set for the condition $3t > 18$. U = I.

2. Harry says that $3t > 18$ is equivalent to $5t > 6$. Henriette says he is wrong. Who is right?

3. Marcia says that $2t > 12$ is equivalent to $3t > 18$. Is she right? How did she do it?

4. Bob found out that $8t > 48$ is equivalent to $3t > 18$. How do you think he did this?

5. George found a method that showed him $1/2\ n > 3$ is equivalent to $3t > 18$. What's his method?

6. See if you can find a method to make three conditions equivalent to $w < 2\ 1/2$.

Following this question is probably an opportune time to ask the class for their methods and discover the property they are using. After agreement is reached on the positive multiplier property or whatever name you choose to call it for the present, the following exercises might prove beneficial.

Which of the following conditions are equivalent to $n > 4\ 1/3$? U = D. [Ans.: (b) (c) (e) (f)]

(a) $2n > 4\ 2/3$

(b) $3n > 13$

(c) $1/6\ n > 13/18$

(d) $5n < 21\ 2/3$

(e) $7n > 30\ 1/3$

(f) $9.23\ n > 40.0659$

Make up three conditions that are equivalent to $2/5\ w < 6$ or $5\ 1/2\ x > 1/3$.

C. The Negative Multiplier Property:

1. Tabulate the solution set of $-2w < 16$. $U = I$.

2. Marlene says that $-2w < 16$ is equivalent to $w < -8$. Do you agree?

3. Gary says he disagrees with Marlene and says that $w > -8$ is equivalent to $-2w < 16$. Who do you agree with?

4. Gary also comes up with the condition $4w > -32$ and says this is equivalent to $-2w < 16$. How do you think he got from $-2w < 16$ to $4w > -32$?

5. Janice got the condition $1/2\ w > -4$ from $-2w < 16$. How did she do it?

6. Can you find a method for showing $6w > -48$ is equivalent to $-2w < 16$?

7. Henry says $10w > 80$. Is he right?

8. Using your method (have you got one yet?) find three conditions equivalent to $-1/2\ c > 3$.

9. What's your method? See how it compares with your classmates' methods. What do you call it?

10. Find out which of the following conditions are equivalent to $-3t > 5$. $U = D$. [Ans.: (c) (d) (e)]

    (a) $-6t < 10$

    (b) $-9t < -15$

    (c) $t < -1\ 2/3$

    (d) $3t < -5$

    (e) $4\ 1/2\ t < -7\ 1/5$

    (f) $2t > -3\ 1/3$

11. Make up three conditions equivalent to $-2/5\ r > 4$ or $-3y < 7\ 1/4$.

After the three "less than" properties have been added to the students' lists, you may wish to resume the previous property bee game or start another one using conditions of the following type in addition to the previous kinds

e.g.	(a) $-4s - 11.3 > 38.9$	Given
	(b) $-4s + -11.3 > 38.9$	DP
	(c) $(-4s + -11.3) = 11.3 > 38.9 + 11.3$	Sum <
	(d) $-4s + (-11.3 + 11.3) > 38.9 + 11.3$	AA
	(e) $-4s + 0 > 38.9 + 11.3$	AIV
	(f) $-4s > 38.9 + 11.3$	IEA
	(g) $-4s > 50.2$	D Sum
	(h) $-1/4(-4s) < -1/4(50.2)$	NM
	(i) $(-1/4 \cdot -4)s < -1/4 \cdot 50.2$	AM
	(j) $1\ s < -1/4 \cdot 50.2$	MIV
	(k) $s < -1/4 \cdot 50.2$	IEM
	(l) $s < -12/55$	D Prod.

Another Property Bee?

(You might prefer this as a substitute or an additional version.)

Arrange your class in groups of 2 or 3, and draw up a single or double knock-out competition. When two teams begin a competition they may write up chains of conditions (involving a specified number, say 10 or 15) and exchange them. The winner is the team with the most correct properties. You might impose a time limit. No books allowed, just the list of properties. Add any other rules you think might help or ask the students for some suggestions on building the game.

ACTIVITY E

CHAIN PUZZLES

As a follow-up to the property bee the following activity should be excellent. The objective is to challenge the student to put in correct order a mixture of equivalent conditions. In essence, it is an intermediate step between identifying the properties used in solving conditions and solving conditions on their own.

For example, consider the following condition and its chain of equivalent conditions leading to the solution set.

$-1/3\ t - 11\ 2/5 > 36\ 3/5$ <u>Given</u>

$-1/3\ t + -11\ 2/5 > 36\ 3/5$

$(-1/3\ t + -11\ 2/5) + 11\ 2/5 > 36\ 3/5 + 11\ 2/5$

$-1/3\ t + (-11\ 2/5 + 11\ 2/5) > 36\ 3/5 + 11\ 2/5$

$-1/3\ t + 0 > 36\ 3/5 + 11\ 2/5$

$-1/3\ t > 36\ 3/5 + 11\ 2/5$

$-1/3\ t > 48$

$-3(-1/3\ t) < -3 \times 48$

$(-3 \cdot -1/3)t < -3 \times 48$

$1\ t < -3 \times 48$

$t < -3 \times 48$

$t < 144$

Cut out each equivalent condition separately, mix them up and place them in an envelope. The student's task is to put these conditions in the correct order starting with the given condition. In addition you might wish your students to name the properties used.

For classroom use, it would be wise to make up 20 to 40 conditions of equality and inequality with varying degrees of difficulty. Label the outside of the envelope with the given condition. You might include an answer key in the envelope to be used after completing the chain or post the answers in some convenient place.

Another use for the envelopes of conditions is to challenge the student to solve the given condition without opening the envelope. The previous activities should have provided a good background for the students solving conditions on their own and, if unsuccessful, the student could seek help from the solution in the envelope, from his classmates or the teacher. In this way, students are free to work and learn on their own and the teacher is involved only with students who need him.

ACTIVITY F

SOLVING CONDITIONS OF EQUALITY AND INEQUALITY IN ONE VARIABLE

The following are additional suggestions you might use as you see fit in developing your students' ability to solve conditions in one variable.

1. At this stage the student should be ready to try solving conditions on his own given the original condition only.

You might write a condition like the following on the board or overhead projector.

e.g. $3n + 11 = 134$

Then ask the students to solve this by developing a complete chain of conditions. You might prefer some easier conditions to start with:

e.g. $\chi - 11 = 13$ or $1/3\ y > -18$

Another approach might be to write the given condition on the board and in a space below the final solution set ask the students to complete the chain of conditions leading to the final solution set.

e.g. $.5\chi < .35$

) Students must
) complete
) the chain.

$\chi < .7$

The teacher must use his own discretion with respect to requiring the students to complete all of the chain. When convinced a student knows the properties for solving a condition satisfactorily or when doubtful as to the benefit of this procedure with a student, short-cuts in solving these conditions are quite in order.

e.g. A student might solve the following condition like so:

$-3y + 9 < 15$

$-3y < 6$

$y > -1/2$

To facilitate this type of condition-solving questions of the following type might be useful.*

(i) Is $-9y + 2y < 8$ equivalent to $-7y < 8$?

(ii) How do you obtain $y > -1\ 1/7$ from $-7y < 8$?

(iii) How can you obtain $2/7\ t = 9$ from $2/7\ t - 17\ 1/4 = -8\ 1/4$?

(iv) How do you get $t = 31\ 1/2$ from $2/7\ t = 9$?

The following are examples of conditions you could assign the students to use with any of the preceding methods or as

---

*Modified after Robert B. Davis, Discovery in Mathematics, A Text for Teachers (Reading, Mass.: Addison Wesley Publishing Co., 1964), pp. 139-144.

you see fit. $U = D$

a) $x + 3\ 1/2 > 4\ 1/3$

b) $-5y < -14.2$

c) $13\ 1/2 - t < 14\ 1/3$

d) $13 = \sqrt{49} + s$

e) $w - 23 > 18\ 1/2$

f) $3/8\ r = -\sqrt{4}$

g) $2.2\ n = -11$

h) $5c - 17.3 = 14$

i) $5c - 17.3 > 14$

j) $-5c - 17.3 > 14$

k) $3r - 5r > -2$

l) $17\ 1/4 - 3/5\ z < 72$

m) $6(x - 3) = 3x$

n) $1/3(s - 2/5) = 7/10$

o) $(15r - .2) = 21r$

p) $14.7 = 3.8 - n$

q) $5/(n - .7) \sim 45/9$

r) $\sim (3k + 9) < 5$

You might wish the students to graph the solution set of some of these on the real number line.

2. You might prefer a more open-ended approach to solving conditions. For example you might write $2t + 5 < 15$ followed by the solution $t < 5$. Then ask the students to show as many different methods as they can of finding the solution. Some methods are trial and error, using chains of conditions, changing the inequality to an equality and solving and then substituting the inequality back again and using an algebraic formula. Compare the different methods and ask the student which he thinks would be the best method for solving all types of conditions accurately.

3. Compound conditions in one variable:*

The following discovery approach illustrates one method of

---

*Modified after Robert B. Davis, Discovery in Mathematics, A Text for Teachers (Reading, Mass.: Addison Wesley Publishing Co., 1964), pp. 242-247.

introducing these types of conditions.

(i) Ask the students to make a graph of the solution set of $y + 7\ 1/2 > 4\ 1/4$ on the real number line.

(ii) Graph the solution set of $-\ 2/3\ y > -4$ on the same number line.

(iii) Now graph the solution set of $y + 7\ 1/2 > 4\ 1/4 \wedge -\ 2/3\ y > -4$.

Since U = D, the solution set of a simple condition of inequality is infinite and this rules out tabulating the solution set. The best pictorial way of displaying the solution set for conditions of inequality in one variable is the number line.

Ask the students to graph the solution sets of the following. If necessary, do one or more examples like the above.

a) $3(\chi + 4) = 12 \vee \sqrt{7} - \chi < \sqrt{7}$

b) $3r + 2\ 1/3\ r < -45 \wedge r - 49.2 > -61.8$

c) $r > -7 \vee 4/3 - r = 8\ 1/3$

d) $3t > -2 \wedge \chi - 14\ 1/4 < 14\ 7/8$

e) $-3y = 21.9 \vee 2y + 3y > -36.5$

f) $12\ 1/2 < \chi + 11\ 7/10 \wedge -6/5 + \chi < 1/2$

ACTIVITY G

The following exercises in this activity should serve to prepare your students for solving word problems in addition to preparing them for magic algebra. The essential feature of algebraic word problems is the use of a variable, letter or placeholder to represent the unknown. Therefore emphasis lies not on solving any condition but in writing the correct mathematical phrase or sentence. Opportunity is also provided in translating mathematical sentences into English.

This activity will be of particular value to your slower students. Some of your more creative students might enjoy the challenge of making their own problem from a condition.

ACTIVITY H

MAGIC ALGEBRA*

The primary purpose of playing magic algebra with your students is to prepare them for solving word problems. It should also give them some valuable practice in solving algebraic phrases and creating their own algebraic tricks.

Have you ever had a friend tell you what number you had calculated after a number of computations, without seeing your answer? How did he do it? Let's find out by trying one of these tricks.

Follow the steps given below:

Step 1. Take a number.

Step 2. Add 5.

Step 3. Multiply by 2.

Step 4. Subtract 8.

Step 5. Divide by 2.

Step 6. Subtract the number you started with.

Your answer should be 1. If not, check your work.

Try another number; your answer will still be 1.

Can you explain why using algebra?

Solution:		Algebraic Solution:	
Step 1.	Take a number.	n	n
Step 2.	Add 5.	$n + 5$	$n + 5$
Step 3.	Multiply by 2.	$2n + 10$	$2(n + 5)$
Step 4.	Subtract 8.	$2n + 2$	$2(n + 5) - 8$
Step 5.	Divide by 2.	$n + 1$	$\frac{2(n + 5) - 8}{2}$
Step 6.	Subtract the number you started with.	1	$\frac{2(n + 5) - 8}{2} - n$

Using a variable (n in this case), these tricks can always be proved. Make up one of your own that will give a different number as the answer.

---

*Modified after W. H. Glenn, and D. A. Johnson, Fun With Mathematics (St. Louis, Mo.: Webster Publishing Co., 1961).

e.g.

		Example:	Algebraic Solution:
Step 1.	Take a number.	12	n
Step 2.	Add the same number plus 2.	26	n + n + 2
Step 3.	Subtract 10	16	2n - 8
Step 4.	Divide by 2.	8	n - 4
Step 5.	Add 20.	28	n + 16
Step 6.	Subtract the original number.	16	16

You should always end up with 16.

Play some black magic on a friend by making him always end up with the number 13, no matter what number he chooses.

<u>Secret</u>:

Design a set of operations that will eliminate the original number chosen and put in the number you wish to end with.

This "magic" can be extended to many other interesting problems where the numbers people choose may be their age, the change in their pockets, the number in their family, the house number, when they were born, and many others you might like to try.

For example:

Your age is ______.

		Example:	Algebraic Solution:
Step 1.	Write down your age.	14	A
Step 2.	Multiply by 5.	70	5A
Step 3.	Add 20.	90	5A + 20
Step 4.	Multiply by 3.	270	15A + 60
Step 5.	Subtract 10 times your age.	130	5A + 60
Step 6.	Divide by 5.	26	A + 12
Step 7.	Subtract 12.	14	A

You might try combining this problem with the amount of change in the person's pocket (less than $1.00). Have the age in the hundreds place and the change in the ones and tens place.

e.g. 1749: Age 17 years, Change 49 cents.

With my crystal ball, I see you were born on ____________, 19__.

Name your friend's birthday.

Assign numbers to each month in order; e.g., January = 1, February = 2, March = 3, etc.

Imagine his birthday is August 23.

	Example:	Algebraic Solution:
	8	M
Multiply the number of the month by 5.	40	5M
Add 6.	46	5M + 6
Multiply by 10.	460	50M + 60
Add 9.	469	50M + 69
Multiply by 2.	934	100M + 138
Subtract 28.	910	100M + 110
Add the day of the month	933	100M + 133

Ask the answer. Then mentally subtract 110 from the answer (933) and you get 823. The 8 stands for August and 23 represents the day of the month.

"Hocus pocus ala kazam," your birthday is August 23.

Get the students to make up one of their own or a different set of directions for one of the examples.

Here is another one with two variables:

	Example:	Algebraic Solution:
Think of a number between 0 and 10.	7	
Multiply by 5.	35	5
Add 7.	42	$5\chi + 7$
Multiply by 2.	84	$10\chi + 14$
Add another number between 0 and 10 (picks 3).	87	$10\chi + 14 + y$
Subtract 3.	84	$10\chi + 11 + y$
Ask friend to tell you his number.		
Then subtract 11 mentally.	73	$10\chi + y$

$10x$ places the first number chosen in the tens place and y, the second number, is in the ones place.

Your friend's numbers are 7 and 3.

ACTIVITY I*

The purpose of this activity is to give the student an opportunity to discover solutions for conditions in two and three variables. Their background to date is based on Lesson III in UNIT 9, STM 2, in which they made tables and graphs of conditions in two variables using ordered pairs of natural numbers in the first quadrant only.

In the following suggested approach, students might work individually or in pairs of equal ability. Distribute graph paper to all and have more handy. Try to tell them as little as possible, using student discoveries to teach when necessary.

(1) Some students might plot ordered pairs of numbers only. Compare these graphs with straight-line graphs; remind them of the universe and ask which graph is correct.

Do not insist on a chart of values. Try and use a discovery approach. Let the students learn from each other. Slow learners might need to start out with a condition like $y = x + 4$.

(2), (3), (4) The purpose of these questions is to emphasize that a point lies on the graph of the solution set only if its coordinates produce a true statement when substituted into the condition.

A geometric approach is emphasized with the simple conditions since it is usually much easier to use the geometric approach to teach compound or simultaneous conditions.

The emphasis you need to use here will depend on your classes' background.

(5), (6), (7) By graphing the solution sets of questions 1 and 5 on the same graph paper, the answer to question 6 should be more easily discovered. Even if some of your students don't come up with a graphical method of solution don't insist on this method of solution. Let your students free to solve question 6 in any

---

*Modified after Robert B. Davis, Explorations in Mathematics, A Text for Teachers (Palo Alto, Calif.: Addison Wesley Publishing Co., 1967), pp. 272-279.

way they can. Some might guess the answer at first, but encourage them to come up with a more systematic method. Other students might try making charts for each condition and looking for an ordered pair common to both charts.

(8), (9) More opportunity and chance to find and use a method for solving compound conditions.

(10) You may get all kinds of methods suggested here and hopefully one of them will be a combination of making charts and graphing each condition to find the point of intersection.

(11) Using Irv's or Harry's or whoever's method [see (10)], students should be able to solve this condition with no problem.

(12), (13), (14) Try and encourage students to look for a simpler non-graphical method if possible here. It is hoped that they might come up with the algebraic method of eliminating one unknown by substituting into a condition to get one unknown only.

(15) Let's hope one of your students comes up with the algebraic method here as this is the best way of solving conditions in three variables which come up in questions 16 and 17.

(16), (17) These are optional and you may leave it that way with your students also.

(18), (19), (20) Again, let your students devise their own methods here. They should be enlightened by a comparison of the graphical solution of conditions like $a = 2b + 1$ and $a > 2b + 1$.

You may have your own ideas on how important it is that <u>all</u> students learn one or more methods of solving compound conditions. There is very little doubt how important it is that this lesson should be approached with a spirit of originality, cleverness, and fun! For this reason you might find it necessary to give some of your slower students some simpler conditions that they can build confidence on.

At the end of the lesson you might ask the students how many methods they know for solving compound conditions. You might also ask how exciting it was?

ACTIVITY J

PIRATES' GOLD* AND THE RED BARON

The purpose of these activities is to provide practice, enrichment and add interest to the graphing of conditions. Pirates' Gold involves the graphing of simple conditions of equality and inequality in two variables. The correct "clue" to locate the hidden gold is the graph of $x + y = 8$. This is the only condition whose graph passes through the rock where the gold is hidden. Shoot the Red Baron involves the graphing of compound conditions of equality in two variables. The solution set of Fire Three is the only compound condition that "hits" the Red Baron.

Both graphs can be reproduced on paper for use as a class activity or as an individual assignment. You might encourage the student to add more "clues" to find another buried treasure or add more "shots" to hit the Red Baron twice. You might also encourage the students to construct one of their own graphs using whatever theme they prefer such as cars racing, throwing a spear, etc. It would be very easy to relate this activity to other subject areas such as literature.

ACTIVITY K

PROBLEMS:

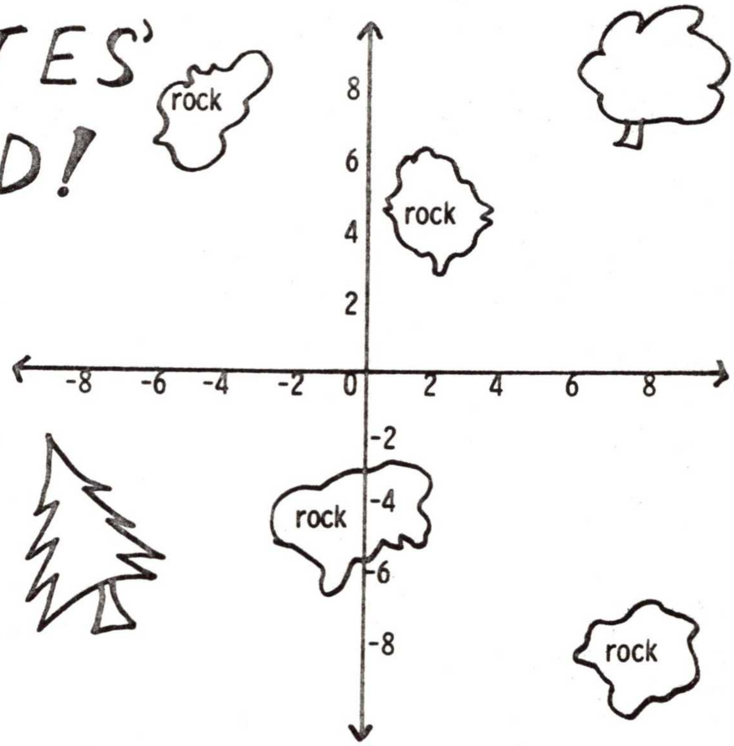
The main student objective in teaching the solution of word problems is to get the students to use an algebraic method in solving these conditions. If the students can find the answer without using algebra then they are not developing the mathematical skills they will need later on.

Rather than tell the students to use algebraic methods of solution problems like the following which are solved so easily when algebra is used might have better results.

---

*Modified after E. M. Turner, Teaching Aids for Elementary Mathematics (New York, N.Y.: Holt, Rinehart and Winston, Inc., 1966), p. 139.

PIRATES' GOLD!



BEWARE!

"Clues"

g_1 $y = -2$

g_2 $x = 5$

g_3 $5x - y = 19$

g_4 $x \leq -7$

g_5 $x - 2y = 4$

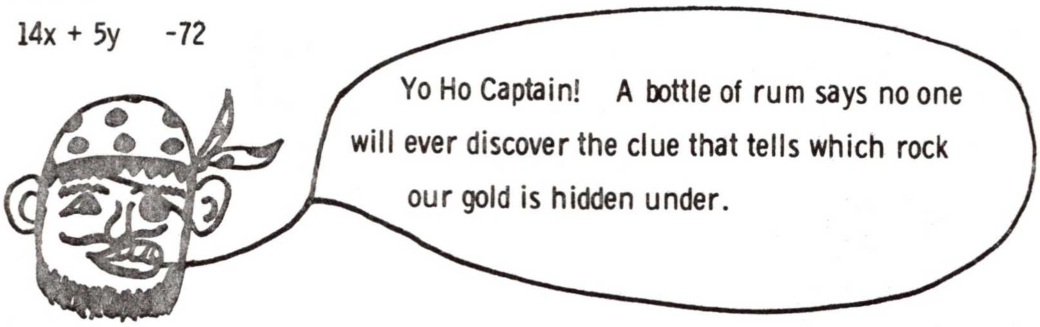
g_6 $x \geq 13$

g_7 $x + y = 8$

g_8 $x + 3y = 3$

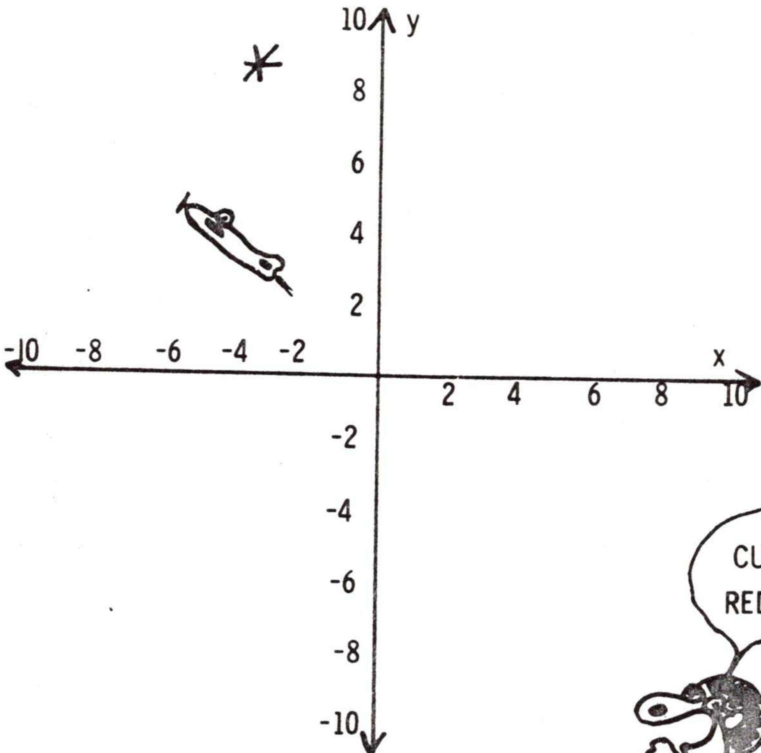
g_9 $y \geq 9\frac{1}{2}$

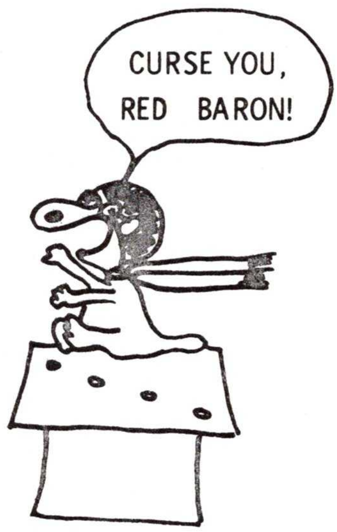
g_{10} $14x + 5y \quad -72$



Modified after Ethel M. Turner, Teacher Aids for Elementary Mathematics (New York, N.Y.: Holt, Rinehart and Winston, Inc., 1966), p. 139.

SHOOT THE RED BARON!





Fire one:	$y = x + 2 \wedge y = 4 - x$
Fire two:	$y = 2x - 3 \wedge y = \frac{1}{2}x + 3$
Fire three:	$y = -2x - 6 \wedge y = \frac{1}{4}x + 6$
Fire four:	$y = -3x + 2 \wedge y = 1/3x - 4$
Fire five:	$y = x + 3 \wedge y = -x - 7$

HELP SNOOPY BRING DOWN THE RED BARON BY FIRING COMPOUND CONDITIONS. THE SOLUTION SET OF ONLY ONE OF THESE WILL VANQUISH SNOOPY'S ARCH FOE.

You might suggest to the student that the use of a variable to represent the unknown would help him solve the problem much more easily or let him solve it completely on his own and see what kind of a method he uses.

Irving Glick gets on a bus at Lacombe. He is the only one to get on there. At Red Deer, seven more people get on and one man gets off. At Penhold, two people get off and four more get on. At Innisfail three more people get on. At Olds two people get off. At Carstairs, one half of the passengers get off. At Airdrie, again one half of the passengers get off. At the Calgary bus depot three people get off and Irving is the only passenger left on the bus.

How many passengers were on the bus when Irving got on?

SOLUTION:

The important algebraic technique here is the use of a variable to represent the unknown--in this case the number of passengers on the bus before Irving Glick got on.

The following table might make the solution more understandable:

	Number of Passengers
Before Irving got on:	P
After Irving got on:	P + 1
After the Red Deer stop:	P + 7
After the Penhold stop:	P + 9
After the Innisfail stop:	P + 12
After the Olds stop:	P + 10
After the Carstairs stop:	1/2(P + 10)

	Number of Passengers
After the Airdrie stop:	$1/4(P + 10)$
After the Calgary stop:	$1/4(P + 10) - 3$
At this point Irving is the only passenger left on the bus, so . . .	$1/4(P + 10) - 3 = 1$
Solving for P:	$1/4(P + 10) = 4$
	$(P + 10) = 16$
	$P = 6$

Therefore there must have been six passengers on the bus when Irving got on.*

A similar kind of problem follows:

On Wednesday, Dianne eats twice as many cherries as she did on Monday. On Thursday she ate twice as many as she did on Wednesday. On Friday she ate 50 cherries. On Saturday she ate twice as much as she did on Thursday. On Sunday and Tuesday she didn't eat any cherries. For the entire week she ate 230 cherries. How many cherries did she eat on Monday?

SOLUTION:

	Number of Cherries
Monday	C
Wednesday	$2C$
Thursday	$2\ (2C) = 4c$
Friday	50
Saturday	$2\ (4C) = 8C$
1 week	$C + 2C + 4C + 50 + 8C = 230$

*Modified after Robert B. Davis, Discovery in Mathematics, A Text for Teachers (Reading, Mass.: Addison Wesley Publishing Co., 1964), pp. 237-241.

Simplifying we obtain:

$$15C + 50 = 230$$

$$15C = 180$$

$$C = 12$$

Dianne ate 12 cherries on Monday.

Problems like the above might be used to introduce the students to an algebraic problem-solving method. When you feel a student is ready to work on his own then he or she should be directed to the activity cards.

ACTIVITY PROBLEM CARDS

These are a varied assortment of problems to provide mathematical experience ranging from easy to difficult and from simple pencil and paper activities to the gathering of scientific data based on classroom experiments. The problems have been printed on individual cards for easy use by individual students or groups of two and three. In most cases, the pupils should be able to read the cards, gather any information or materials necessary, and solve the problems with little or no help from the teacher. It is essential, however, that the teacher discuss the results of many of these problems with the group to ensure real understanding of the problem.

The purpose of the activity card approach is to allow children to:

1. discover for themselves the essential methods of solving problems.
2. work at their own ability level and rate of learning.
3. work on problems of particular interest to them.

4. learn from each other and work together co-operating.

Class teaching of mathematics is undesirable when it comes to solving problems since the abilities and interests of your students have such a wide range. But on occasion, class discussion might be valuable to draw attention to something well done or point out a common fault that needs to be corrected.

Organization of Cards

This can be done according to teacher preference. One suggestion is to arrange these cards in three groups:

a) problems with one variable

b) problems with two variables

c) enrichment problems (brain-teasers and brain-busters)

Preparing and Starting the Class

A good introduction is necessary to initiate a high quality standard of thinking, working, and discussion in addition to arousing interest. You might select a particular problem, discuss it thoroughly with the class and make the following points clear so that the students have no doubt about what is expected.*

1. The card is their problem, their challenge.

2. The card should be read carefully and discussed by both partners.

3. If the students are clear about what is to be done they should proceed on their own (including any collection of materials and information).

4. Ask the teacher for assistance only as a last resort.

---

*Modified after R. A. J. Pethen, The Workshop Approach to Mathematics (Macmillan Co. Canada, Ltd., 1968).

5. Show and record your work on paper.

6. Make sure your card is completely answered.

7. Check with the teacher when finished. If the teacher is busy, then proceed with another card and check your previous problem later.*

Once you feel your students are ready to proceed on their own, let them loose. You might find it wise to post a set of instructions similar to the above for handy student reference.

The clue to a good activity card approach in the classroom is to use it at the right time and for proper lengths of time to sustain progress and interest.

The choice of content and materials for these cards is not intended to limit their use in any way. Please feel free to adapt, add to, or delete from the set of cards as best suits your classes and circumstances.

---

*Modified after R. A. J. Pethen, The Workshop Approach to Mathematics (Macmillan Co. Canada, Ltd., 1968).

ACTIVITY CARDS

The following problems are intended for classroom use as mentioned in the preceding teacher's guide. The problems may be keyed according to degree of difficulty but there are still sufficient of all kinds to provide student choice. Teacher choice is allowed for also in that each teacher may delete, add to, or modify any of the problems as appropriate for the needs of the students.

Problem 1:

A freight train left Vancouver for Calgary averaging 55 m.p.h. At the same time a passenger train left Calgary for Vancouver averaging 70 m.p.h. If they left at 9:00 a.m., what time would they pass each other? At what distance from Calgary?

Problem 2:

During its flight to Mars, Mariner 6 was at a point 104,637,260 straight-line miles from earth. The total distance between earth and Mars was calculated at 112,660,617 miles. If Mariner 6 was travelling at a velocity of 55,000 m.p.h., how long did it have to travel before reaching Mars? (Answer to nearest hour.)

Problem 3:

Smash-It-Hard and Company charge 10¢ for renting a badminton racquet, plus 2¢ per day.

a) Complete the following table

Number of Days (n)	1	2	3	4	5	7	10	n
Charge (cents) (c)	12	14	16					

b) If C cents is the charge for n days, write a condition connecting C and n.

c) Make a graph from the table and use this to find out how long you could rent a racquet with 50¢.*

Problem 4:

The tide is rising at the rate of 16 inches per hour. Four rungs of the ladder on the boat are below the water. Each rung is 1 1/2 inches thick. There are 8 inches between rungs.

HOW MANY RUNGS OF THE LADDER WILL BE SUBMERGED 5 1/2 HOURS LATER?**

---

*Modified after F. Lord, The Language of Mathematics (London, Ont.: John Murray Publishers Ltd., 1960).

**Modified after D.G. Seymour, and R. Gidley, EUREKA (Palo Alto, Calif.: Creative Publications, 1968), p. 120.

Problem 5:

Beginning from the smallest, write the following real numbers in order.

$\sqrt{12}$ $\quad$ $3.4\overline{6}$ $\quad$ 3.47 $\quad$ 3.465 $\quad$ $\sqrt{11}$ $\quad$ $3.46\overline{3}$

Starting with the largest, write the following real numbers in order.

$-\sqrt{7}$ $\quad$ -3 $\quad$ $-\sqrt{15}$ $\quad$ -2 $\quad$ $-\sqrt{11}$ $\quad$ -8

Problem 6:

The Wizard from Poppy Land has dwarfs and dragons. These creatures have 50 heads and 140 legs. How many dwarfs and how many dragons does the Wizard have?

Problem 7:

Which is greater?

1. 1/3 or 0.343343334

2. -5.3 or -5.4

3. $-2.73\overline{2}$ or $-2.732\overline{1}$

4. $1.7\overline{6}$ or $1.76\overline{1}$

Problem 8:

Name an irrational number

a) greater than $4.1\overline{2}$

b) less than $1.2\overline{3}$

c) greater than $7.1\overline{2}$ but less than $7.1\overline{3}$

d) between $-3.\overline{3}$ and $-3.\overline{2}$

e) between 3.14159 and 3.14160

f) between $2.13\overline{42}$ and $2.13\overline{41}$

g) between 0.231231123111123. . . . and 0.2312123121112. . . .

Problem 9:

	Number of Table Napkins	Thickness
1st Time	1	.003 in.
2nd Time	2	.006 in.
3rd Time	4	.012 in.
4th Time	8	?

Continue this table by doubling the number of table napkins each time and calculating the total thickness. Continue the table up to 32 times. BUT FIRST MAKE A GUESS AS TO HOW HIGH THE PILE OF NAPKINS WILL BE. Do you think it will be 1 foot high? As high as your room from the floor to the ceiling? As high as the Husky Tower? As high as ___? ? ?___.*

Problem 10:

In a barn there are some hens and pigs. There are altogether 13 heads and 36 legs. How many hens and how many pigs are there in the barn?

---

*Modified after L. R. Lieber, and H. G. Lieber, The Education of T. C. Mits (New York, N.Y.: W. W. Nanton and Co., Inc., 1944).

Problem 11:

Irving ran up a gas bill of $8.46 on his Bloopo Credit Card, the number of gallons used was 18. How much did Irving pay per gallon of gas?

If Irving averaged 51.7 miles per gallon, how many miles did he put on his scooter?

Problem 12:

The first stage rocket of Venus 14 carries 19,000 lbs. of solid fuel. If it uses this fuel at a rate of 2,000 lbs. per minute, how long will the first stage fire before it burns out?

Problem 13:

While watching a woodchopper, Sam hears the axe strike the tree 1 2/5 seconds after he sees it strike the tree. How far is Sam from the woodchopper?*

Problem 14:

In Sunalta Junior High School there are 210 girls. There are twice as many blue-eyed girls as green-eyed girls and there are 10 more brown-eyed girls than blue-eyed girls. How many blue-, green-, and brown-eyed girls are there?

---

*Modified after D. G. Seymour, and R. Gidley, EUREKA (Palo Alto, Calif.: Creative Publications, 1968), p. 110.

Problem 15:

A mattress is 4 feet 6 inches wide and 6 inches deep. How much of a 90-inch wide sheet is available on each side of the bed for tucking under the mattress?*

Problem 16:

Using a measuring tape, find out how long it would take you to walk 55 yards.

Using this figure, find out how long it would take you to walk a mile; 25 miles; The Miles for Millions March.

---

*Modified after F. Land, The Language of Mathematics (London, Ont.: John Murray Publishers Ltd., 1960).

Problem 17:

Lorne built a fence around his 19 girl friends. The enclosed region was square shaped and he used 27 fence poles on each side of the square. How many poles did he use altogether?

Problem 18:

If you had $5 billion and you gave away a $500 bill every minute, how long would it take you to give away all your money?*

*Modified after D. G. Seymour, and R. Gidley, EUREKA (Palo Alto, Calif.: Creative Publications, 1968), p. 121.

Problem 19:

The express bus averages 56 m.p.h. on a trip to Edmonton.

The express train averages 64 m.p.h. on a trip to Edmonton. How much time do you save by taking the train from Calgary to Edmonton?

Problem 20:

Richard wants to buy a new Super Dooper Scooper Motor Bike for $750. One dealer offered him $150 for his old bike as a trade-in. Another dealer offered him a 6 1/2% cash discount off the purchase price but no trade-in. Richard could sell his bike privately for $100.

Which dealer is giving him the best offer?

How much does he save by going to this dealer?

Problem 21:

THE PHEASANT COOP

Herbie caught a young pheasant with an injured wing. He decided to keep it in his back yard. He had 50 feet of wire fence, and wanted to make a pen with the largest area possible. Work out the length and width which will enclose the largest space.

Problem 22:

The area manager for a local ski hill says 3.5 inches of fresh powder snow are needed for every inch of packed base. If a ski hill has a base of 5 feet 3 inches, how much fresh snow has fallen?

Problem 23:

Apollo astronaut Armstrong walked 885 yards in 177 steps. What were the average size of his "small steps" or were they "giant steps"?

Problem 24:

I see two numbers. If you add them together you get 29. If you subtract the second number from the first you get 3. What two numbers do I see?

Problem 25:

How fast can you run 100 yards?

Could you break the 4-minute mile if you could maintain your top speed?*

Problem 26:

Vancouver's population is greater than Calgary's by more than 860,000. What is Vancouver's population?

---

*Modified after R. A. J. Pethen, The Workshop Approach to Mathematics (Toronto, Ont.: The Macmillan Co. of Canada Ltd., 1968), Complete Set: Cards 1-224.

Problem 27:

Find the area of one of your feet to the nearest square inch. Find your weight.

Calculate the weight per square inch of foot when you are standing on two feet and one foot.

Problem 28:

Write these mathematical sentences in words by making up a problem or a story:

$$n + 183 = 490$$

$$47 + X < 78$$

$$s \pm 24\ 1/2 > 50 \wedge s < 31$$

Problem 29:

If Dianne were 3 years older, she would be twice as old as Susan. If Dianne were two years younger, she and Susan would be the same age. How old is each girl?

Problem 30:

See if you can find a condition to express the relationship between the height of bounce of a ball and the height of the drop.

(Hint: make a table of values and graph the results.)

Problem 31:

Make up a problem that involves the number indicated:

a) your age

b) the total number in your family

c) your weight

Problem 32:

Suppose we send you to the planet Pluto in 1980 in a space ship that travels 85,000 m.p.h. How long will it take you to get there?

Problem 33:

Before Christmas, every girl in Mrs. Makway's class had 3 wigs, except for Heather, who had 5 wigs (one was orange). On Christmas Day one half of the girls received a new wig. As a result the class had 16 per cent more wigs than they had before. How many girls are there in Mrs. Makway's class?

Problem 34:

Allan had a dream about girls and sports cars. When all the girls in Allan's dream got into sports cars, there was one girl per car, and five cars were empty. Then one half of the girls got into a space ship and blasted off for Jupiter. After that, whenever all of the remaining girls got into the sports cars, there was one girl per car, but 23 cars were empty. How many girls were there in Allan's dream?*

---

*Modified after Robert B. Davis, Discovery in Mathematics, A Text for Teachers (Reading, Mass.: Addison Wesley Publishing Co., 1964).

Problem 35:

The Apollo space capsule is travelling at a speed of 1,000 m.p.h. The lunar module is trying to catch up to the space capsule at a speed of 1,200 m.p.h. If they are 2,300 miles apart, how long will it be before they meet? How many miles will the lunar module travel before it catches the space capsule?

Problem 36:

WOOD SCREWS*

The sizes of wood screws are indicated by numbers. The size of a screw indicates the diameter of its shank. Examine the table below.

Diameter of shank of screw in thousandths of an inch (y)	66	80	94	108	122	136	150	164	178	192	206
Standard size of screw (x)	1	2	3	4	5	6	7	8	9	10	11

Plot these numbers on a graph and find the algebraic condition for the line which passes through these points.

*Modified after F. Land, The Language of Mathematics (London, Ont.: John Murray Publishers Ltd., 1960), p. 113.

Problem 37:

Last winter, the F. E. Osborne Junior High School Ski Club went skiing on four Saturdays. On the first three Saturdays, two-thirds of those who went skiing each weekend were injured. Each member skiied every Saturday until he or she was injured. After the injury the member did not ski again.

The Club bought 100 day tickets for the ski tow at the beginning of the season. Each ticket was good for one member for one Saturday. After the fourth weekend, the Club had 20 tickets left. The Club voted unanimously to sell these tickets to the Colonel Irvine Ski Club and rename theirs the F. E. Osborne Curling Club.

How many members of the F. E. Osborne Ski Club went skiing on the first weekend?*

Problem 38:

Find out how far your family's car will travel on a gallon of gasoline on the highway. How many gallons of gasoline (to nearest tenth of a gallon) will be needed to travel to Edmonton and back; Vancouver and back? How much will the return journey from Vancouver cost at 49.9¢ per gallon of gasoline?

---

*Modified after Robert B. Davis, Discovery in Mathematics, A Text for Teachers (Reading, Mass.: Addison Wesley Publishing Co., 1964), p. 240.

Problem 39:

The sum of two numbers is 20. If one number is doubled and the other is multiplied by 4, the sum of the two new numbers is 66. What are the two numbers?

Make up a similar problem.

Problem 40:

How would you convince your friend that if there are 367 boys and girls in the auditorium, then at least two of them must have the same birthday?*

---

*Modified after E. D. Nichols et al., Elementary Mathematics 7, Patterns and Structure, Teacher's Edition (New York, N.Y.: Holt, Rinehart and Winston, Inc., 1966).

Problem 41:

Calculate your weight in pennies.

Problem 42:

A new Jumbo Jet flies over your school at a speed of 575 m.p.h. at 8:30 a.m., on a trip to Bermuda. If it maintains this speed, about what time should it be over Bermuda?

Problem 43:

Kathleen bought a 6-foot roll of stamps for $6.00. If one stamp measures 5/8 inches, how many stamps did she get in the roll?

Problem 44:

Lorne and Doug climbed Mt. Assiniboine in the summer. Mt. Assiniboine has an altitude of 12,093 feet above sea level. The difference in altitude between the top of Mt. Assiniboine and their base camp at Wonder Pass was 6,117 feet. What was the elevation of their camp?

Problem 45:

How far can you travel in 1/4 minute:

walking?

running?

How far in an hour?

Could you run a 4-minute mile if you could maintain your top speed?*

Problem 46:

The sonic boom from some jets flying over Kelowna last summer took 5.5 seconds to reach town and crack a number of store windows. How far above town were the jets?

---

*Modified after R. A. J. Pethen, The Workshop Approach to Mathematics (Toronto, Ont.: The Macmillan Co. of Canada Ltd., 1968), Complete Set: Cards 1-224.

Problem 47:

Miss Moog handed out three sheets of graph paper and had 31 sheets left. She then handed out 1 more sheet to each student and had 8 sheets left. How many sheets of graph paper did Miss Moog have to begin with and how many students are there in the class?

Problem 48:

The number of bacteria enclosed in a bottle doubles each minute. If the bottle is filled completely with bacteria after 30 minutes, how long ago was the bottle half full?*

---

*Modified after E. D. Nichols et al., Elementary Mathematics 7, Patterns and Structure, Teacher's Edition (New York, N.Y.: Holt, Rinehart and Winston, Inc., 1966).

Problem 49:

My age this year is a multiple of 7 and next year it will be a multiple of 5. If I am not yet 50 and over 30, can you say how old I am?

Problem 50:

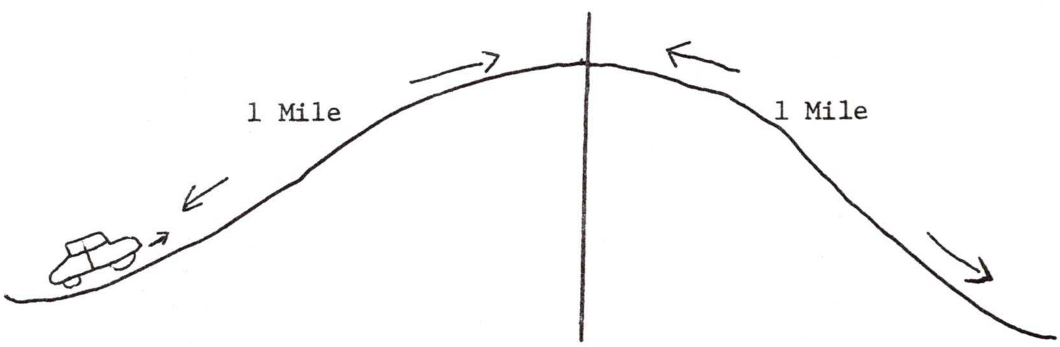
Make up your own problem. See if one of your classmates can solve your problem.

Problem 51:

Jill and Jenny start out on a hike near Moraine Lake at 1:00 p.m. and return at 7:00 p.m. If their speed is 4 m.p.h. on level land, 3 m.p.h. uphill and 6 m.p.h. downhill, how far did they walk?

Problem 52:

A car travels 1 mile uphill at 30 m.p.h. How fast should it travel 1 mile downhill in order to have an average speed of 60 m.p.h. over the entire 2 mile stretch?

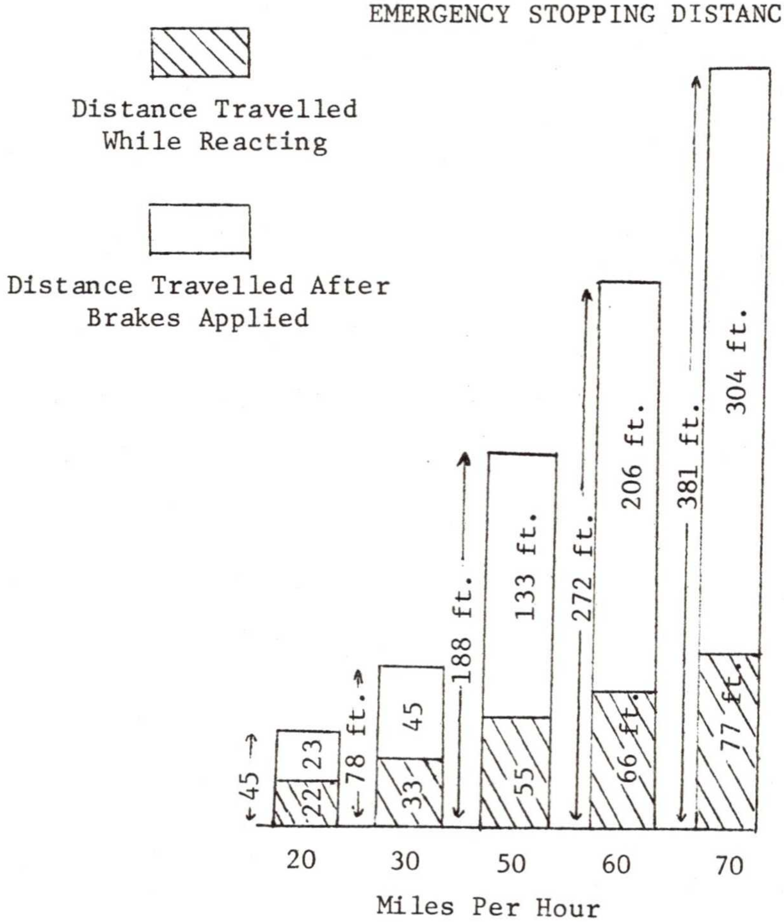


Problem 53:

Each Boston player received $9,000 as a member of the Stanley Cup team. Assuming each player played an average of 25 minutes in each game of a total of 14 games, how much did each player earn per minute of playing time?

Problem 54:

EMERGENCY STOPPING DISTANCES



Can you figure out the equation they used?

Using these figures calculate a driver's reaction distance while travelling at the world record land speed.*

---

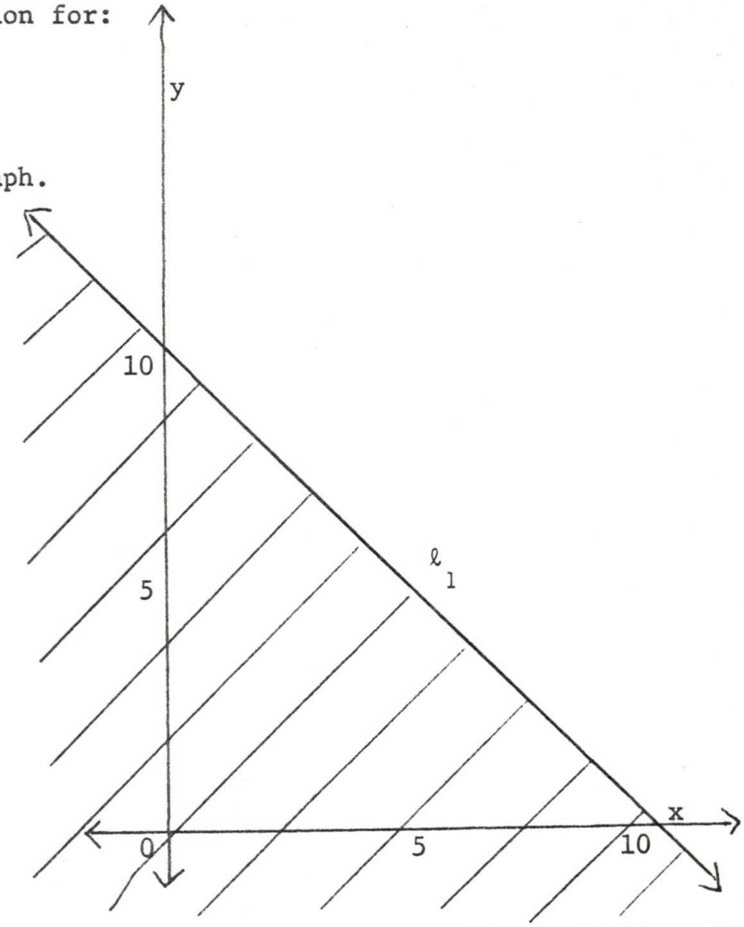
*Department of Highways and Transport, Alberta Operators' Manual (Edmonton, Alta.: L.S. Wall, Queen's Printer, 1969), p. 46.

Problem 55:

See if you can write a condition for:

a) ℓ_1

b) the shaded part of the graph.



Problem 56:

Joe Bananas bought a ball and a bat for his son, Lurch. The bat and ball together cost $3.75. The ball cost $.75 more than the bat. What was the cost of each?*

Problem 57:

A number is 2 more than a second number and is also 6 less than twice the second number. What's the number?

---

*Modified after D. G. Seymour, and R. Gidley, EUREKA (Palo Alto, Calif.: Creative Publications, 1968), p. 114.

ABOUT THE WRITER

Dale Fisher has taught junior high school mathematics and science in Calgary and Victoria. He is presently teaching at the Kaduna Polytechnic Institute in Nigeria on a Canadian Department of External Affairs program.

He developed his "Active Learning Unit on Real Numbers" as part of his M. Ed. thesis project in the Department of Curriculum and Instruction at The University of Calgary during the 1969 - 70 academic term.