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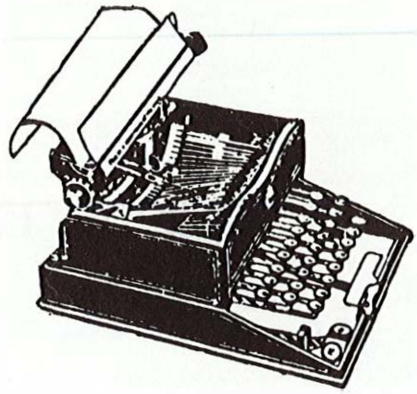
Inside This Issue

From the Editors

Don Kapoor



The Canadian Mathematics Teacher is a co-operative venture of the associations of mathematics teachers in Alberta, British Columbia, New Brunswick, and Saskatchewan, and the Ontario Association for Mathematics Education. Its primary purpose is to act as a clearing house across Canada and promote the cause of mathematics education at all levels. This publication is now into its third year, and we do hope that with your continued support, it will become a truly Canadian magazine with representation from all provinces across Canada.

Volume 3 has a wide variety of articles, ranging from general interest to hands-on experiences and practical tips for actual classroom use. According to our accepted practice, each participating provincial editor submits one article. All articles are carefully reviewed and edited. For all intents and purposes, the opinions of the provincial editors override the referee team. We welcome your comments and suggestions to improve the quality of this professional journal.

Since this journal is a joint effort, I thank the following provincial editors:

Gordon Nicol
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A Canadian Link with Down-Under: The Australian Mathematics Competition

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Within eight years, the Australian Mathematics Competition has grown from a small local competition with 1300 entrants to a competition spanning Australia with 250,000 entrants, 23% of the school population from 80% of Australian secondary schools, or 1.8% of the Australian population, plus another 25,000 entrants from 11 South Pacific countries. A significant stimulus for the competition was the Canadian Mathematics Competition observed by one of the authors when on sabbatical leave in 1973 in Waterloo University, Ontario. This paper gives a brief description and history of the competition and surveys some interesting qustions. —Editor

Early Days

During the early 1970s, the tendency in educational circles was to view flexibility as a panacea. Unfortunately that point of view was linked with a move both to downgrade so-called "difficult" school subjects (such as mathematics) and also to discourage excellence (the word *elitism* acquired some nasty overtones and was used to justify attacks on scholarship and the pursuit of excellence).

As a reaction to this lowering of standards, a group of mathematicians at the Canberra College of Advanced Education (CCAE) and teachers from the Canberra Mathematical Association (CMA) created in 1976 a small local mathematics competition, which attracted 1300 entrants from 30 local schools in Canberra.

The competition was designed to encourage secondary school students to strive for excellence at all levels; to encourage a high mastery of the basic numeracy skills; and to create, over the years, a pool of interesting, rewarding, and challenging problems to extend, supplement, and enrich regular school work.

The 1976 paper was set so that conscientious students of average ability would sense achievement from attempting the competition. The top 45% of entrants in each year would receive at least a certificate of achievement.

The success of the 1976 competition encouraged the committee to extend the competition. After a pilot scheme in 1977, the competition became national in 1978—the Australian Mathematics Competition. It was jointly sponsored by the CCAE, the CMA, and Australia's largest bank, the Westpac Banking Corporation (then called the Bank of New South Wales). The entry statistics in Table One gauge the success of the AMC.

TABLE ONE
The AMC Entry Statistics

Year	Number of Australian entries	Number of overseas entries	Total number of entries	Number of schools	Average per school
1976	1,300	—	1,300	33	39
1977	4,600	—	4,600	73	63
1978	60,854	890	61,744	708	87
1979	101,236	917	102,153	1,283	80
1980	141,057	14,246	155,303	1,776	87
1981	180,775	16,547	197,322	2,007	98
1982	210,021	20,823	230,844	2,112	109
1983	248,034	27,423	275,457	2,222	124

Some Special AMC Features

Features that have contributed to the success of the AMC include the following:

- Three separate competition papers, a junior paper for school years 7 and 8, and intermediate paper for years 9 and 10, and a senior paper for years 11 and 12. (Year 12 is the final year of secondary schooling in Australia.)
- Students compete, within their own state and school-year groupings, for prizes and certificates. This minimizes the effect of syllabus and age differences.
- Special school and summary reports can be produced, because all entrants' responses are centrally processed.

For example:

(a) *The School Result Report* gives a complete list of results for each entrant from the school. The report details results achieved by each entrant in the three sections of the papers and the percentile ranking of the entrant within the state grouping, and it concludes with the cut-off marks for prizes and other awards for the appropriate state;

(b) *School Question Analysis Report* gives for a particular school the percentage of responses to the five options for each question. Because the corresponding state results are also given, the report provides the school with an aid to identifying strengths or weaknesses of the curricula at the school level.

Competition Questions and Statistics Examples

The competition committee has available student response rates to questions at both the state level and overall in the competition. These statistics greatly assist the committee in setting future papers. Consider the following set of questions and statistics categorized broadly as examples of misplaced questions, questions with varying male/female response rates, and examples of challenging questions.

Since the participating schools and the students within these schools do not constitute a random sample, any analysis of these statistics must be interpreted with caution. However, the authors believe that these statistics can be of great value to the thoughtful teacher.

Examples of Misplaced Questions

So that all entrants may gain a sense of achievement from attempting the papers, the committee has tried over the years to ensure, not only that the first part of each paper is within the experience and competence of most entrants, but also that the questions are well graded. This has been very much a learning process for the committee. For example, questions 1, 2, and 3 below, though reasonable questions within their own right, were mistakenly (in retrospect) placed too early in the papers. Questions 4 and 5 are perhaps examples of more subtle misjudgements. The blank column in the response-rate tables identifies the percentage of students who do not code any option of the multichoice answers for the particular question.

1. Question 1 (Grades 12 and 13) 1978

If $\log_2 \{ (\log_{16} 2)^{(\log_5 125)} \} = -a$, the value of a is

(A) 0 (B) 1 (C) −3 (D) 6 (E) ¼

Response Rate

Grade	A	B	C	D	E	Blank
12	3	5	16	*24*	9	44
13	2	3	12	*40*	9	33

1978 was the first year of the AMC, and this first question in the Grades 12 and 13 paper is a classic example of a poorly placed question as is evidenced by the high percentage of students (44% and 33%) who did not mark an answer.

2. Question 2 (Grades 8 and 9) 1979

$\frac{3}{8}$ expressed as a percentage is

(A) 60% (B) 62.5% (C) 42.5% (D) 40% (E) 37.5%

Response Rate

Grade	A	B	C	D	E	Blank
8	3	4	7	9	*58*	18
9	2	3	5	4	*79*	8

The question was poorly placed, as is evidenced by the high percentage of students (18%) who did not mark an answer. This statistic greatly surprised the committee until it was discovered that some Australian states do not include percentages as a topic in their Grade 8 curriculum.

3. Question 2 (Grades 12 and 13) 1978

A plane figure consists of a circle and a pair of parallel lines tangent to the circle. The number of points in the plane equidistant from the circle and the two lines is

(A) 1 (B) 2 (C) 3 (D) infinite (E) none of these

Response Rate

Grade	A	B	C	D	E	Blank
12	18	24	*12*	29	6	10
13	20	24	*21*	23	4	8

The statistics here reflect students' inability to succeed in even straightforward geometry questions, a fact which makes the committee hesitant in setting such questions too early in any paper.

4. Question 5 (Grades 12 and 13) 1982
The square of an integer is called a perfect square. If n is a perfect square, then the next largest square greater than n is

(A) $n + 1$ (B) $n^2 + 1$ (C) $n^2 + 2n + 1$
(D) $n^2 + n$ (E) $n + 2\sqrt{n} + 1$

Response Rate

Grade	A	B	C	D	E	Blank
12	10	5	34	6	*33*	11
13	7	3	33	3	*43*	9

The poor response rate for the correct option here surprised the committee. Further investigation revealed that the more able student had an "above-expected" response rate for option C. Such students tend to rush through the first two-thirds of a paper to allow more time to solve the more difficult and challenging questions at the end of the paper. Response (C) here is a very tempting feasible distractor.

5. **Question 26 (Grades 8 and 9) 1981; Question 23 (Grades 10 and 11) 1981; Question 17 (Grades 12 and 13) 1981**

When $3^{1981} + 2$ is divided by 11,
the remainder is

(A) 5 (B) 0 (C) 7 (D) 6 (E) 3

Response Rate

Grade	A	B	C	D	E	Blank
8	*16*	10	8	16	13	37
9	*18*	10	8	15	12	39
10	*13*	10	8	11	11	48
11	*11*	11	7	10	9	52
12	* 9*	9	7	7	10	57
13	* 9*	8	8	6	8	61

The statistics for this question, which was set for all grades, are interesting. One would have expected that both a higher correct response rate and a lower blank response rate would be obtained by the older students. That the correct answer can be obtained by a totally erroneous answer and that that possibly only occurred to the younger students, may be the explanation here.

(On division by 11, 1981 gives a remainder of 1. The $3^1 + 2 = 5$.)

Questions with Varying Male/Female Response Rates

For many mathematics competitions in the world, more boys than girls participate, but this is not so in the Australian Mathematics Competition. Overall the numbers of boys and girls entering are very close, with more girls at Grades 8, 9, and 10. While the competition committee is heartened by the participation of girls, it is conscious that boys are more successful in the competition than girls.

Over the years, boys have predominated in the medal and prize listings. For example, in the 1983 papers, each with 30 questions, boys obtained 1 to 1½ more correct answers than girls. At the same time, girls tended to omit more questions than boys. Girls may be more cautious than boys in guessing, especially if there is a penalty for wrong guessing as there is in the Australian competition.

The following examples of questions and the response-rate statistics for boys and girls should be of value to teachers. It can be most instructive to teachers to talk to students and discuss their responses to the questions. Some of the questions may provide definite starting points for an attack on their misunderstandings and approach to problem-solving.

6. Question 20 (Grades 10 and 11) 1983; Question 15 (Grades 12 and 13) 1983
In a race of 2000 m, Raelene finishes 200 m ahead of Marjorie, and 290 m ahead of Betty. If Marjorie and Betty continue to run at their previous average speeds, by how many metres will Marjorie finish ahead of Betty?

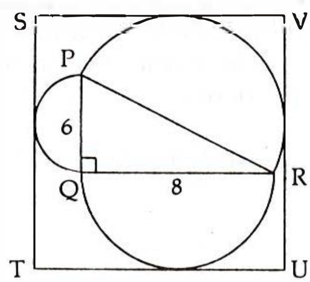
(A) 90 (B) 100 (C) 120 (D) 180 (E) 200

Response Rate

Grade	Boys A	B	C	D	E	Blank	Girls A	B	C	D	E	Blank
10	53	*22*	5	6	3	12	71	* 7*	4	4	2	12
11	47	*28*	5	4	2	13	68	* 9*	4	4	1	14
12	38	*39*	7	4	2	12	60	*15*	5	5	2	14
13	28	*48*	6	3	1	14	50	*20*	5	4	1	18

In Grades 10 and 11, about three times as many boys as girls responded correctly. Girls were more likely to respond with option A, a plausible distractor.

7. Question 23 (Grades 10 and 11) 1983; Question 17 (Grades 12 and 13) 1983
The triangle PQR, right angled at Q, has semi-circles drawn with its sides as diameters. The sides of the rectangle STUV are tangents to the semi-circles and parallel to PQ or QR, as drawn.



If PQ = 6 cm and QR = 8 cm, then the area of STUV, in square centimetres, is

(A) 121 (B) 132 (C) 144 (D) 156 (E) 192

Response Rate

Grade	Boys A	B	C	D	E	Blank	Girls A	B	C	D	E	Blank
10	6	9	*18*	10	10	47	6	8	*14*	10	9	54
11	6	8	*22*	10	9	44	4	8	*17*	10	10	51
12	5	11	34	13	10	28	4	9	20	12	11	44
13	5	12	41	9	8	26	4	10	25	11	9	42

The difference in correct response rate can be explained by the percentage difference of the boys and girls actually attempting the question (note the response for the blank column).

8. **Question 17 (Grades 10 and 11) 1983**

The wheels of a truck travelling at 60 km/h make 4 revolutions per second. The diameter of each wheel, in metres, is

(A) $\frac{25}{12\pi}$ (B) $\frac{6\pi}{25}$ (C) $\frac{25\pi}{6}$ (D) $\frac{100}{6\pi}$ (E) $\frac{25}{6\pi}$

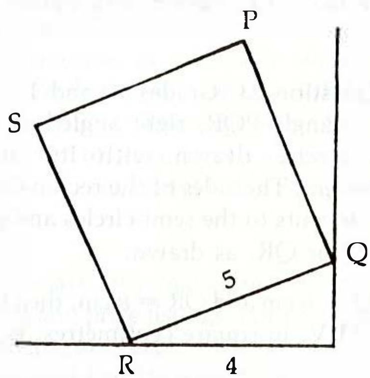
Response Rate

Grade	Boys A	B	C	D	E	Blank	Girls A	B	C	D	E	Blank
10	6	10	8	8	* 9*	58	6	9	6	8	*5*	66
11	5	8	8	7	*12*	59	5	8	7	7	*6*	68

9. **Question 6 (Grades 12 and 13) 1981**

A square box of side 5 cm is leaning against a vertical wall as shown with R 4 cm from the wall. The height of P, in centimetres, above the floor is

(A) $\sqrt{50}$ (B) 7 (C) 8 (D) $3 + \sqrt{5}$ (E) 6



Response Rate

Grade	Boys A	B	C	D	E	Blank	Girls A	B	C	D	E	Blank
12	4	*62*	13	4	4	12	3	*38*	23	7	7	21
13	3	*73*	10	3	2	9	3	*47*	20	4	5	20

This question has resulted in one of the largest AMC differences of boy/girl correct responses rates over the years. This is due to differences in response rates of both the blank and the C distractor.

10. Question 18 (Grades 8 and 9) 1983

Two missiles are initially 5000 km apart. They travel along a straight line directly toward one another, one travelling at 2000 km/h and the other at 1000 km/h. How many kilometres are they apart 1 minute before impact?

(A) 3000 (B) 1000 (C) 500 (D) 100 (E) 50

Response Rate

	Boys						Girls					
Grade	A	B	C	D	E	Blank	A	B	C	D	E	Blank
8	10	17	12	11	*25*	25	16	26	8	6	*13*	31
9	10	13	12	10	*31*	25	11	20	10	9	*20*	30

The differences in response rates here are spread across the blank response and a couple of distractors.

11. Question 10 (Grades 12 and 13) 1983

Of $a + b = 1$ and $a^2 + b^2 = 2$ then $a^4 + b^4$ is

(A) 4 (B) 8 (C) 1 (D) 3 (E) 3½

Response Rate

	Boys						Girls					
Grade	A	B	C	D	E	Blank	A	B	C	D	E	Blank
12	34	26	3	4	* 5*	30	44	21	3	3	* 3*	27
13	20	19	3	4	*15*	39	31	16	3	3	*10*	37

The statistics here are very surprising on a number of counts: the disappointing correct-response rate for such a familiar type of exercise, the high response rate of girls for the distractor A, and the higher blank rate for the Grade 13 students.

12. **Question 7 (Grades 12 and 13) 1982**

A certain substance doubles its volume every minute. At 09:00, a small amount is placed in a container, and at 10:00 the container just fills. The time at which the container was one-quarter full was

(A) 09:15 (B) 09:30 (C) 09:45 (D) 09:50 (E) 09:58

Response Rate

Grade	Boys A	B	C	D	E	Blank	Girls A	B	C	D	E	Blank
12	18	5	2	1	*67*	6	36	6	4	2	*45*	7
13	12	4	2	2	*71*	9	29	6	4	1	*49*	12

In this question, approximately the same percentage of boys and girls responded to the options. Again one of the distractors, A, attracted a large number of girls away from the correct response. Teachers and researchers should be asking themselves why.

13. **Question 10** (Grades 8 and 9) 1983

The five tires of a car (four road tires and a spare) were each used equally on a car that had travelled 20,000 km. The number of kilometres of use of each tire was

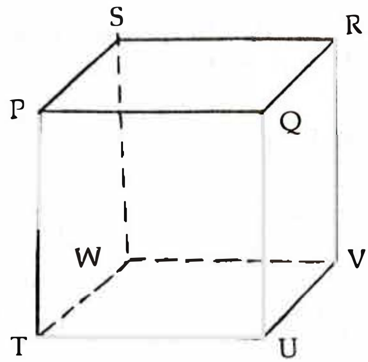
(A) 4,000 (B) 5,000 (C) 16,000 (D) 20,000 (E) 100,000

Response Rate

Grade	Boys A	B	C	D	E	Blank	Girls A	B	C	D	E	Blank
8	70	9	*10*	3	4	2	82	9	*3*	3	4	1
9	69	7	*18*	3	2	1	81	7	*6*	3	2	2

Note the large response rate for the distractor A.

14. **Question 26 (Grades 12 and 13) 1983**
Given the cube PQRSTUVW as shown, the plane that passes through P and the centres of faces TUVW and UQRV intersects UV at X. The ratio $\frac{UX}{XV}$ is



(A) 2 (B) $\frac{3}{2}$ (C) 3 (D) $\frac{5}{4}$ (E) $\frac{5}{2}$

Response Rate

Grade	Boys A	B	C	D	E	Blank	Girls A	B	C	D	E	Blank
12	*10*	14	5	5	5	61	*12*	13	4	4	4	64
13	* 7*	13	6	6	6	64	* 9*	14	3	4	4	66

The result here is somewhat unusual, for the literature on sex differences in mathematical performance alerts us to possible problems with geometry and spatial visualization.

15. **Question 24 (Grades 12 and 13) 1983**
We can devise a short-hand notation for large numbers by letting d_n stand for the occurrence of n consecutive d's where n is a positive integer, and d is a fixed digit $(0 \leq d\ 9)$. Thus, for example, $1_4 9_5 8_2 3_6$ denotes the number 1111999998833333. Find the ordered triple (x, y, z) if

$$2_x3_y5_z + 3_z5_x2_y = 5_37_28_35_17_3$$

(A) (4, 5, 3) (B) (3, 6, 3) (C) (3, 5, 4) (D) (5, 3, 4) (E) (5, 4, 3)

Response Rate

Grade	Boys A	B	C	D	E	Blank	Girls A	B	C	D	E	Blank
12	5	7	6	7	*26*	49	6	5	6	5	*31*	47
13	4	6	5	5	*32*	47	4	4	5	4	*36*	46

This is an example of a question where the success rate favors the girls. These usually occur when the success rates are low overall. Incidentally, a similar result occurs in (9) earlier in this article.

16. **Question 29 (Grades 12 and 13) 1979**
If $P = (1 + 4)(1 + 4^2)(1 + 4^4)(1 + 4^8)(1 + 4^{16})(1 + 4^{32})$
then P equals

(A) $\frac{2^{128}+2^{64}-5}{3}$ (B) $\frac{2^{127}+2^{63}+5}{3}$

(C) $\frac{2^{126}-1}{3}$ (D) $\frac{2^{126}-1}{3}$

(E) none of these

Response Rate

Grade	A	B	C	D	E	Blank
12	3	7	*3*	5	19	63
13	3	6	*4*	5	14	69

This is an example of one of the questions that has an extremely low correct response rate. Often such questions have a brute force time-consuming solution or a short insightful solution. For example, P above can be expanded in terms of all its powers of 4 from 1 to 63:

$$P = 1 + 4 + 4^2 + 4^3 \ldots + 4^{63}$$

Thus P is a G.P. with first term 1, common ratio 4, and 64 terms. Therefore

$$P = \frac{1(1-4^{64})}{1-4} = \frac{4^{64}-1}{3} = \frac{2^{128}-1}{3}$$

17. **Question 25 (Grades 8 and 9) 1980; Question 25 (Grades 10 and 11) 1980**
A beetle crawls around the outside of a square of side 1 metre, at all times keeping precisely 1 metre from the boundary of the square. What is the area enclosed, in square metres, in one complete circuit by the beetle?

(A) $\Pi + 4$ (B) 5 (C) $2\Pi + 4$ (D) $\Pi + 5$ (E) 9

Response Rate

Grade	A	B	C	D	E	Blank
8	15	11	11	*4*	21	37
9	16	9	13	*4*	30	28
10	12	9	13	*4*	33	29
11	10	7	12	*6*	38	27

This is a typical question that challenges most students—a question that requires some modelling skills.

18. **Question 27 (Grades 8 and 9) 1981; Question 28 (Grades 10 and 11) 1981**
In how many different ways can a careless office boy place four letters in four envelopes so that no one gets the right letter?

(A) 4 (B) 9 (C) 12 (D) 6 (E) 24

Response Rate

Grade	A	B	C	D	E	Blank
8	26	*6*	42	7	10	10
9	19	*7*	50	7	9	7
10	12	*9*	52	7	10	10
11	8	*9*	55	7	9	11

Many challenging questions involve interpretive and systematic enumerative skills. This question is typical of the genre. Valuable understanding of and insight into students' difficulties in problem-solving can be obtained by investigating why so many students choose the distractors, for example, options C and A above.

19. Question 20 (Grades 10 and 11) 1981

If $x + \frac{1}{x} = 3$ then $x^2 + \frac{1}{x^2} =$

(A) 9 (B) 10 (C) 27 (D) 11 (E) 7

Response Rate

Grade	A	B	C	D	E	Blank
10	43	3	9	2	*7*	35
11	44	3	8	2	*8*	35

This poor correct-response rate is surprising. Teachers can obtain insight into the students' thinking by noticing the high response rate to distractor A.

20. Question 25 (Grades 10 and 11) 1982
The area of a circle circumscribed about a regular hexagon is 2π. The area of the hexagon is

(A) 6 (B) $3\sqrt{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\sqrt{3}$ (E) $6\sqrt{3}$

Response Rate

Grade	A	B	C	D	E	Blank
10	13	*4*	8	4	10	61
11	12	*6*	8	3	10	61

21. Question 27 (Grades 12 and 13) 1979
If x and y are integers such that
$(x-y)^2 + 2y^2 = 27$,
then the only numbers x can be are

(A) 3, 5 (B) −6, 4 (C) 0, 4, 6
(D) 0, −4, 4, −6, 6
(E) 0, −2, 2, −4, 4, −6, 6

Response Rate

Grade	A	B	C	D	E	Blank
12	7	5	8	*7*	6	68
13	6	5	8	*9*	5	66

Conclusion

The success of the Australian Mathematics Competition has been due to many factors. Not insignificant was the inspiration and generous support that the Mathematics Faculty at the Waterloo University in Ontario, Canada gave the AMC Committee during the early planning days of the competition. The authors dedicate this paper to their colleagues at Waterloo as an acknowledgement of the support that they have given AMC.

References

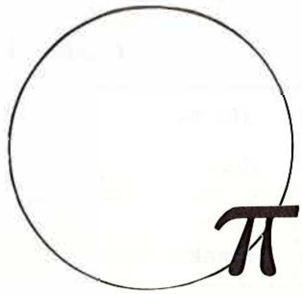
Edwards, J.D. and P.J. O'Halloran (1978). "The Genesis of a Competition." In *Learning and Applying Mathematics*, AAMT, Acacia Press, Melbourne.

O'Halloran, P.J., *et al.* (1980). "The Growth of National Competitions in the U.S.A., Canada, and Australia." In *Mathematics —Theory into Practice*. AAMT, Acacia Press, Melbourne.

Edwards, J.D. and P.J. O'Halloran (1982). "The Australian Mathematics Competition." *Math Scientist* 7, 93-104.

The AMC Solutions and Statistics have been published annually since 1978 by CCAE, Canberra.

"Mathematical Competitions, Contests, Olympiads." (1980). *ICME Report*, 557-567 (Birkhouse-Boston Inc., 1983).



Finding the Area of a Circle Without Pi

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As teachers, we probably rush too quickly to formulas and spend too little time to develop understanding of concepts. The authors of this article suggest a number of instructive activities to help students understand the concept of circular area without using Pi. —Editor

In a Grade 7 mathematics classroom, a student was asked how to find the area of the circular face of the clock on the wall. The student replied, "It doesn't have area. It's not a rectangle." The incident indicates that we probably rush too quickly to formulas and spend too little time to develop understanding of the concept of *area*. The problem is more severe with circles than with rectangles. Students begin using the area formula, $A = \pi r^2$, without any idea what π means, and the general concept of *area* becomes obscured by meaningless rote calculations.

A number of activities can help students to develop a better concept of the area of a circle without reference to π. The activities help students to reinforce skills and understanding with areas of rectangles. These activities, at different levels of sophistication, might be given over an extended period of time or in successive lessons. Most of the activities use simple materials. The activities are most suitable with students working in pairs. All these activities are suggested for use with circles of the *same size*. By using the same size circles, students can compare their results using their different methods. In practice, the similarity of results reinforces students' faith in the methods. At the same time, differences in results give the students opportunities to consider why variations occur and why some methods are more accurate than others. Students also become more aware of the inexactness of all measurements and the need for exercising care in measuring.

The activities are all based upon students' previous experiences with areas of simple geometric figures. The activities and the previous experiences upon which they are based are as follows:

Activity Number	Method	Based upon
1	Area by counting unit squares	Counting, unit squares and average
2	Area by comparison with inscribed and circumscribed squares	Linear measurement, area of square and average
3	Area by weighing	Counting, unit squares and larger squares
4	Comparison with areas of square and octagon	Linear measurement, area of squares and right triangles
5	Area by random numbers	Counting, random numbers, area of square, co-ordinates
6	Area by cutting circles into sectors	Linear measurement, area of parallelogram
7	Area by covering surface of circle and transferring to a rectangle	Linear measurement, area of a rectangle, radius of circle

By using areas of geometric figures with which students are already familiar as a means of approximating areas of circles, the mystifying concept of π is postponed until later when students have had the experiences and time to assimilate the *concept* of area of a circle. Activities that involve weighing and probability reinforce the *area* concept and demonstrate how widely different, seemingly irrelevant activities may be connected with a single concept.

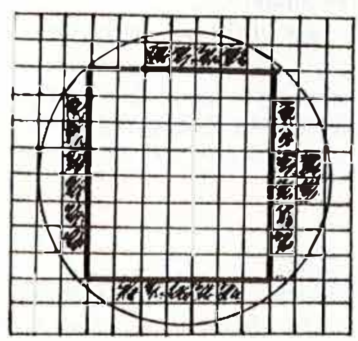
The last activity concludes with directions for students to calculate the area of a circle using the formula $A = \pi r^2$. By this time students should have acquired a better foundation and understanding of the *area* concept to enable them more sensibly to compare the measurements, approximations, and results with the area derived by formula.

1. Finding the Area of a Circle: Counting-Squares Method

Estimate the area of a circle by counting squares of size 1 cm^2 on squared paper.

Materials
Circular cake pan; centimetre squared paper.

Procedure
On a sheet of centimetre squared paper, trace the base of a circular cake pan.



Count the number of squares that lie *entirely within* the circle. *Hint:* Lightly shade the region. (This will make it easier to count other squares later.) It will also be faster and easier to count the squares entirely within the circle if you first draw the largest possible square or rectangle enclosing centimetre squares.

Number of squares entirely within circle:

______

Count the number of *additional* squares that touch the boundary (circumference) of the circle. These squares will be outside the shaded region that includes only the centimetre squares entirely inside the circle.

Number of squares on boundary of the circle: ______
Total number of squares either inside the circle or on boundary or the sum of the above two counts: ______

The number of squares *entirely inside* the circle represents an area less than the area of the circle. The total of squares that are either inside the circle or on the boundary represents an area larger than the area of the circle. Estimate the area of the circle by taking the *average* of the total count of squares covering the circle and the count of squares inside the circle.

Squares entirely inside circle (underestimate): ______
Squares either inside circle or on boundary (overestimate): ______
The average of the two estimates: ______

Why will the average of the under- and overestimates give a good estimate of the area of the circle?

2. Finding the Area of a Circle: Averaging Areas of Squares[1]

Estimate the area of the cake-pan circle by obtaining an average of the areas of two squares. One square, inscribed in the circle has an area smaller than the circle does. Another square, circumscribed about the circle, has an area larger than the circle does.

Length of the side of the small square:

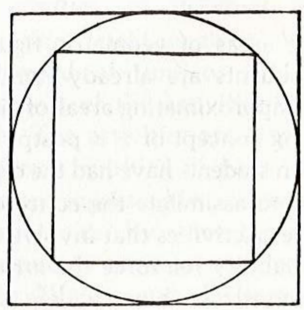
______

Length of the side of the large square:

______

Find the areas of the two squares. Then find the average of the two areas. Is this average a reasonable estimate of the area of the circle. Explain.

Compare your result with the area of the same circle found by other methods. Which method is most accurate?



[1]*Archimedes used a similar method, starting with hexagons and ending with polygons having 96 sides, to obtain an excellent approximation for pi.*

3. Finding the Area of a Circle: Weighing Method

Estimate the area of a circle by weighing.

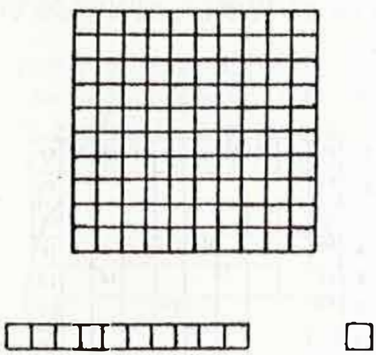
Materials

Cutouts of heavy sheet material or linoleum floor covering; balance scale.

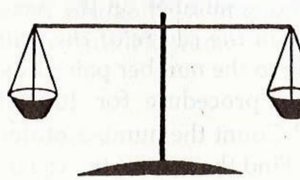
Procedure

Trace the cake pan and carefully cut the circle from the piece of linoleum. Cut pieces of linoleum with sizes as follows:

(a) 10 cm x 10 cm — 3 pieces
(b) 10 cm x 1 cm — 4 pieces
(c) 1 cm x 1 cm — 9 pieces



Place the circle on one pan of the balance. Now balance the scale by placing your other linoleum cutouts on the other pan until the two pans balance.



Determine how many square centimetres of linoleum pieces it takes to balance the pans.

Number of square centimetres to balance weight of circle: ________ cm^2

Reflection

Does the number of 1 cm squares give us a good estimate of the area of the circle?

Would it make any difference if we used different materials for the circle and the linoleum pieces?

Does it make sense to weigh to find area? Explain.

Compare your area result using this procedure with the resulting area from other procedures for the same size circle.

4. Finding the Area of a Circle: Egyptian Method[2]

Estimate the area of the circle by finding the area of an octagon (eight sides). Directions for constructing the octagon are given below.

Materials

Circle traced from cake pan.

Procedure

Trace the circle on a sheet of paper.

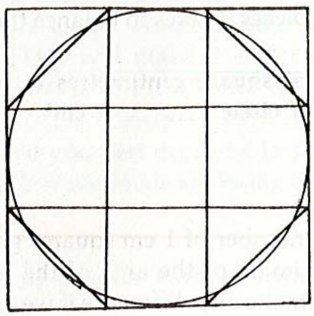
Draw a square circumscribing the circle. (All four sides touch the circle.)

Draw an octagon within the square as follows:

(a) Divide each side of the square into three equal parts.

(b) Connect the points of division to form an octagon as shown below.

[2] *It is believed that the ancient Egyptians used a similar method to obtain an approximate formula for the area of a circle.*



(c) The area of the octagon nearly equals the area of the circle.
(d) Draw horizontal and vertical line segments to divide the square into nine congruent squares as shown.

From the figure, note that the area of the circle appears to consist of five squares (excepting very small parts) and four right triangles (not counting very small parts). Would it make sense to find the area of the squares and triangles as a reasonable estimate of the area of the circular region?

If we join the four right triangles, we get two full squares. If we add these two squares to the other five squares, we have a total of seven squares as the estimate of the area of the circle. The area of the circle is about 7/9 of the area of the large square.

Measure the side of the square: ________ cm
Find the area of the square: ________ cm²
Find 7/9 of this area: ________ cm²

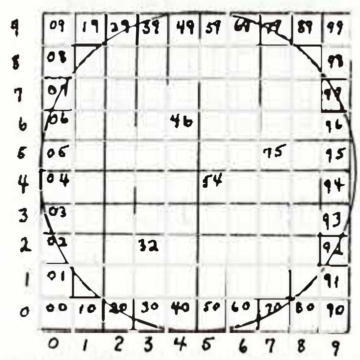
Compare your answer with results from other methods. Is one method more accurate than the other? Explain.

5. Finding the Area of a Circle: Random-Number Method

Estimate the area of a circle by using random numbers.

Materials
Circle; book of random numbers or telephone book.

Procedure
Use a tracing of the cake-pan circle and carefully circumscribe a square about the circle. Carefully divide the sides of the square into 10 equal parts. Connect the points of division by vertical and horizontal lines. You now have what looks like a grid for a graph. Mark the numerals from 0 to 9 on the bottom and left side of the square.



Now take any page of the telephone book at random. Take the last two digits of the first telephone number on the page. Make a clear dot *in the centre of the square* corresponding to the number pair just selected. Follow this procedure for 100 telephone numbers.[3] Count the number of dots inside the circle. Find the area of the square. Using

[3]*Note: If a point occurs in the same square twice, it counts twice! You can reduce the time considerably by having four groups plot 25 points each and pool the results.*

the ratio of squares with dots inside the circle to the ratio of squares on the entire grid (100), we can obtain an estimate of the area of the circle. For example, if there are 73 dots inside the circle and the area of the square is 400 cm^2, we estimate the area of the circle to be 0.73 x 400 or 292 cm^2.

Now continue as follows:

Number of points entirely within circle: ________

Total number of points plotted: ________

Measure the side of the square: ________ cm

Find the area of the square: ________ cm^2

What decimal fraction of the area of the square should you take to get a reasonable estimate of the area of the circle? ______

Compute the area of the circle: ______ cm^2

Compare your result with your results for the same circle obtained by other methods. Which method is *most* accurate? Why? Which method is *least* accurate?

6. Finding the Area of a Circle: by Transformation into a Parallelogram

Estimate the area of a circle by cutting up a circle and reshaping it into a figure that looks like a parallelogram.

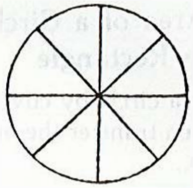
Materials

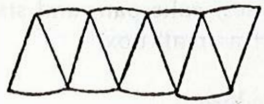
Cake-pan circle.

Procedure

Divide the circle into eight congruent parts (pie pieces).

Cut out the eight parts and assemble them to form a "parallelogram" as shown below.





Of course, a figure with arcs is not a parallelogram, but it approximates a parallelogram closely enough so that we can use the area of a parallelogram as a reasonable estimate of the area of the circle.

The base of the parallelogram consists of four arcs. The four arcs make up half of the circumference of the circle.

The height of the parallelogram is not constant. However, the length of the radius would be a reasonable estimate of the height.

To find the area of this figure, measure the length of the base and use the radius for the height.

Measure the length of the base: ________ cm

Measure the radius length: ________ cm

Calculate the area of the parallelogram:

Area = base x height or

A = ______ cm x ______ cm = ______ cm^2

Compare the result with what you obtained for the area of the circle by using other methods.

Which method seems most accurate? Why? Which method seems least accurate? Why?

If you cut the circle into 16 parts and formed a parallelogram, would your answer be more or less accurate for the area of the circle?

7. Finding the Area of a Circle: Transferring to a Rectangle

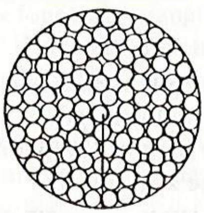
Estimate the area of a circle by covering it with marbles and then transfer the marbles to a rectangular box.

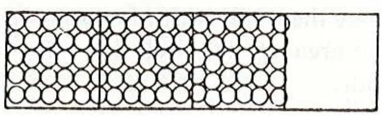
Materials

Marbles, cake pan, and stiff sheet to construct a small box.

Procedure

For this activity, we require a shallow container with a circular base (cake pan) and a shallow rectangular box with length equal to four times the radius of the circle and width equal to the radius of the circle. A sturdy box can be made with stiff sheeting. The box should have a depth sufficient to contain a layer of marbles, beans, or some other uniform material. This method of estimating the area of the circle will consist of loading a layer of marbles on the circular cake pan just enough to cover as much of the base of the cake pan as possible, and then transferring the marbles to the box also in a single layer. The marbles will not make a complete layer pack in the box.





We can *measure* the length of the rectangular region covered by the marbles. We *know* the width is the length of the radius. When performing this activity, answer the following questions:

What is the length of the rectangle? ______ cm

What is the measure of the radius or height of the rectangle? ______ cm

Compute the area of the rectangle: ______ cm^2. This, of course, gives the area of the circle as well.

Compare this result with the estimate obtained by using other methods for finding the area of the same circle. Which method do you think is best?

Take another look at the box containing the marbles. Is the length of the rectangle covered by the marbles a little more or a little less than three times the length of the radius?

If you draw squares in the box as shown in the diagram, what is the area of each square? ______ cm^2 Each square has an area equal to the square of the radius. That is, each square has area r^2. About how many r-squares are covered with the marbles?

The usual formula for the area of a circle is $A = \pi \times r^2$, approximately $A = 3.14 \times r^2$. Use this formula to calculate the area of the circle. Compare all your various results with this number. Which method gave an answer closest to the result by formula?

Compare this strategy for estimating the area of a circle with the strategy of using a parallelogram. If a great number of very thin congruent sectors are used, the parallelogram would look more like a rectangle. Do you see that the two methods would become very much the same, using a rectangle (parallelogram) with width (height) equal to the radius of the circle and length (base) equal to $\pi \times r$ (one-half the circumference)?

Notes for Teachers

Have various circles, grids, etc., drawn ahead of time. Make copies for students. Examples are as follows:

Activity 1:
Have copies of cake-pan tracing on centimetre-squared paper already prepared for distribution.

Activity 2:
A copy of the same size circular region with the inscribed and circumscribed squares drawn will save time and eliminate confusion and improperly drawn squares.

Activity 4:
Copies of an octagon drawn on the circle of the same size as earlier activities will eliminate the problem of making a proper octagon. For most students, dividing the region into nine equal parts is not trivial.

Activity 5:
A copy of the circle with 100 squares numbered will eliminate problems and get the students onto the task of estimating area.

Activity 6:
Prepare copies of same-size circle split into eight regions for immediate cutting with scissors.

Activity 7:
Make a rectangular box in advance with length equal to four times the radius and width equal to the radius of the cake pan. Draw r-squares in the box in advance.

The Powers That Be

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For what values of a and b (a, b positive), is $a^b > b^a$? An interesting investigation for mathematics teachers. —Editor

Which is the larger of the two numbers e^π and π^e? This was one of the problems in the Fourth Annual Mathematics Contest of Atlantic Math Days, which was held at the University of Moncton in October 1980. In this article, I would like to investigate the more general problem; that is, for what values of a and b (a, b positive), is $a^b > b^a$?

A quick look at the pairs 2^3 and 3^2, 2^4 and 4^2, 3^4 and 4^3, is enough to convince one that the solution is not simply $a^b > b^a$ for $a > b$, or $a^b > b^a$ for $b > a$.

Next, since $a^b < 1$ for $0 < a < 1$, b positive, whereas $a^b > 1$ for $a > 1$, b positive (see Figure 1 and Figure 2), a subdivision of the values of a and b at the number one is reasonable in our investigations.

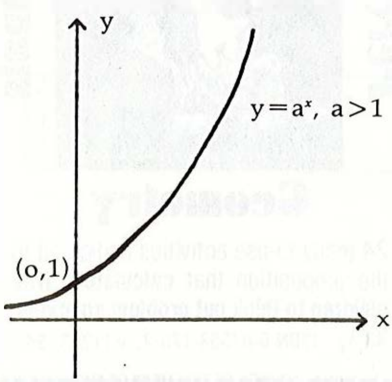

Figure 1

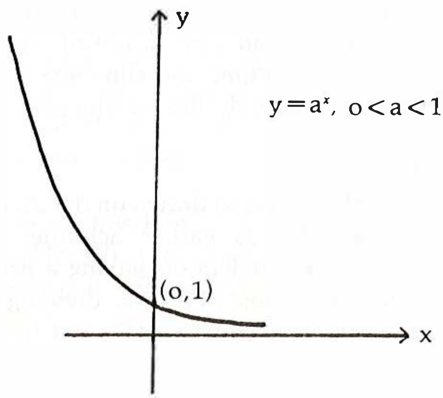

Figure 2

This leads to the following table (Figure 3) in which the entries are easy to make and to justify except in the two cases that are left blank and that will be investigated in detail below.

b \ a	$0<a<1$	$a=1$	$a>1$
$0<b<1$		$a^b>b^a$	$a^b>b^a$
$b=1$	$a^b<b^a$	$a^b=b^a$	$a^b>b^a$
$b>1$	$a^b<b^a$	$a^b<b^a$	

Figure 3

Case I

For $0<a<1$, $o<b<1$, which is the greater of the two numbers a^b and b^a?

In this case, the inequality, $a^b>b^a$, is equivalent to the inequality $\frac{b}{a} < \log_a b$, as can be seen by taking logarithms to the base a of $a^b>b^a$ (reversing the inequality sign, since $y = \log_a x$, $o<a<1$, is a decreasing function) and finally dividing the result by a.

Considering the functions $y = \frac{x}{a}$, $x>o$ and $y = \log_a x$, $x>o$ where $o<a<1$ (Figure 4) and noting that the point (a, 1) satisfies both equations, the following result is obtained.

Results:

$a^b>b^a$, for $o<b<a<1$

$a^b<b^a$, for $o<a<b<1$

and trivially, $a^b=b^a$, for $a=b$

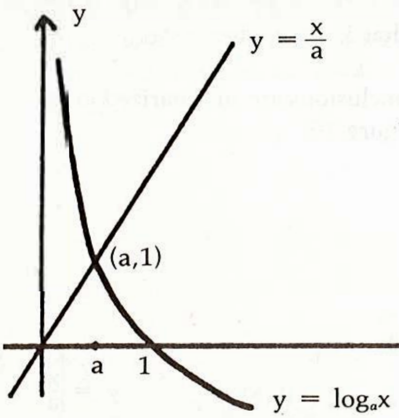

Figure 4

This result can now be filled into the first blank left in the table of Figure 3 (see Figure 5).

b \ a	$o<a<1$
$o<b<1$	$a^b>b^a$, $b<a$ $a^b=b^a$, $b=a$ $a^b<b^a$, $b>a$

Figure 5

Case II

For $a>1$ and $b>1$, which is the greater of the two numbers a^b and b^a?

Now the inequality, $a^b>b^a$, is equivalent to the inequality, $\frac{b}{a} > \log_a b$. (Note the difference in the sign of the inequality from Case I due to the fact that $y = \log_a x$ is an increasing function for $a>1$.)

Again we consider the functions $y = \frac{x}{a}$, $x>o$ and $y = \log_a x$, $x>o$ and note that the point (a, 1) satisfies both equations (Figure 6 and Figure 7).

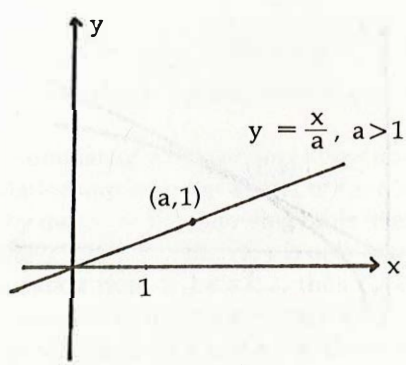

Figure 6

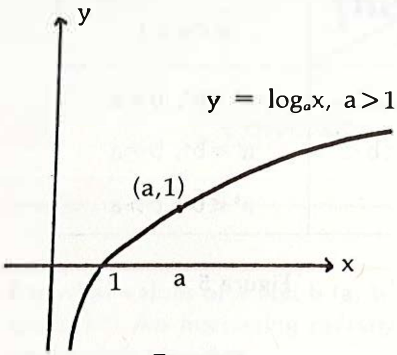

Figure 7

If we superimpose these two curves and realize that $\frac{x}{a} > \log_a x$ for x large (L'Hôpital's Rule), it can be seen that either Case II (a), $y = \frac{x}{a}$ is a tangent line to $y = \log_a x$ at the point (a, 1), or Case II (b), $y = \frac{x}{a}$ is a secant line to $y = \log_a x$ and there is a second point, with x-co-ordinate r, say, which is common to the two curves. (There are at most two intersection points since $y = \log_a x$ is always concave downward and $y = \frac{x}{a}$ is a straight line.)

Let us treat these two cases separately.

Case II (a)
In the tangent situation, the slopes of $y = \frac{x}{a}$ and $y = \log_a x$ must be equal at the point (a, 1). That is, $\frac{1}{a} = \frac{1}{a\ln a}$; hence $\ln a = 1$ and $a = e$ (see Figure 8).

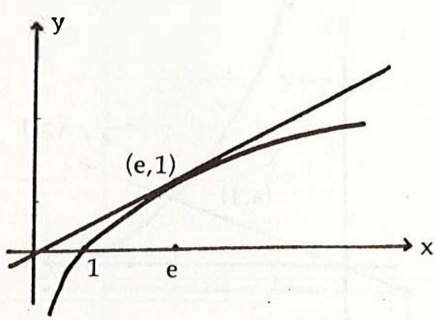

Figure 8

Result:
$a^b > b^a$ for $a = e$, b positive and $b \neq e$ and trivially $a^b = b^a$ for $a = b = e$.

This proves that $e^\pi > \pi^e$ (the contest problem mentioned earlier).

Case II (b)
Here $a>1$, $b>1$, $a \neq e$ and the curves $y = \frac{x}{a}$, $x>o$ and $y = \log_a x$, $x>o$ intersect at the two points (a, 1) and $(r, \frac{r}{a})$.

At the point (a, 1), the slopes of $y = \frac{x}{a}$ and $y = \log_a x$ are respectively $\frac{1}{a}$ and $\frac{1}{a\ln a}$.
If $1<a<e$, then $o<\ln a<1$ and $\frac{1}{a\ln a} > \frac{1}{a}$;
whereas if $a>e$, then $\ln a>1$ and $\frac{1}{a\ln a} < \frac{1}{a}$.
Therefore, if $1<a<e$, then $a<r$, but if $a>e$, then $a>r$.

Also, if $1<a<e$, then $\log_a e > \frac{a}{e}$ and therefore $r>e$. In the same way, it can be shown that if $a>e$, then $r<e$.

These conclusions are summarized in Figure 9 and Figure 10.

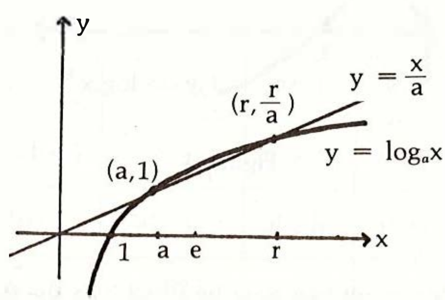

Figure 9

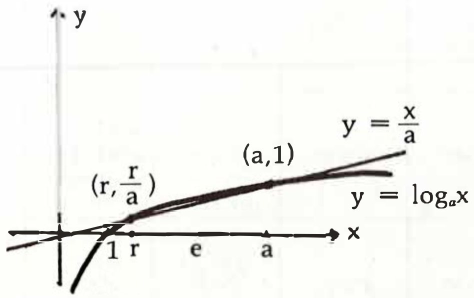

Figure 10

Result:
If $1<a<e$, then $a^b>b^a$ for $b<a$ or for $b>r$ and $a^b<b^a$ for $a<b<r$; whereas if $a>e$, then $a^b>b^a$ for $b<r$ or for $b>a$ and $a^b<b^a$ for $r<b<a$ ($a^b=b^a$ if $b=a$ or if $b=r$).

If r were known, these results would complete our investigations. However, to find r, in most cases, a numerical methods technique is required. There is one case, however, when r is known. Since $2^4=4^2$, then for $a=2$, $r=4$ (or for $a=4$, $r=2$). See Figure 11.

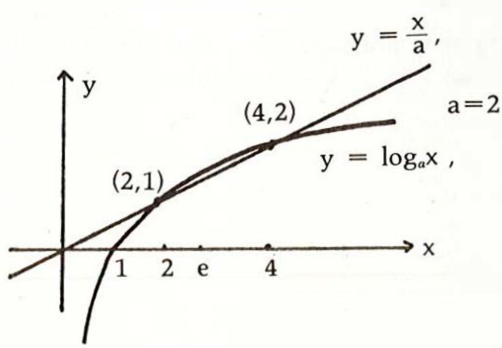

Figure 11

Comparing Figures 9, 10, 11 causes one to ask the following questions: If $1<a<2$, is $r>4$? If $a>4$, is $1<r<2$? If $2<a<e$, is $e<r<4$? If $e<a<4$, is $2<r<e$?

The answer to all four questions is yes. Let us justify one of the four here, say, if $1<a<2$, then $r>4$. The others can be done similarly.

Lemma:
If $1<a<2$, the curves $y = \log_a x$, $y = \frac{x}{a}$ that intersect at the points (a, 1) and $(r, \frac{r}{a})$ are such that $r>4$.

Proof:
It is sufficient to show that $\log_a 4 > \frac{4}{a}$ for $1<a<2$ (see Figure 12)

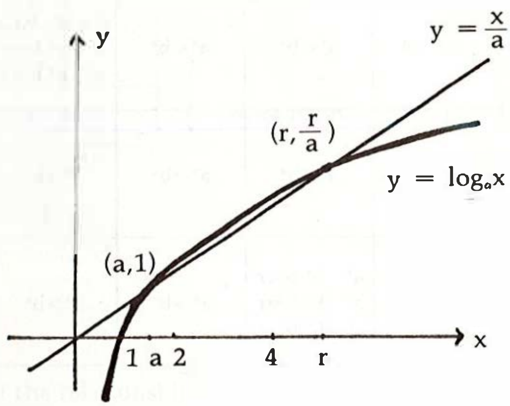

Figure 12

or since $\log_a 4 = \frac{1}{\log_4 a}$, that $\log_4 a < \frac{a}{4}$ for $1<a<2$ and this is evident from Figure 11.

Let us summarize what we now know about the relationship between a^b and b^a for $a>1$, $b>1$ by means of the following table (Figure 13), where, as shown above, r is only known to the extent that, if $1<a<2$, then $r>4$; if $a=2$, then $r=4$; if $2<a<e$, then $e<r<4$; if $e<a<4$, then $2<r<e$; if $a=4$, then $r=2$; and if $a>4$, then $1<r<2$.

$b \backslash a$	$1<a<1$	$a=2$	$2<a<e$	$a=e$	$e<a<4$	$a=4$	$a>4$
$1<b<2$	$a^b>b^a, b<a$ $a^b=b^a, b=a$ $a^b<b^a, b>a$	$a^b>b^a$	$a^b>b^a$	$a^b>b^a$	$a^b>b^a$	$a^b>b^a$	$a^b>b^a, b<r$ $a^b=b^a, b=r$ $a^b<b^a, b>r$
$b=2$	$a^b<b^a$	$a^b=b^a$	$a^b>b^a$	$a^b>b^a$	$a^b>b^a$	$a^b=b^a$	$a^b<b^a$
$2<b<e$	$a^b<b^a$	$a^b<b^a$	$a^b>b^a, b<a$ $a^b=b^a, b=a$ $a^b<b^a, b>a$	$a^b>b^a$	$a^b>b^a, b<r$ $a^b=b^a, b=r$ $a^b<b^a, b>r$	$a^b<b^a$	$a^b<b^a$
$b=e$	$a^b<b^a$	$a^b<b^a$	$a^b<b^a$	$a^b=b^a$	$a^b<b^a$	$a^b<b^a$	$a^b<b^a$
$e<b<4$	$a^b<b^a$	$a^b<b^a$	$a^b<b^a, b<r$ $a^b=b^a, b=r$ $a^b>b^a, b>r$	$a^b>b^a$	$a^b<b^a, b<a$ $a^b=b^a, b=a$ $a^b>b^a, b>a$	$a^b<b^a$	$a^b<b^a$
$b=4$	$a^b<b^a$	$a^b=b^a$	$a^b>b^a$	$a^b>b^a$	$a^b>b^a$	$a^b=b^a$	$a^b<b^a$
$b>4$	$a^b<b^a, b<r$ $a^b=b^a, b=r$ $a^b>b^a, b>r$	$a^b>b^a$	$a^b>b^a$	$a^b>b^a$	$a^b>b^a$	$a^b>b^a$	$a^b<b^a, b<a$ $a^b=b^a, b=a$ $a^b>b^a, b>a$

Figure 13

As was mentioned above, to pinpoint r more closely, one would need to use numerical analysis. The table below (Figure 14) gives consecutive integer bounds for r, for certain values of a. Such values could be used to start an iterative procedure that would evaluate r to the desired accuracy.

a	bounds for r
1.1	43, 44
1.2	19, 20
1.3	12, 13
1.4	9, 10
1.5	7, 8
1.6	6, 7
1.7	5, 6
1.8	4, 5
1.9	4, 5

a	bounds for r	
2	r = 4	
2.1	3, 4	
2.2	3, 4	
2.3	3, 4	
2.4	3, 4	
2.5	2, 3	better bounds are e and 3
2.6	2, 3	
2.7	2, 3	
2.8-3.9	2, 3	better bounds are 2, e

Figure 14

An interesting side result of this investigation of the relationship between a^b and b^a can also be drawn from our knowledge of r.

Theorem:
The only solution to the equation $x^y = y^x$ in which both x and y are positive integers and $x \neq y$ is the solution $x = 2$, $y = 4$ (or $x = 4$, $y = 2$).

Can Teachers Develop the Ability To Think on Their Feet?

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All teachers should have the ability to answer questions from their students while "thinking on their feet." This paper is an attempt to help teachers develop such competence. —Editor

Questions play an important role in teaching. But we seldom realize just how important questions are. Research indicates that four-fifths of school time is occupied with question-and-answer recitation. Thus, in light of their sheer frequency of occurrence, questions form a substantial part of a teacher's instructional repertoire.

Research literature gives the following criteria for a "good" question:

> A good question should stimulate thought, lead to inquiry, and result in understanding and mastery. A good question may be judged by how much thought response it arouses, by the discussion interest that people show, and by the expression of thought in an adequate manner showing that the answer has been assimilated.

Do teachers ask good questions? Research indicates that they do not. Most questions teachers ask fall into the knowledge, memory, or recall category. Such questions do not meet the above criteria. Let us direct our attention to types of questions teachers *should* ask rather than to questions they currently ask.

In teaching methodology classes several years at the University of Regina, I found that student teachers had great difficulty answering mathematical questions their pupils raised. I began to collect the kinds of questions that puzzled my university students, as well as teachers in service.

This paper is a brief sampling of my compilation of those questions, together with suggested solutions.

Questions are placed in two broad categories as follows:
Type I are questions that teachers usually face when teaching specific topics (content questions).
Type II are questions that occur occasionally, but are difficult for many mathematics teachers to answer (incidental questions).

Most questions have come from actual classes. No attempt is made to look for the so-called best response. Seeking answers to these questions can be valuable for teachers.

Question #1 (Type II)
A student hands in the following solution to this problem.

"Factor $15\,x^2 - 31\,x + 10$"

$$15 \times 10 = 150$$
$$25 \times 6 = 150$$

Note: To have two parts of 150 such that their sum or difference is −31

$$15\,x^2 - 31\,x + 10$$

$$\rightarrow 15\,x^2 - 25\,x - 6\,x + 10$$
$$\rightarrow 5 \cdot (3x - 5) - 2(3x - 5)$$
$$\rightarrow (5x - 2)(3x - 5)$$

Is this method correct? Would it always work? Can you justify?

Answer #1
Yes, this method will always work. In fact this technique takes the guesswork out of factoring trinomials by trial and error. Motivation for this method goes back to the sum and product of roots of a general quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.

Question #2 (Type II)
A student hands in the following solution to this problem. "Solve the following equation by factoring.

$$4\,x^2 - 23\,x + 15 = 0"$$
$$4\,x^2 - 23\,x + 15 = 0$$
$$\acute{x}^2 - 23\,\acute{x} + 60 = 0$$
$$(\acute{x} - 20)(\acute{x} - 3) = 0$$
$$\acute{x} = 20 \text{ or } \acute{x} = 3$$

Therefore

$$x = \frac{20}{4} = 5 \text{ or } x = \frac{3}{4}$$

Is this method correct? Will it always work? Can you justify?

Answer #2
Yes, this method will always work.

Justification:
Let $ax^2 + bx + c = o$, $a \neq o$ be the general quadratic equation in variable x.

Using the quadratic formula, we have

$$x = \frac{-b \pm \sqrt{b^2 - 4aC}}{2a} \qquad (1)$$

Using the student's method, let $\acute{x}^2 + b\,\acute{x} + ac = o$ be the general quadratic equation in variable $\acute{x}$.

Using the quadratic formula, we have

$$\acute{x} = \frac{-b \pm \sqrt{b^2 - 4 \cdot 1 \cdot ac}}{2} \qquad (2)$$

Comparing (1) and (2) we have $\left(\frac{\acute{x}}{a}\right) = x$

Now consider
$4\,x^2 - 23\,x + 15 = 0 \; . \; . \; . \; .$ (A)

Change $x \rightarrow \frac{x}{4}$ in (A)

We have:

$$4\left(\frac{\acute{x}}{4}\right)^2 - 23\left(\frac{\acute{x}}{4}\right) + 15 = 0$$

$$\frac{\acute{x}^2}{4} - 23\,\frac{\acute{x}}{4} + 15 = 0$$

$$\acute{x}^2 - 23\,\acute{x} + 60 = 0$$

$$(\acute{x} - 20)(\acute{x} - 3) = 0$$

$$\acute{x} = 20 \text{ or } \acute{x} = 3$$

$$\text{i.e. } x = \frac{20}{4} \text{ or } x = \frac{3}{4}$$

$$x = 5 \text{ or } x = \frac{3}{4}$$

Question #3 (Type II)
One of your students asks the following question.

"Is there a difference between *equal* and *equivalent* in mathematics?" How would you answer?

Answer #3
Yes, there is a difference between *equal* and *equivalent* in mathematics. Usually *equal* is used to mean that whatever is on one side of the equals sign names the same object as that on the other side of the equals sign. Thus, $1 + 2 = 3$ means that "$1 + 2$" and "3" are names for the same number. Thus, we say that the expressions "$1 + 2$" and "3" are equivalent expressions, but the numbers that they stand for are equal. The expressions themselves are not equal since they are not identical expressions. Likewise, the numerals 5/2, 2½, 2.5 and 250% are equivalent numerals since they name the same number. Thus the numerals are equivalent but not equal; i.e., the symbol 5/2 is not the same symbol as 2½, etc., but the numbers represented by these numerals are equal.

Equivalency and equality are related but not the same concept. Equivalency is in many instances a generalization of equality and usually means that objects share a common property or attribute. Thus if $A = \{1,2,3\}$ and $B = \{3,4,5\}$, then A is equivalent to B since there is a one-to-one correspondence between the elements of A and B. If $C = \{2,3,1\}$, then we say that A is equal to C since they contain exactly the same elements. Thus, in this illustration with sets, we can say that if two sets A and B are equal then they are equivalent, but the converse statement need not be true.

As another illustration, consider the following equation: $3y^2 + 5y + 4y + y^2 = 4y^2 + 9y$. If we replace y by any member of its replacement set, we have a true statement. Thus the members of this equation are called equivalent expressions. For any value of y in the replacement set, the equivalent expressions represent the same number, and thus the equation is called an identity. The expressions "$3y^2 + 5y + 4y + y^2$" and "$4y^2 + 9y$" are not equal expressions; i.e., the same expressions.

As a last example, consider the statements $p \rightarrow q$ and $\sim p \vee q$ where p and q are statements. These expressions are clearly not the same expressions. However, if a truth table is made for each expression, the truth tables are identical, i.e., equal, and thus the expressions are called logically equivalent statements.

Space prevents us from considering other examples in detail, but it could be a very interesting exercise for the reader to go through secondary mathematics material to find many other different places where the idea of equivalence is used. The following is an incomplete listing to start you off:

Equivalent expressions
Equivalent equations
Equivalent inequalities
Equivalent rational expressions
Equivalent systems of equations
Logically equivalent statements
Equivalent triangles
Equivalent polygons
Equivalent vectors
Equivalent matrices
Equivalent polynomials
Equivalent fractions
Equivalence classes

The name *equivalent* comes from the notion that when objects are equivalent, this relation usually turns out to be an equivalence relation. In many instances, the distinctions concern the difference between an object (equality) and the names of the objects (equivalence).

Reference

Walter Sanders, "Equivalence and Equality," *The Arithmetic Teacher,* April 1969, pp. 317-322.

Question #4 (Type I)

You make the statement that $.\overline{9} = 1$. Some of your students protest, saying that the above statement is impossible because you have nines all the time after the decimal point. How do you handle this?

Answer #4

The usual method seen in most algebra texts is:

$$\begin{aligned} \text{Let}\quad N &= .9999\ldots. \\ 10N &= 9.9999\ldots. \\ 9N &= 9.0000\ldots. \\ N &= 1. \end{aligned}$$

So, .9999 . . . . is 1. Ask the students emphatically, "Have I made a mathematical mistake?" The usual answer from the students is "no, but . . . ."

Now, tell the students that the following arguments are not proofs but good intuitive reasonings.

(a) $\frac{1}{9} = .111111\ldots$

Since

$$\begin{array}{r|l} & .1111\ldots \\ \hline 9 & 10000 \\ & \underline{9}\downarrow \\ & 10 \\ & \underline{9}\downarrow \\ & 10 \\ & \underline{9}\downarrow \\ & 10 \\ & \underline{9} \\ & 1 \end{array}$$

Similarly,

$\frac{2}{9} = .222222\ldots$

$\frac{3}{9} = .333333\ldots$

$\frac{4}{9} = .444444\ldots$

$\frac{5}{9} = .555555\ldots$

$\frac{8}{9} = .888888\ldots$

Then say, "Intuitively speaking, then, what is .9999 . . .?" Students acknowledge that the answer is $\frac{9}{9} = 1$

(b) Ask students if they agree that

$\frac{4}{9} = .4444\ldots$ and

$\frac{5}{9} = .5555\ldots$

When they agree, then ask if you might add equals to equals. Adding, we get:

$\frac{9}{9} = .9999\ldots$, or $1 = .9999\ldots.$

(c) Sometimes a very good student will suggest that if .9999 . . . does equal 1 or $\frac{9}{9}$, then you should be able to change $\frac{9}{9}$ to .9999 . . . by dividing the denominator into the numerator as follows:

$$\begin{array}{r|l} & .9999\ldots \\ \hline 9 & 9.0000 \\ & 8.1\downarrow \\ & .90 \\ & \underline{.81}\downarrow \\ & .090 \\ & \underline{.081}\downarrow \\ & .0090 \\ & \underline{.0081} \\ & .0009 \end{array}$$

(d) .9999 . . . can be written as the sum of an infinite geometric progression as follows:

$$
\begin{aligned}
.9 &= \frac{9}{10} \\
.09 &= \frac{9}{10^2} \\
.009 &= \frac{9}{10^3} \\
.0009 &= \frac{9}{10^4} \\
.00009 &= \frac{9}{10^5} \\
.000009 &= \frac{9}{10^6} \\
&\vdots \qquad \vdots
\end{aligned}
$$

So .9999 . . .

$$= \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \frac{9}{10^5} + \ldots$$

$$= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \ldots\right)$$

$$= \frac{9}{10}\left(\frac{1}{1-\frac{1}{10}}\right)$$

$$= \frac{9}{10}\left(\frac{1}{\frac{9}{10}}\right)$$

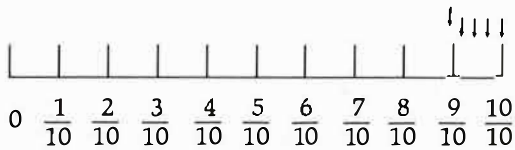
$$= \frac{9}{10} \cdot \frac{10}{9} = 1$$

Note: Since sum $= \frac{9}{1-\gamma}$

for $s = a + a\gamma + a\gamma^2 + \ldots$

$\gamma < 1$

A natural follow-up at this stage is introducing the concept of limit as shown below:



Note that the sum approaches 1 as a limit.

A problem of this type has endless opportunities for teachers to lay the foundations of high-level concepts such as geometric progressions, limits, etc.

References

Lucien T. Hall, "Persuasive Arguments: .9999 . . . = 1," *The Mathematics Teacher,* December 1971, pp. 749-750.

W. Peterson, "Beware of Persuasive Arguments," *The Mathematics Teacher,* December 1972, p. 709.

Question #5 (Type I)

"Why isn't 1 a prime number?"

Answer #5

(a) At one time in history, the number 1 was considered to be a prime number. If 1 is allowed to be prime, then the uniqueness property of the Fundamental Theorem of Arithmetic is violated. Consider factoring 72 into a product of primes:

$$72 = 3^2 \cdot 2^3$$

However, if 1 is considered a prime, then one can write

$$72 = 3^2 \cdot 2^3 \cdot 1$$
$$3^2 \cdot 2^3 \cdot 1^2, \text{ etc.,}$$

and the uniqueness is lost.

(b) Another way of answering this question is to ask students to write factor sets of natural numbers from 1 to 100 and categorize them into the following table:

F [n] means factor set of a natural number n. Consider:

F [1] = {1} Since $1 = 1 \times 1$
F [2] = {1,2} Since $2 = 1 \times 2$

I.e., 1 and 2 are

A	B	C
Exactly one factor	Exactly two factors	More than two factors
1	2, 3, 5, 1, 11, . . .	4, 6, 8, 9, 10, 12, . . .

F [3] = {1,3} Since $3 = 1 \times 3$
F [4] = {1,2,4} Since $4 = 1 \times 4$, $4 = 2 \times 2$

Similarly,
F [5] = {1,5}
F [6] = {1,2,3,6}
F [7] = {1,7}
F [8] = {1,2,4,8}
F [9] = {1,3,9}
F [10] = {1,2,5,10}
⋮

We are now ready to make the following definitions:

(1) A natural number n is a prime number if and only if the factor set of n has exactly two elements in it.

(2) A natural number is a composite number if and only if the factor of n has more than two elements in it.

This approach helps the student understand why 1 is not prime. Category B is the set of all prime numbers, Category C contains all the composite numbers, and 1 in Category A stands all by itself.

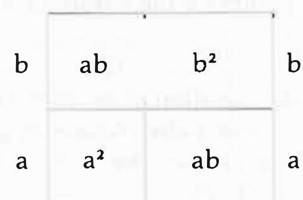
Question #6 (Type I)
In your Grade 9 class, a student hands in the following solution. "$(a + b)^2 = a^2 + b^2$ because, except for the operation's being different, this problem is just like $(a \cdot b)^2$, which is equal to $a^2 \cdot b^2$. Therefore, it follows that $(a + b)^2 = a^2 + b^2$." How would you help this student?

Answer #6
Two different ways to explain this appear below:

(a) If $(a + b)^2 = a^2 + b^2$, then this equality should hold for all values of a and b.

Let $a = 3$ and $b = 4$, then $(a + b)^2 = (3 + 4)^2 = 7^2 = 49$, while $a^2 + b^2 = 3^2 + 4^2 = 25$, and $49 \neq 25$.

(b) A geometric consideration of $(a + b)^2$ could also help students. $(a + b)^2$ is the area of the square shown below:



It is easily seen that

$(a + b)^2 = a^2 + ab + ab + b^2$

$\qquad = a^2 + 2\,ab + b^2$

Therefore,

$(a + b)^2 \neq a^2 + b^2$

Question #7 (Type II)

A student asks, "What is the difference between a quadratic function and a quadratic equation?" How would you answer?

Answer #7

A quadratic function f over the set R of real numbers is a function whose domain is R and whose values are given by

$y = f(x) = ax^2 + bx + c$

Where a, b, $c \epsilon R$ and $a \neq o$. That is,

$f = \{(x, y) \mid y = ax^2 + bx + c,\ a \neq o$ and $a, b, c \epsilon R\}$

A quadratic equation is given by $ax^2 + bx + c = 0$ where a, b, $c \epsilon R$ and $a \neq o$. That is a quadratic equation consists of all ordered pairs

$\{(x, o) \mid ax^2 + bx + c = o,\ a \neq o$ and $a, b, c \epsilon R\}$

Let's consider $y = x^2 - 1$ over the set R of real numbers.

The graph of $y = f(x) = x^2 - 1$ consists of all the points on the parabola as shown in Figure 1.

Whereas the equation $x^2 - 1 = 0$ consists of precisely those points where the graph of the parabola crosses the x-axis. That is, (1, 0) and (−1, 0).

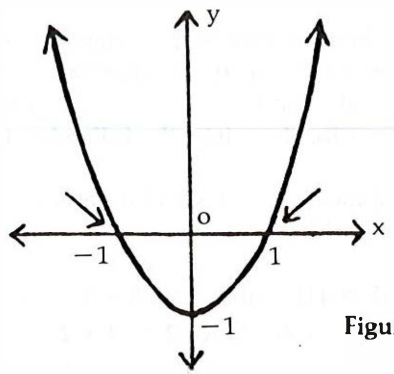


Figure 1

Question #8 (Type II)

A student asks, "Are all non-terminating decimals irrational numbers?" What reply would you give?

Answer #8

This question has the potential of characterizing rational and irrational numbers as:

1. γ is rational $\leftrightarrow$ γ has an infinite periodic decimal representation.
2. α is irrational $\leftrightarrow$ α has an infinite non-periodic decimal representation.

For example, .323232 . . . is non-terminating, but it represents the rational number 32/99 in fractional form.

1.010010001500001205007 . . . is also non-terminating, but represents an irrational number.

Question #9 (Type II)

How do you explain that division by zero is not defined? What about 0/0?

Answer #9

Recall the definition of a divisor:

Definition:

b divides a $\leftrightarrow$. There exists a unique number c such that $a = bc$

i.e., $\frac{a}{b} = c \quad \leftrightarrow a = b\,c$

$\frac{12}{4} = 3 \quad \leftrightarrow 12 = 4 \times 3$

Now consider the following cases:

Case #1

$a \neq o,\ b = o$

$\frac{a}{o} = \square \leftrightarrow a = o\square$

Obviously there is no solution for $a = o\square$ in the set of real numbers.

Case #2

$a = o,\ b = o$

$\frac{o}{o} = \square \leftrightarrow o = o\square$

$o = o\square$ has infinite numbers of solutions, since any real number will satisfy this equation.

So, both cases violate the above definion, hence are not allowed.

Microcomputers in the Elementary Mathematics Classroom

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This article surveys some ways to use computers in elementary mathematics. Computers are a powerful teaching tool, and we teachers should use them to improve mathematics learning. —Editor

In the classroom, the computer can be:
1. an *object* of instruction
2. an *aid* to instruction
3. a *medium* of instruction

Object of instruction
The students learn about the computer itself—what it can do, how it affects us, etc. This article will not deal with computer literacy directly.

Aid to instruction
The teacher uses the computer as an aid in teaching a subject. This article will deal with several uses of the computer to aid instruction.

Medium of instruction
The computer does the teaching in tutorials and the like. This article will deal with some mathematics material available in this area.

The computer is being used as a teaching aid or as a tool much like other pieces of A-V equipment that have been at the disposal of teachers. We have a powerful device; the question is how can we make good use of it to improve mathematics learning.

The teacher can use one or more computers in the following ways:
1. Computer-assisted instruction
2. Student programming
3. Problem-solving
4. Games and simulations

Computer-assisted instruction
Computer-assisted instruction could probably include all of the categories mentioned above, as the computer is assisting in the teaching of mathematics. However, for this article, computer-assisted instruction includes (a) drill and practice, (b) tutorials, and (c) classroom management.

Drill and practice is the easiest application of a microcomputer. Most of the original software produced fell into this category. The material for drill and practice is reasonably easy to write, but it does not create a great deal of interest with students. Drill and practice on a computer does, however, have two positive aspects: (a) the student generally receives immediate feedback on tasks completed, and (b) the computer demonstrates a great amount of patience, as it will wait any length of time for responses.

Two packages are approved by the Alberta Provincial Clearinghouse: Millikan Math Sequences and SRA Computer Drill and Instruction. They are both listed in the School Book Branch Catalogue. Both packages have a heavy emphasis on drill; however, both have an instruction or tutorial component. The concept is explained, and examples are given if the student requests help. There are some problems with these packages: (a) they are expensive, (b) they require one machine per student for extended periods of time, and (c) they deal mainly with numeration and operations.

Records and management include markbook programs, spreadsheets, attendance records, item banks, inventories, and objective management systems. Computers (micros to mainframes) do the clerical tasks that normally consume a teacher's time or that just by their magnitude are impossible to do without a computer. Several markbook programs on the market run on microcomputers and keep student records very well. Spreadsheet programs such as Visicalc can also be used to keep records.

Larger minicomputers can be very useful in management. The CML Q-Math project is an example of this. An objective-based mathemathics item bank is being piloted in several Calgary Board of Education schools. It also has a record-keeping program as part of the CML programs.

Two microcomputer-based objective managed mathematics packages are also being piloted in Calgary Board of Education schools. West Dalhousie Elementary School is using the Holt ECCO program, and Marlborough Elementary School is using the Ginn Computer Managed Math Program. Both programs are tied to the respective elementary mathematics series. The computer scores tests, keeps track of student progress, and suggests or prescribes additional work for any deficiencies.

Student programming

Students can begin to write procedures or programs in elementary school. They can use either LOGO or BASIC. LOGO is being introduced to children as young as Kindergarten. Children begin with a concrete situation and move to a more formal situation involving the computer. Robots can be used at this level so that the situations are kept very concrete. Older students can write more elaborate programs using LOGO. Each program is developed in a problem-solving mode. LOGO begins with graphics, which are of great interest to students in this age group.

BASIC should be used sparingly in elementary schools. It is often criticized because its format can quickly lead to bad structure and bad programming habits. However, it is present in virtually every machine and is easy to learn and use at a very simple level.

Problem-solving

Writing a program can be thought of as solving a problem, because one should go through all of the steps as outlined in Polya's Model of solving a problem.

Understand the problem
Make a plan
Carry out the plan
Look back
Plan the program
Write and run the program
Revise the program

Using LOGO, students can explore and solve various problems. The environments that are set up are what Papert calls "microworlds" (in *Mindstorms*). The students do not write programs in this mode; instead they explore and solve problems in a situation set by the teacher.

Games and simulations

Games and simulations referred to here are educational not arcade games. Students are encouraged to develop strategies in playing the games. They should understand the rules, develop a strategy, play the game and adjust the strategy, and continue. Games can reinforce good problem-solving techniques and a curriculum concept at the same time.

Simulations can place the student in situations impossible actually to experience because of time, expense, or danger. The consequences are imaginary, so the student can try something, go back, change the strategy, and try again.

Computers have a great potential, if used appropriately, to enhance the learning that takes place in our schools.

Y84-0023
April 1984

HELP!

We need articles for *The Canadian Mathematics Teacher*. This is your publication, and it is only as good as the members want to make it. Please let us share activities or methods you have found to be successful in your class. Provide us the ideas, and we will put them in final form for publication. Handwritten copies are acceptable. Send you ideas to your provincial editor.

LET'S HEAR FROM YOU.

DATE DUE SLIP

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