delta-k

Journal of the Mathematics Council of the Alberta Teachers' Association

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Mathematics Scholarship and Classroom Applications

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom or a scholarly focus. They may include

- letters to the editor;
- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the National Council of Teachers of Mathematics (NCTM).

Suggestions for Writers

- *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
- If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association (MCATA) for the republication, representation and distribution of the original and derivative material.
- Peer-reviewed articles are normally 8–10 pages in length.
- All manuscripts should be submitted electronically (in Microsoft Word format) and double-spaced. All pages should be numbered.
- The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identity to the reviewers.
- Pictures or illustrations should be submitted as separate files (such as JPEG or GIF) and clearly labelled. Their placement should be indicated in the text of the article. A caption and photo credit should accompany each.
- All sources should be properly referenced. Entries in the reference list and in-text citations should be formatted consistently, using the author–date system.
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MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-k

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From the Editor's Desk

Lorelei Boschman

The scholarship of teaching and learning (SoTL) is a crucial part of keeping current in the field of education. Ultimately, educators are looking for a deeper and more comprehensive understanding of the teaching and learning process, with the goal of improving educational practices and student outcomes.

SoTL can take many forms, including the following:

- Collaborative practices (such as engaging in hallway conversations, sharing effective pedagogical practices and learning activities, and exploring challenges and developing solutions)
- Reflective practices (such as reading with the purpose of learning and developing as an educator, journalling, and examining narratives)
- Research and case studies that investigate and draw conclusions about effective teaching and learning practices
- Dissemination (such as presentations, conferences and publications)

This issue of *delta-K* includes many SoTL articles. These articles represent the authors' significant time, effort and dedication directed toward the research or teaching topics that the articles explore. Choosing to share one's research and work is a purposeful and important activity, and for that we thank the contributors.

These articles also represent a desire to add to our collective knowledge to help other mathematics educators, with students at the forefront and as beneficiaries of this work. The practical applications of this work—and, ultimately, all mathematics teaching and learning—have brought us to this point in mathematics education and will help us progress into the future together.

Let's support the continuation of this study, research and work. We all have valuable insights to offer, whether informally within our educational institutions or more formally through published works reaching a wider audience.

What can you share with others today? Let's keep SoTL alive in our mathematics environments, our schools and the broader education community.

Engaging Girls in Mathematics: University of Calgary Math Outreach Programs and Camps for Girls

Lauren DeDieu

According to the Canadian Nuclear Safety Commission (2024) website, less than 25 per cent of Canadians working in science and technology are women.

I see the underrepresentation of women in my undergraduate math classes, and when I joined the University of Calgary in 2018, I noticed the disproportionately small percentage of girls in our math outreach programs. This motivated me to develop math outreach initiatives that create safe spaces for girls to engage with mathematics and begin to see themselves as part of the mathematical community.

These programs target Grades 6–10 students who identify as girls. Part of my motivation for choosing this age group was that research shows that as girls get older, they become increasingly likely to associate mathematics and science with men (Aguilar et al 2016; Picker and Berry 2000).

For example, a meta-analysis found that when asked to draw a scientist, 70 per cent of 6-year-old girls drew women, whereas only 25 per cent of 16-year-old girls did (Miller et al 2018).

Another study found that female undergraduate students described mathematicians as “exceptionally intelligent, obsessed with mathematics, and socially inept” (Piatek-Jimenez 2008, 633).

Negative views about mathematicians and stereotypes linking science with men may make women less likely to enter science, technology, engineering and mathematics (STEM) fields and careers. By exposing girls to female role models and peers who are enthusiastic about mathematics, I hope to combat this gender bias and inspire girls to pursue their passion for mathematics.

Girls Excel in Math

In spring 2019, I created a program called Girls Excel in Math (GEM).

GEM is a Saturday-morning program for students in Grades 6–8 who identify as girls. The program aims to combat gender bias and stimulate girls’ interest in mathematics by allowing them to have fun participating in collaborative, noncompetitive mathematical activities and by exposing them to female role models.

In fall 2023, over 180 participants from eight Calgary schools participated in GEM. The sessions ran twice per semester at the University of Calgary, from 9 AM to 12 PM on Saturday. Students first participated in a session led by me, in a large lecture hall. They then moved into classrooms to work in small groups, with support from teachers from their schools and undergraduate volunteers.

That fall, we explored cryptology and graph colouring. Past topics include fractals, origami, the prisoner’s dilemma, voting theory, mathematical card tricks, taxicab geometry and Eulerian graphs.

I am the sole organizer of GEM and coordinate all aspects of the program, including designing the curriculum, advertising to schools, and recruiting and training undergraduate volunteers. Our fall 2023 program had 13 undergraduate volunteers, as well as 16 volunteer teachers from participating schools, who facilitated group activities and recruited students from their schools.

GEM has received positive feedback from both teachers and students. One participant said, “With GEM I was able to discover a new passion for mathematics and make new friends who shared the same interest in math.”

For more information about GEM, go to https://science.ucalgary.ca/mathematics-statistics/engagement/educational-outreach/girls-excel-math.

Math Attack Summer Camp for Girls

I also organized the Math Attack Summer Camp for Girls, held in July 2022 and August 2023.

These eight-day camps, held at the University of Calgary and the Banff International Research Station (BIRS), brought 21 Grades 6–10 students who identified as girls together from across the province to engage in fun mathematical activities and build connections. The participants stayed in the university residence for the first five nights and at BIRS for the last two nights.

The camps aimed to encourage girls to pursue their passion for mathematics and make connections with peers who shared similar interests.

At the 2022 camp, participants engaged in mathematical sessions exploring topics such as cryptology, data science, probability paradoxes and actuarial science. They investigated the spread of disease by modelling a zombie outbreak, learned what they could do with a math degree at the Women in Math Panel and competed in the Amazing (Math) Race. The 2023 camp program was very similar.

The camps exposed participants to over 20 female role models, including recent high school graduates, undergraduate and graduate math students, mathematics faculty, and mathematicians in industry.

Participants also had opportunities to have fun and build friendships. They engaged in evening activities such as sports, swimming, board games, karaoke and a walk along the Bow Falls Trail. They also took some time to explore the town of Banff and hike up Tunnel Mountain.

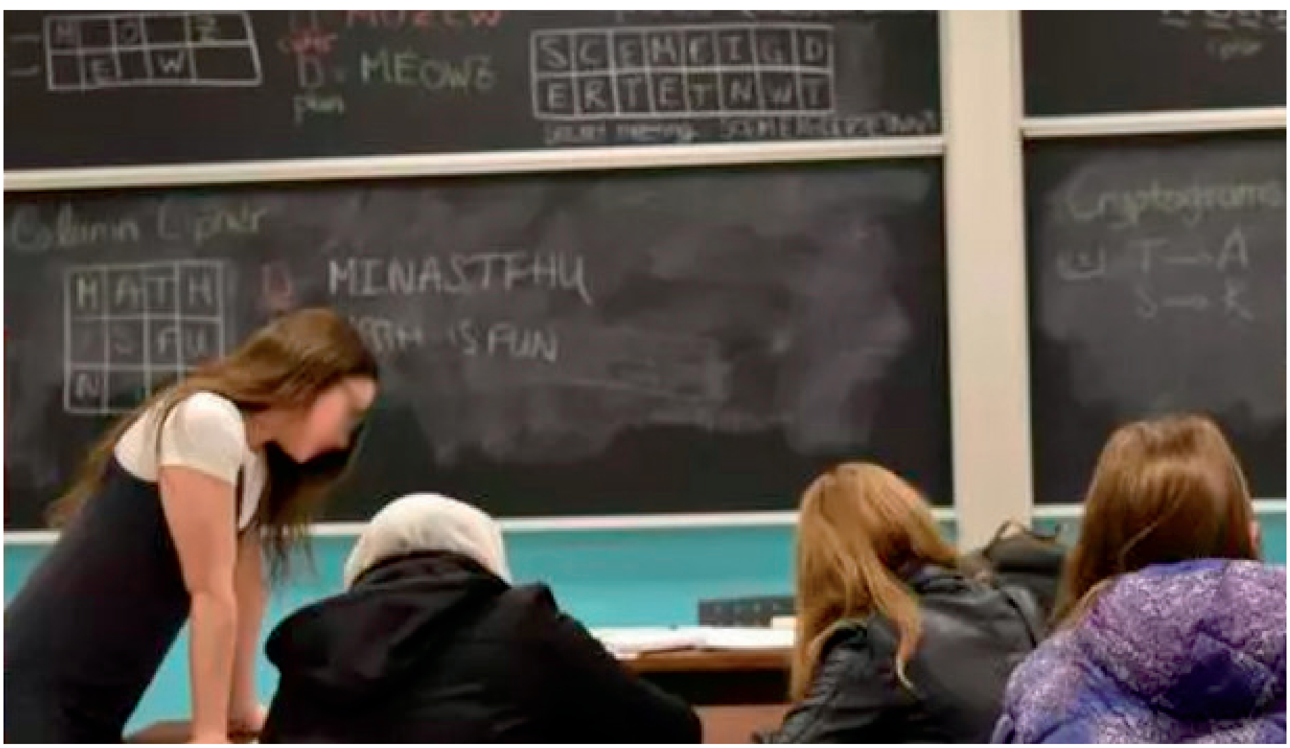
Participants were selected based on their demonstrated passion for mathematics. There was no registration fee, and all meals and accommodations were provided.

I served as the camp director and lead organizer and coordinated all the day-to-day activities. This involved applying for grants so that we could keep the camp free for participants, putting together an academic program, leading sessions, managing chaperones and volunteers, and overseeing all administrative aspects (such as risk management, food and accommodations, communication with parents, and the purchase and printing of materials). Our amazing head chaperone, Dami Wi, supervised the participants and the two junior chaperones.

Leading this camp was an incredibly rewarding experience. I'll never forget the joy on the girls' faces and how tightly knit our community became in just one week. I witnessed girls jumping in celebration after figuring out a proof, carefully debating the validity of a mathematical argument and eagerly presenting their work to their new friends. I believe that this camp had a significant impact and truly made a difference in the lives of the participants.

The feedback from participants was overwhelmingly positive. One participant said,

> This was a once-in-a-lifetime experience, and I really loved it. The supportive attitude of everyone made it feel like home, and in only a week I've gotten so attached to everyone. It feels like we've been together for so long now. There are so many new concepts and unique ways of thinking that are going to stick with me for the rest of my life, as well as precious school advice from dozens of professionals and current students. If I could go back in time and do it again, I would.



For more information about the Math Attack Summer Camp for Girls, including schedules, pictures and feedback, see the final reports (available at https://science.ucalgary.ca/mathematics-statistics/engagement/educational-outreach/math-attack).

Girls Excel in Math Summer Camp

The Girls Excel in Math (GEM) Summer Camp was a three-day camp, held in 2022 and 2023, that brought Grades 6–10 students who identified as girls together to engage in fun mathematical activities and build connections.

I created the GEM Summer Camp as a result of the high demand for the 2022 Math Attack Summer Camp for Girls. We received over 150 applications, but the camp had only 21 spots. So I launched the GEM Summer Camp and invited the remaining 130 applicants so that they, too, could engage in the sessions. The GEM Summer Camp and the Math Attack Summer Camp had many of the same speakers, and the Math Attack girls joined the GEM girls for one session per day.

The 2022 and 2023 camps were held at the University of Calgary and had 99 and 121 participants, respectively.

I was the sole organizer of the GEM Summer Camp and oversaw all aspects, including recruiting and training 12 undergraduate volunteers. It was amazing to see such a high level of interest for a girls' math camp in Calgary.

A highlight of the 2022 GEM Summer Camp was the Women in Math Panel (which was also attended by the Math Attack Summer Camp participants). The panel was hosted by my colleague, Kristine Bauer, and I recruited four female panellists with math backgrounds who work in industry (for example, data science and biostatistics). The panel explored the topic of what women can do with a math degree. The participants were excited about this session all week and prepared questions in advance.

The girls offered extremely positive feedback about the panel and the camp overall. One participant said, "*Fun, interesting, engaging, role modelling* and *inspiring* are words I would use to describe this camp. I love how women are inspiring young girls in math and how camps like Girls Excel in Math are becoming a thing."

For more information about the GEM Summer Camp, including schedules and photos, see the final reports (available at https://science.ucalgary.ca/mathematics-statistics/engagement/educational-outreach/girls-excel-math-camp-2023).

If you want to be notified about future offerings of these programs and other math outreach events at the University of Calgary, join our Math Outreach Email List at https://science.ucalgary.ca/mathematics-statistics/engagement/educational-outreach.

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Lauren DeDieu is an associate professor (teaching) in the Department of Mathematics and Statistics at the University of Calgary. She regularly teaches first-year calculus and second-year linear algebra, as well as a mathematics appreciation course. Through her teaching and her K–12 outreach initiatives, she works to foster excitement about mathematics and promote a sense of inclusion in the mathematical community. She is also passionate about helping students develop mathematical communication skills. She serves as the Department of Mathematics representative on the Mathematics Council of the Alberta Teachers' Association (MCATA) executive.

Integrating Career Education into Alberta's Mathematics 20-3 and Information and Communication Technology Curricula

Emily Kaun, Kerry B Bernes and Karissa L Horne

For many high school students, the overarching objective involves completing the required courses and earning enough credits to graduate. What they want their life to look like after graduation, however, is often unclear to them.

Teachers are in a position to inform, educate and prepare students for their future careers (Bloxom et al 2008). However, most subject curricula do not address how or why learning the content might be relevant to a career. Students often ask, "Why am I learning this?" or "How does this apply to my life?"

It could be argued that students (even in postsecondary school) most often question the relevance of mathematics to their lives and careers. When connections are not developed, they may avoid pursuing science, technology, engineering and math (STEM) careers (Leyva et al 2022).

Moreover, many students are not able to bridge the gap between having a plan for the future and knowing what is needed to make that plan a reality (Gibbons et al 2006). Making mathematics more relevant to their career plans may encourage them to complete specific mathematics courses before postsecondary so that they can pursue the career paths they are interested in.

The short unit described in this article integrated career education into Alberta's high school mathematics and information and communication technology (ICT) curricula. It allowed students to research and explore the vast number of careers available and to find out exactly what education and experience they would need in order to attain employment in their desired field.

Although career and technology studies (CTS) already exists in the Alberta curriculum, this unit was about more than helping students develop skills for entry into the workforce and preparing them for the future with the use of technology. It also aimed to make the mathematics curriculum more meaningful and relevant to them. Also, given that the ICT curriculum is intended to be implemented into other courses, this unit used ICT outcomes to enhance students' overall learning and research in career planning.

Background

The following is an overview of how career planning became integrated into Alberta's curriculum.

Career Needs Survey

First, Magnusson and Bernes (2002) developed a comprehensive career needs survey (CCNS) to gain a better understanding of students' career needs. The CCNS was a collaborative initiative between the Southern Alberta Centre of Excellence for Career Development, Faculty of Education, University of Lethbridge; the Chinook Regional Career Transitions for Youth Project; and the South-Western Rural Youth Career Development Project.

The CCNS aimed to capture students' perceptions of their career-development and career-planning needs, as well as any perceived gaps in existing services (Magnusson and Bernes 2002; Witko et al 2006). The survey included both quantitative and qualitative responses and was distributed by classroom teachers in 54 junior high and senior high schools in southern Alberta (Witko et al 2006).

The results indicated that the most pressing needs for students were as follows (Magnusson and Bernes 2002):

- Finding their interests and abilities
- Discovering their passions
- Gaining support for their career plans and post-secondary education
- Gaining financial information

Additionally, the results implied that beginning career planning earlier (in junior high or even before) could be more effective in assisting students through the process of career decision-making (Witko et al 2006).

Career Education Pilot Project

Given these results, it was evident that career planning was an important component lacking in students' educational experiences in southern Alberta.

Thus, members of the Faculty of Education, University of Lethbridge, created and implemented a career education pilot project—Career Coaching Across the Curriculum (Slomp, Gunn and Bernes 2014). Alberta Education and the Canadian Career Development Foundation (CCDF) supported the project by providing funding.

The pilot project aimed to train 50 preservice teachers in career education, which allowed them to go into schools in Alberta and implement career education across the K–12 curriculum.

As Slomp, Gunn and Bernes (2014) outline, the pilot project consisted of two components: a career education course and an internship experience.

First, the preservice teachers attended a career education course delivered over four weekends. The first three weekends provided them with the knowledge and skills necessary for integrating career interventions into the regular curriculum. On the fourth weekend, they shared with their classmates the lessons, unit plans and schoolwide interventions they had developed.

After successfully completing the career education course, the preservice teachers completed a 12-week internship in which they transferred their newly acquired knowledge and skills into elementary, middle and high schools in southern Alberta.

The larger data set from this pilot project has already been published (Slomp, Gunn and Bernes 2014). This article details a specific classroom implementation from the larger study.

Context of the Teaching Environment

The unit was administered to students in a Catholic school in a small rural town in south central Alberta.

The town's economy was focused on agriculture and trading within a wide radius of villages and hamlets.

The school was a junior/senior high school consisting of Grades 7–12.

The unit was taught in a Mathematics 20-3 classroom of twelve students (four girls and eight boys), aged 15–18.

The students were of Caucasian and First Nations descent.

Cross-Curricular Integration

The unit was integrated into Mathematics 20-3 and included ICT learning outcomes.

Mathematics 20-3

The mathematics program of studies lends itself to learning outcomes related to budget analysis and creation and, therefore, to the integration of career planning.

The unit required students to complete research on various careers and create a monthly budget based on the average income for a specific career.

The unit addressed the following learning outcomes for Mathematics 20-3 (Alberta Education 2008, 36):

- “Develop number sense and critical thinking skills” (general outcome).
- “Solve problems that involve personal budgets” (specific outcome).

ICT

Many learning outcomes in the mathematics program of studies directly reference the ICT program of studies.

The unit addressed the following ICT learning outcomes (Alberta Learning 2000–03, 9):

- “Use technology to investigate and/or solve problems” (general outcome C6).
- “Investigate and solve problems of organization and manipulation of information” (specific outcome 4.2).
- “Generate new understandings of problematic situations by using some form of technology to facilitate the process” (specific outcome 4.4).

Detailed Description of the Unit

The unit consisted of three 90-minute lessons during the Mathematics 20-3 class.

The teacher created a handout for students that outlined the objectives, goals and procedures of the career budget assignment. The assignment was described as follows:

> This project requires you to research career options that are meaningful and of interest to you. You will provide information about two different career choices as well as create a conservative monthly budget based on the salary/wage of one of those choices.

Lesson 1: Online Personality Tests and Related Careers

The first lesson took place at the end of the personal budgeting unit in Mathematics 20-3.

In this lesson, students completed questionnaires about their interests, preferences and hobbies, as well as their strengths and weaknesses, to determine potential careers for them. The questionnaires also gave the teacher insight into how the students saw themselves and the types of careers that might interest them.

To begin, students were given the Identifying Your Interests handout.[1] This self-assessment was based on Holland’s (1973) occupational codes.[2] Students completed a short checklist of interests in six occupational profile groups:

- Realistic (R)
- Investigative (I)
- Artistic (A)
- Social (S)
- Enterprising (E)
- Conventional (C)

They then determined the three categories they had scored the highest in. Upon receiving their occupational profiles, they looked at the careers on the Holland Codes Career Interests handout and listed all the careers that interested them.[3]

Students then accessed two online personality tests:

- O*NET Interest Profiler (www.mynextmove.org/explore/ip)
- Jung Typology Test (www.humanmetrics.com/personality/test)

These tests asked students questions about their personality and preferences and then aggregated their responses to create a list of careers that might suit them. Students added any careers they were interested in to their list.

The teacher advised students to list two to five careers that appealed to them, for the purpose of gathering more information about those careers in subsequent lessons. If they already had a career in mind, they could include that career.

Lesson 2: Career Research and Written Summary

In the second lesson, students researched careers on the Alberta Learning Information Service (ALIS) website (https://alis.alberta.ca), which is now called simply Alis.

In the previous lesson, students generated a list of careers of interest to them, based on their personality and interests. In this lesson, they read the in-depth career descriptions on the ALIS website and used that information to narrow down their list to two careers.

Students then wrote a summary for each career, based on a checklist provided by the teacher (Appendix A). They were to include a description of the career, the required education and the average salary, as well as why the career was one they might wish to pursue.

Students were to hand in their summaries for marking at the end of the third lesson.

Lesson 3: Career Budget Creation

The Mathematics 20-3 program of studies includes knowledge of personal budgets as a learning outcome (Alberta Education 2008, 36). In the third lesson, students created a conservative monthly budget based on their findings about one of their career choices.

Students were given a budget checklist (Appendix B). They chose one career and stated the average monthly income for that career. They then listed their monthly expenses (with the assumption that they were out of high school, were living alone and had met all the requirements for their chosen career).

Students calculated their budgets according to the guidelines in Table 1.

TABLE 1. Spending Guidelines

Expense	Percentage of monthly income
Housing (including utilities)	35–45%
Transportation (including car insurance and gas)	5–15%
Food	10–20%
Entertainment	5–10%
Other	15–20%
Savings	5–20%

They were given resources to determine the approximate cost of car insurance and rent:

- Automobile Insurance Rate Board (AIRB) grid rate calculator (https://airb-applications.alberta.ca/Grid/Calculator)
- Kijiji's rental section (www.kijiji.ca/b-for-rent/canada/c30349001l0)

For other expenses (such as groceries, clothing, cosmetics, electronics and entertainment), they were asked to estimate values.

In the same document as their career summary, students stated their unrounded monthly income and expenses and created a conservative budget. They were to calculate their spending percentages and compare them with the provided spending guidelines, showing their work.

At the end of this lesson, students were asked to hand in the following written assignments:

- Written career summary
- Conservative monthly budget
- Bibliography of all websites used
- Career summary checklist
- Budget checklist

Appendix C is a sample student assignment.

Evaluating the Unit's Effectiveness

Formative Assessment

Student engagement and interest in learning about various careers was assessed through ongoing observation and dialogue as students

- completed the questionnaires,
- used technology to research careers and
- created a monthly budget.

Having students explain in writing why a career was important or interesting to them was another means for the teacher to assess their grasp of the content and of their own personality and interests.

The teacher made sure that every required task, for both the written career summary and the monthly budget, was clear and was allotted a particular number of marks. For example, listing the high school requirements for entering a career was worth three marks. This made it clear to students what they were being given marks for and what they needed to include in their research.

Summative Assessment

At the end of the third lesson, students completed an evaluation form.

Parts 1 and 2 asked questions related to three activities students completed throughout the unit:

- Online personality tests and related careers (lesson 1)
- Career research and written summary (lesson 2)
- Creation of a monthly budget based on the income for a specific career (lesson 3)

Part 1 asked students whether they had completed each activity. Part 2 aimed to capture students' perceptions of the helpfulness of the unit by asking them to indicate whether each activity was *not good at all, good* or *great*. Students were also asked what they liked about the unit and how the unit could be improved.

Part 3 asked students to indicate *I don't agree, I'm not sure* or *I agree* with regard to each of the following statements:

- This unit helped me to learn a lot about myself.
- This unit helped me to learn a lot about careers.
- This unit made me excited about what I could do with my life.
- This unit made me want to learn more about different careers.

Results

Formative Assessment

Throughout the unit, students were engaged and interested in learning about themselves and the careers that would best suit them.

During the first lesson, students stayed on task and were immersed in answering the questions in the personality tests and looking through the careers recommended for them. They openly shared their results and were genuinely excited when a career they had already had in mind appeared as a recommendation.

For the entirety of each activity, students remained attentive and focused, and they expressed their curiosity and excitement about the future.

Summative Assessment

Completion of the Activities

As Table 2 shows, all 12 students participated in all three activities, except for one student who did not complete the career budget creation activity.

Table 2. Completion of the Activities

Activity	I didn't do it	I did it
Lesson 1—Online personality tests and related careers	0 (0%)	12 (100%)
Lesson 2—Career research and written summary	0 (0%)	12 (100%)
Lesson 3—Career budget creation	1 (8%)	11 (92%)

Note: Overall, on average, 97% of the students completed all the activities.

Helpfulness of the Activities

Table 3 shows students' perceptions of the helpfulness of each activity in terms of learning about themselves and careers.

Table 3. Helpfulness of the Activities

Activity	Not good at all	Good	Great
Lesson 1—Online personality tests and related careers	0 (0%)	5 (42%)	7 (58%)
Lesson 2—Career research and written summary	0 (0%)	4 (33%)	8 (67%)
Lesson 3—Career budget creation	0 (0%)	4 (33%)	8 (67%)

Note: 100% of students rated the activities as either *good* or *great*.

Students also provided feedback on what they liked and disliked about the unit, as well as how the unit could be improved. This feedback will be discussed in the Discussion section.

Objectives Fulfilled

Table 4 shows to what extent students believed the unit fulfilled its objectives.

TABLE 4. Objectives Fulfilled

	I don't agree	I'm not sure	I agree
This unit helped me to learn a lot about myself.	2 (17%)	4 (33%)	6 (50%)
This unit helped me to learn a lot about careers.	0 (0%)	0 (0%)	12 (100%)
This unit made me excited about what I could do with my life.	0 (0%)	1 (8%)	11 (92%)
This unit made me want to learn more about different careers.	4 (33%)	2 (17%)	6 (50%)

Note: Overall, on average, 73% of the students agreed that this unit met all the objectives.

Discussion

Learning About Careers and Themselves

In written feedback, seven students (58 per cent) stated that their favourite part of the unit was learning which careers suited them best. They expressed great interest in learning more about themselves and careers.

The integration of technology allowed students to find additional information on their desired careers.

In their written feedback, five students (42 per cent) stated that they enjoyed conducting research about specific careers to determine how much money they could earn and the type of lifestyle they could live.

These same students, upon creating their budget, realized that they had held unrealistic expectations. Many were surprised to discover the amount of money a particular career earned, as well as the education required.

Student Engagement and Motivation

Allowing students time to complete questionnaires about themselves and discuss their findings with the teacher and their peers evoked a surprising amount of interest and enthusiasm.

Most students had a spare class period before the class, and most (if not all) arrived at class 20 minutes early to discuss their career directions among themselves and with their teacher. This was unexpected, since these students had had numerous problems with attendance in other classes. Even after concluding the unit, they continued to come to class early to discuss careers.

All students who completed the unit were in Mathematics 20-3. There is an underlying belief that dash-3 classes comprise relatively low-achieving students and that these students are limited in their scholastic abilities.

In their discussions, many students disclosed that their families were generally apathetic about, or even forthrightly unsupportive of, their career aspirations and that most adults implicitly (and sometimes explicitly) treated them like low-achieving people. One student aptly stated, "I will act the way I am being treated."

Moreover, in both the assignments and the discussions, many students expressed surprise that a school subject had been made personally relevant to them.

A particularly interesting outcome was that, upon completing the unit, many of the students' grades began to improve. These students reported that, as a result of the unit and their teacher's interest in their life and career, they felt school was beginning to feel meaningful to them.

Mathematics 20-3 is geared toward students interested in pursuing careers in the trades or those who will likely go straight into the workforce upon graduating. In conversation with the students, the teacher observed that it was evident that many of them had already been thinking about possible career options.

Many students (approximately half) expressed excitement when their self-assessments listed a career they had already been considering. They noted that the unit had not helped them discover a potential career path, because they had already decided on a career. However, the unit did reinforce their career direction, and they greatly enjoyed the unit's emphasis on pursuing a career they found meaningful.

This explains why half of the students selected *I don't agree* or *I'm not sure* when asked whether the unit helped them learn more about themselves or made them want to learn more about different careers, even though almost all of them indicated that the unit helped them learn a lot about careers (100 per cent) and made them excited about what they could do with their life (92 per cent).

Half of the students noted that they especially appreciated learning about the educational requirements for their career of choice, particularly the courses they would need to take and the grades required for admission into a training program. As a result, these students revised their schedule to accommodate more courses.

This was an especially significant outcome, since these students had previously expressed that they wished to complete only the minimum requirements for graduation. This unit helped them discover that the minimum requirements would not be enough for them to be admitted into their desired training programs.

For example, one student desired to open a daycare and was especially surprised at the amount of education required. Consequently, she revised her academic objectives, deciding that instead of attaining a two-year diploma in child development, she should also attain a business degree.

Time Limitations

In written feedback, four students (33 per cent) stated that they wished the unit had allowed more time for learning about various careers. Many students sought career coaching from the teacher outside of class.

Time was a tremendously limiting factor in implementing this unit. Including topics beyond the already-crowded math curriculum is especially difficult.

Allowing more time for research and more options for completing assignments could improve the unit. Since many students sought career coaching outside of class, perhaps career-related homework assignments would allow for more career exploration.

Conclusion

In Alberta, a learning outcome related to budgeting is already part of the mathematics curriculum. This learning outcome allows for an easy segue into a career-themed research project in which students create a budget based on information gathered from researching careers.

Ideally, teachers should aspire to integrate amenable mathematics topics with career topics that are personally relevant to students. They must make it a priority to give students time to explore their personalities, to find dreams that fit with their interests and abilities, and to learn how to achieve their goals.

The purpose of the unit discussed here was to allow students to learn about themselves, identify career choices that would best fit them and discover what they need in high school (and beyond) to fulfill their career goals.

The unit integrated career education seamlessly into a regular mathematics classroom. Considering its profound effect over only three lessons, it would be interesting to see the effects of career education delivered over a longer period of time and integrated into topics other than budgeting.

Integrating this career education unit into mathematics, with the assistance of the ICT curriculum, enhanced students' understanding of the relevance of mathematics and technology in their daily and future lives, as well as how technology could help them research and answer questions that arose as they planned for their future.

Moreover, the unit allowed students to learn more about themselves, their interests and their goals early enough to allow them to adjust their high school plans to fit with what they wanted to do later in life.

By giving students the tools they need to make their dreams a reality, teachers can better prepare them to be successful, no matter what they want to achieve.

Appendix A: Written Summary Checklist

For each of your top-two career choices, state the following:

1. Name of career. (*1 mark each*)

2. Description of all responsibilities related to the job, in your own words. (*3 marks each*)

3. Daily location of job. (*2 marks each*)

    a) Environment (outside at a job site, in an office, in a forest). (*1 mark*)

    b) Job availability in certain areas (for example, most likely a marine biologist would not work in Taber, Alberta). (*1 mark*)

4. High school education needed. (You may need to choose a postsecondary institution to find this information. If so, state which you chose.) (*3 marks each*)

    a) Required courses. (*1 mark*)

    b) Level of courses (for example, English 20-1 versus English 20-2). (*1 mark*)

    c) Minimum grades required (average, per course). (*1 mark*)

5. Any further education needed and requirements (university degree, apprenticeship). (*1 mark each*)

6. Describe the number of expected working hours per week. (*1 mark each*)

7. Describe the type of person (attributes) most suitable for the career, in your own words. (*2 marks each*)

8. Explain the form of income. (*2 marks each*)

    a) Range of income per year. (*1 mark*)

    b) Whether it is hourly, salary, etc. (*1 mark*)

9. Provide at least one fact about your career choice that you found interesting, exciting, strange, etc. (*2 marks each*)

10. Write a brief summary, at least three reasons, explaining why your career choice is something you may wish to pursue. (*3 marks each*)

Appendix B: Budget Checklist

Choose one of your top-two career choices and create a conservative monthly budget that includes the following:

1. State the form of income. (*2 marks*)

   a) Average monthly income. (*1 mark*)

   b) Payment interval (hourly, salary, monthly, bimonthly paycheque). (*1 mark*)

2. List unrounded amount of all expenses per month. (*10 marks*)

   a) Include monthly rent, car insurance and groceries. (*3 marks*)

   b) Include other expenses, such as entertainment, clothing, etc. (*3 marks*)

   c) Include all research and calculations completed. (*4 marks*)

3. Create a chart of your conservative monthly budget. (*8 marks*)

   a) Include income amount (variable income, biweekly paycheques, etc). (*2 marks*)

   b) Include all expenses per month. (*4 marks*)

   c) Calculate total of both income and expenses. (*2 marks*)

4. Calculate your monthly savings. (If you have no savings or are in a deficit, adjust your budget accordingly.) (*2 marks*)

5. The attached spending guidelines represent how much you should be spending in each category. (*8 marks*)

   a) Compare your budgeted spending with these guidelines. Show your calculations. (*6 marks*)

   b) Are there any categories where you should adjust your spending? If so, suggest a way in which your budget could change. (*2 marks*)

Appendix C: Sample Student Assignment

Written Summary

1. Radio announcer

2. Job responsibilities:
   - Comment on the music, local events and interest stories.
   - Use humour, small talk and background information to fill spaces.
   - Offer giveaways and prizes to listeners.
   - Follow a predetermined schedule for airing commercials.
   - Answer calls from listeners.

3. a) In a studio, or in the community at events.
   b) It is hard to find a job, because there is only so many radio stations that need more people. Generally, there is only a few people needed per station.

4. Mount Royal University: 65% average, five appropriate Grade 12 subjects (including English Language Arts 30-1 or 30-2 and Social Studies 30-1).

5. Two-year broadcasting diploma. Four-year communication degree program in journalism.

6. Announcers generally work shifts that include evenings, weekends and holidays and may include early mornings or late nights.

7. Good-sounding voice; good English language, reading and writing skills; self-confidence; ability to relate to an audience; ability to make small talk; high level of interest and enthusiasm.

8. a) $35,704/year
   b) $23.74

9. I think that this career is interesting because you interact with people and at the same time you sit in studio.

10. I'm good at talking and speaking English, and I have a lot of enthusiasm.

Monthly Budget for Radio Announcer

1. a) $2,975.33
   b) Monthly

2. Rent (two-bedroom condo): $660/month
   Car insurance: $100/month
   Groceries: $200/month
   Entertainment: $50/month
   Clothing: $50/month

3. Monthly Income and Expenses

Income		**Expenses**	
Net pay	$2,500	Rent	$700
		Car insurance	$200
		Groceries	$200
		Entertainment	$75
		Clothing	$75
	Total $2,500		Total $1,250

4. $2,500 – $1,250 = $1,250. Total savings is $1,250.

5. I was under in all my categories. No changes need to be made.

Notes

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1. This handout, originally posted on the University of Manitoba website, was archived April 13, 2016, at https://web.archive.org/web/20160413225521/http://umanitoba.ca/student/employment/media/List_of_Occupations_based_on_Holland_Codes.pdf.

2. This handout, originally posted on the University of Manitoba website, was archived April 13, 2016, at https://web.archive.org/web/20160413225509/http://umanitoba.ca/student/employment/media/Holland_Codes_Descriptions.pdf.

3. See note 1.

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The Role of Spatial Visualization in Primary Mathematics and Science: A Content Analysis of Alberta's Curriculum

Josh Markle, Evelyn Matthews, Mijung Kim and P Janelle McFeetors

Alberta's mathematics and science curricula are in transition. In the 2023/24 school year, K–6 teachers were expected to implement the new mathematics curriculum, and K–3 teachers also undertook the implementation of the new science curriculum.

This article reports on a content analysis of the outgoing and incoming curriculum documents for both mathematics and science. Our aim is to identify and analyze changes in the curricula that may have an impact on how teachers interpret and implement them in the classroom.

To narrow our discussion, we focus on spatial visualization, which is fundamental to teaching and learning in both mathematics and science.

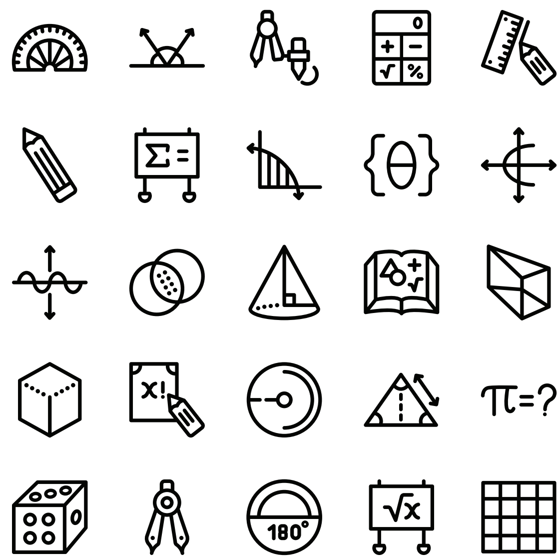
In what follows, we provide

- an interdisciplinary comparison of how visualization is taken up differently in the mathematics and science curricula,
- an intradisciplinary analysis of how the role of visualization has changed in the incoming curriculum for each subject and
- windows of opportunity for primary (Grades 1–3) teachers to attend to learning outcomes through visualization in their mathematics and science classrooms.

Context

This study was occasioned by our recent work with elementary teachers to support them in using visualization in their mathematics and science lessons.

Throughout our professional learning sessions with the teachers, we noticed that all four curricula for Grades 1–3 mathematics and science—the two outgoing curricula and the two incoming curricula—present both opportunities and challenges for incorporating visualization, but each discipline has unique opportunities and challenges.



In mathematics, the outgoing curriculum (Alberta Education 2007) includes visualization (which involves using mental imagery and external representations, such as graphs) as one of seven mathematical processes. Moreover, this mathematical process is explicitly linked to learning outcomes. However, specific guidance for teachers is not provided. Rather, linking the process of visualization to learning outcomes seems to suggest only the opportunity to employ it.

In contrast, the incoming mathematics curriculum (Alberta Education 2022) does not include mathematical processes but explicitly incorporates visualization into some learning outcomes.

The incoming (Alberta Education 2023) and outgoing (Alberta Education 1996) science curricula are more consistent, but both focus exclusively on working with external visualizations (such as drawings, graphs and models).

There are opportunities to spatialize these curricula, as called for by Davis, Okamoto and Whiteley (2015).

First, though, we felt that we needed a clearer picture of how visualization manifests in all four curriculum documents.

Visualization in Mathematics and Science

Before we proceed with our content analysis, it is worth discussing the importance of visualization itself and providing a rationale for using visualization to guide our analysis.

In mathematics education, visualization is fundamental to spatial reasoning, which occurs when students engage in "locating, orienting, decomposing/recomposing, shifting dimensions, balancing, diagramming, symmetrizing, navigating, transforming, comparing, scaling, feeling, and visualizing" mathematical objects in the classroom (Davis, Okamoto and Whiteley 2015, 140). Spatial reasoning also involves the capability to "recall, generate, manipulate, and reason about spatial relations" (Gilligan-Lee, Hawes and Mix 2022, 1).

Research in cognitive psychology and mathematics education points to the deep connection between spatial reasoning and mathematical understanding.

Spatial reasoning is associated with students' current levels of mathematical achievement (Atit et al 2022), their future mathematical achievement (Verdine et al 2017) and their achievement in other domains, such as reading comprehension (Hanline, Milton and Phelps 2010).

Moreover, spatial reasoning is susceptible to intervention, and spatial training in the classroom is associated with gains in overall achievement in mathematics (Uttal et al 2013).

In an extensive meta-analysis of the literature, Gilligan-Lee, Hawes and Mix (2022) further established evidence that this association may be causal: effective spatial training has been observed to translate into improved outcomes in mathematics.

Visualization, which we focus on here, is fundamental to spatial reasoning and, more generally, mathematical thinking.

Visualization is often defined as the capacity for mentally transforming shapes and objects, but that does not limit its usefulness to spatial contexts. Consider the role of the number line, an inherently spatial object, in teaching and learning number sense.

Moreover, visualization is often evoked through the metaphor of the mind's eye, but we have reason to appreciate the role of the entire body in spatial reasoning, generally, and in visualization, specifically. Markle (2021), for example, describes visualization as a sensorimotor phenomenon that involves bodily movement, through both physical actions (such as gesture) and imagined movement. In short, visualization draws on all our senses, not just the visual, and this presents a host of instructional and assessment opportunities for the classroom teacher.

In science education, visualization has been emphasized as an important skill for reasoning and communication. Just as visualization plays a critical role in meaning-making, explanation and communication in scientific communities, visualization in the science classroom is critical to enhancing students' reasoning and learning (Gilbert 2005). Teachers and students use visualization in diverse ways, such as shifting from physical material to abstract models and from pictorial to symbolic representations (Olson 2013).

Although visualization is critical to scientific meaning-making and communication, it has not gained much attention in the science classroom with respect to the use of mental imagery.

Traditionally, visualization in science education has been limited to interpreting or creating tables, graphs or diagrams or drawing real objects or phenomena. In this approach, students' visualization or visual representations are examined as the products or outcomes of their knowledge.

Visualization has also been recognized as a process of reasoning that explores the role of visual and spatial modalities (Gilbert 2005). Research has shown that when students engage in constructing visual representations together, they question, speculate, refine and develop scientific knowledge collectively (Tytler et al 2020; Yoon, Kim and Lee 2021). For example, while drawing the movement of air molecules in a heated container, students make meaning of the unseen and abstract phenomenon. Knowledge is discussed and negotiated through drawing collectively.

Despite the current momentum behind the use of visualization in science education, visualization is often perceived as a subordinate tool in the process of developing scientific language and knowledge.

Defining *Visualization* in an Interdisciplinary Context

Visualization is fundamental to both science and mathematics, but as we have noted, it is taken up differently in each discipline.

In mathematics education, visualization is an integral aspect of spatial reasoning and is typically conceived of as the capacity to mentally transform shapes and objects (Davis, Okamoto and Whiteley 2015).

In science education, visualization is typically associated with external representations (such as graphs, diagrams, drawings and other models) that help students explain complex or abstract scientific concepts and process their thinking as functional elements of the collective reasoning system (Tytler et al 2020).

We think of visualization as encompassing not only mental imagery and working with external representations but also the interactions between our minds, bodies and the material environment. To conduct a content analysis of Alberta's mathematics and science curriculum documents, we needed to operationalize a definition of *visualization* that reflected these commitments. Our definition is as follows:

> Visualization entails (1) the process of developing, interpreting and using mental images and (2) the process and products of developing, constructing and using spatial inscriptions.

In general, our definition of *visualization* aligns with Arcavi's (2003) influential definition. The first part of the definition preserves the importance of mental imagery. The second part attends to what might be best captured by the term *external visualizations,* such as graphs and diagrams. We use the term *spatial inscriptions* instead of *images* or *representations* because, following others (Presmeg 1986; Roth and McGinn 1998), we view spatial inscriptions (such as diagrams) as emerging out of interactions between individuals, collectives and the material world.

Analytical Process

Using our definition of *visualization,* we conducted a qualitative content analysis of Grades 1–3 learning outcomes in the outgoing mathematics (Alberta Education 2007) and science (Alberta Education 1996) curriculum documents and the incoming mathematics (Alberta Education 2022) and science (Alberta Education 2023) curriculum documents.

Content analysis involves interpreting textual data "through the systematic classification process of coding and identifying themes or patterns" (Hsieh and Shannon 2005, 1278).

We separated our definition of *visualization* into two parts: mental imagery and spatial inscriptions. We then categorized each learning outcome as one, both or neither. Only learning outcomes that explicitly reflected an element of our definition received a code. Many learning outcomes would naturally lend themselves to visualization in the classroom, and we will share some of those later, but we focused exclusively on what was present in the learning outcome. To ensure consistency in our coding, we engaged in several rounds of coding subsets of learning outcomes and continued to do so until the coders reached unanimous consensus.

As noted, our study was in part occasioned by a major shift in the curricula that Alberta's teachers and students engage with in the classroom. We highlight here some important differences between the outgoing curricula and the incoming curricula that had an impact on our analysis.

Of particular relevance, the outgoing curricula are organized at the most granular level by specific outcomes (SOs) in mathematics and specific learner expectations (SLEs) in science. This level of granularity does not exist in the incoming curricula. What are called learning outcomes (LOs) in the incoming curricula are often equivalent to what the outgoing curricula call general outcomes (GOs) in mathematics and general learner expectations (GLEs) in science. The most granular level of organization in the incoming curricula consists of the knowledge, understandings, and skills and procedures (KUSPs) for each learning outcome.

We sought to maintain consistency with regard to the curricular components, despite these differing structures. To this end, we applied codes to the SOs/SLEs in the outgoing curricula and the KUSPs in the incoming curricula. When we discuss both together in this article, we use the lowercase *outcomes.*

Our choice to code these outcomes requires three important caveats.

First, we are aware that the SOs/SLEs and the KUSPs do not align perfectly in terms of granularity.

Nevertheless, we argue that they provide comparable reflections of the extent to which visualization is present in the curricula as a tool, process or object of learning.

Second, while we acknowledge the importance of the mathematical processes and their critical role in the outgoing mathematics curriculum, no comparable structure exists in the incoming curriculum. Therefore, we focused only on whether our definition of *visualization* was explicit in the outcome, regardless of whether the process of visualization was explicitly linked to that outcome.[1]

Third, and perhaps most important, we acknowledge that a curriculum is more than a collection of outcomes. For example, most curriculum documents include front matter that provides insight into the intentions that inform the curriculum. This front matter is critical to how teachers attend to curricular objectives. One reason we restricted ourselves to analyzing outcomes is that substantive front matter is absent in the incoming curricula. We also wanted to maintain our focus on the role of visualization in how teachers teach and assess specific topics and outcomes.

Results and Discussion

Table 1 provides an overview of the results of our content analysis, in terms of visualization, of the four curriculum documents.

TABLE 1. Number of Outcomes Explicitly Reflecting Visualization in Alberta's Mathematics and Science Curricula

	Mathematics		**Science**	
	Outgoing (n = 71)	Incoming (n = 448)	Outgoing (n = 180)	Incoming (n = 419)
Mental imagery	0	2	0	0
Spatial inscriptions	14	45	8	14
Both	0	1	0	0
Neither	57	400	172	405

Interdisciplinary Comparison

Our content analysis found that visualization is prominent in neither the mathematics curricula nor the science curricula.

Mental imagery is virtually absent from all the curriculum documents. Only the incoming mathematics curriculum contains explicit references to mental imagery and even there in only 3 of the 448 KUSPs.

It is important to note that the outgoing mathematics curriculum attends to mental imagery as a process, but even had we included those curricular links, the results of our analysis would be unchanged due to our requirement that the use of mental imagery be explicit in order to be categorized as visualization.

In science, as noted, visualization is associated with external representations, not mental imagery. That said, we were surprised by the low number of outcomes in the science curricula categorized as spatial inscriptions.

Intradisciplinary Comparison: Mathematics

As noted, the structure and content of the outgoing and incoming mathematics curricula are significantly different, but in terms of explicit reference to visualization at the outcome level, the curricula are similar.

A potentially positive aspect of the incoming curriculum is that it explicitly incorporates language addressing visualization into the KUSPs for geometry, measurement, number and statistics—four of the incoming curriculum's organizing ideas (which are comparable to strands in the outgoing curriculum). Unfortunately, some of that language is ambiguous, as we discuss below.

Statistics

The incoming mathematics curriculum uses the verb *visualize* in at least three ways.

One way is in the context of statistics—"the science of collecting, analyzing, visualizing, and interpreting data" (Alberta Education 2022, 12). Data visualization does indeed draw on a variety of the skills and capabilities that fall under the umbrella of spatial reasoning, in particular spatial visualization (for example, imagining various ways to represent data). However, in the context of statistics, data visualization is about organizing and interpreting numerical information in pictorial form (for example, displaying data using a graph).

Number

Another way the incoming mathematics curriculum invokes visualization is through number, which deals with quantities "measured with numbers that enable counting, labelling, comparing, and operating" (Alberta Education 2022, 1).

As noted, spatial visualization is an important aspect of spatial thinking, in particular, and mathematical understanding, more generally. However, the verb *visualize* is again used in this context in an ambiguous and potentially limiting way.

Consider the four instances of the verb *visualize* under the organizing idea of number (Table 2). In each case, *visualize* seems to be used synonymously with *pictorial depiction*, which is connected to spatial visualization but is not constitutive of it. In fact, all four outcomes in Table 2 are explicitly linked to spatial inscriptions and were coded as such.

TABLE 2. Visualization in the Context of Number in Alberta's Incoming Mathematics Curriculum

Grade	Skills, procedures and understandings
1	Visualize quantities between 10 and 20 as compositions of 10 and another quantity.
2	Visualize 100 as a composition of multiples of 10 in various ways.
3	Visualize and model products and quotients as arrays.
3	Visualize fractions as compositions of a unit fraction.

Although there is great potential for incorporating spatial visualization exercises in support of these outcomes (and others under the organizing idea of number), the results of our coding suggest that using spatial language more precisely would help make specific spatial actions explicit in the curriculum and aid teachers in implementing spatial practices in the classroom.

Geometry and Measurement

The incoming mathematics curriculum also references visualization in geometry and measurement. Since visualization is fundamentally involved in thinking about shapes and objects in space (for example, through mental rotation of shapes and objects, perspective-taking, navigating and so on), this is to be expected.

Table 3 shows the relevant outcomes.

TABLE 3. Visualization in the Context of Spatial Reasoning in Alberta's Incoming Mathematics Curriculum

Grade	Organizing idea	Skills, procedures and understandings
2	Geometry	A shape can be visualized as a composition of other shapes.
2	Geometry	Create a picture or design with shapes from verbal instructions, visualization, or memory.
2	Measurement	Estimate length by visualizing the iteration of a referent for a centimetre.
3	Measurement	Estimate length by visualizing the iteration of a referent for a centimetre or metre.

It is worth taking up these outcomes in detail. Although they all refer to visualization, we coded them differently: the first outcome received no code, and the other three were identified as attending explicitly to mental imagery.

Table 3 also makes it evident that visualization as it pertains to spatial reasoning in the curriculum is restricted to estimating length. These outcomes provide the clearest and most coherent link to spatial visualization as a mathematical process: students must imagine a referent for a centimetre (say, a finger's width), and then imagine the number of referents associated with a given length. Visualizing length and area in this way is a valuable practical skill and a powerful spatial reasoning exercise that can enhance students' capabilities in other mathematical domains.

The other outcomes for geometry in Grade 2 are not as clear in the context of spatial visualization as a mathematical process.

In the first outcome in Table 3, for example, *visualized* could be interpreted as being synonymous with *pictorial representation* or *concrete representation.* For example, a student could use pattern blocks to show a hexagon as the composition of two trapezoids or six triangles. Of course, this activity involves the mathematical process of spatial visualization, but it does so only implicitly. To capitalize on the importance of spatial visualization in mathematical understanding, a task must be designed to explicitly elicit spatial visualization (for example, "I'd like you to imagine what shapes you might put together to make a hexagon") and open up opportunities for assessment (for example, "Will you tell me what you imagined?").

The second outcome in Table 3 introduces yet another ambiguity through its distinction between *visualization* and *memory.* On the one hand, it is likely that *visualization* is intended to be synonymous with *pictorial representation* or *concrete representation.* On the other, it raises questions about the role of memory in spatial visualization as a mathematical process.

Intradisciplinary Comparison: Science

Given how often teachers and students use spatial inscriptions in the science classroom, visualization is surprisingly not explicit in the outgoing and incoming science curricula. Only 8 outcomes (out of 180) in the outgoing curriculum and 14 outcomes (out of 419) in the incoming curriculum explicitly reflect spatial inscriptions. Considering the total number of outcomes in each curriculum, visualization is even less present in the incoming curriculum.

In both curricula, visualization is explicit in outcomes related to modelling, scientific process and scientific communication.

Modelling

Visualization for modelling includes illustrating, diagramming or modelling Earth systems or living systems (such as life cycles, water cycles or structures). The science concepts include the large scale of time and space and the complexity of interactions and changes. These concepts are challenging for young students to understand. Visualizing the interconnected time- and space-related concepts is effective for conceptual understanding.

Table 4 shows the relevant outcomes.

It is clear that there is more emphasis on modelling through visualization in the incoming science curriculum than in the outgoing curriculum.

TABLE 4. Examples of Modelling Systems in Alberta's Science Curricula

Outgoing curriculum	Incoming curriculum
Using a variety of materials and techniques, design, construct and test structures that are intended to: . . . • serve as models of particular living things, objects or buildings. (Grade 3)	Life cycles can be represented in many ways, such as • illustrations • diagrams • models • stories. (Grade 2) Describe and diagram the changes of state of water using the water cycle. (Grade 3)

Scientific Process

Scientific process skills are critical for scientific investigations and problem solving.

The subject introduction to the incoming science curriculum states that students are expected to "deepen knowledge and understanding through collaborative conversation, recording and analyzing data, and interpreting scientific texts, including diagrams, models, or digital simulations" (Alberta Education 2023). Therefore, visualizing data collection and analysis is explicitly emphasized as part of the scientific process.

Table 5 shows examples of scientific process in the science curricula.

TABLE 5. Examples of Scientific Process in Alberta's Science Curricula

Outgoing curriculum	Incoming curriculum
Record observations and measurements, using captioned pictures and charts, with guidance in the construction of charts. Computer resources may be used for record keeping and for display and interpretation of data. (Grade 3)	Observations can be recorded as data in many ways, such as • words • drawings • photographs • numbers and counts • sound and video recordings. (Grade 1)

Scientific process through visualization is explicit in both science curricula, but there is more emphasis on it in the outgoing curriculum. In the outgoing curriculum, four out of eight outcomes focus on recording observation through visual inscriptions, whereas one out of eight outcomes focuses on that skill in the incoming curriculum.

Scientific Communication

Scientific communication is multimodal, with diverse forms of communication (such as language, visual inscriptions and gestures). The outgoing and incoming science curricula both refer to visualization for communication, though the emphasis is minimal.

Table 6 shows examples of scientific communication in Alberta's science curricula.

TABLE 6. Examples of Scientific Communication in Alberta's Science Curricula

Outgoing curriculum	Incoming curriculum
Communicate results of construction activities, using oral language, captioned pictures and simple graphs (pictographs and bar graphs). (Grade 2) Communicate results of construction activities, using written and oral language and pictures. (Grade 3)	Work individually or in groups to create instructions using precise words, pictures, or diagrams. (Grade 2)

The emphasis on visualization as communication is more explicit in the outgoing science curriculum than in the incoming curriculum. The only outcome with a communication emphasis in the incoming curriculum is in Grade 2, under the organizing idea of computer science: "Students apply creativity when designing instructions to achieve a desired outcome" (Alberta Education 2023, 12).

Windows of Opportunity: Implications and Strategies for the Classroom

As discussed, our content analysis revealed gaps and ambiguities with respect to visualization in Alberta's primary mathematics and science curriculum documents.

One significant gap involves mental imagery. While visualization overall tends to be underrepresented in Alberta's curricula, the explicit use of mental imagery in outcomes is very limited compared with references to spatial inscriptions (such as diagrams and graphs).

There are also ambiguities that could lead teachers to overemphasize or neglect important aspects of visualization (for example, the use of the word *visualize* to refer to creating pictorial representations).

Despite these gaps and ambiguities, we believe there is great potential for spatializing both curriculum and pedagogy. The following are practices to consider adopting in the classroom.

Use Spatial Language Frequently, Purposefully and Precisely

Regardless of age or level of formal mathematical training, we all share the experience of movement—we are all spatial beings. But we often take this shared experience for granted. One way we can acknowledge it is to make it explicit through the effective use of spatial language.

In their wonderful resource for thinking spatially in early years mathematics, Moss et al (2016) note that spatial language includes precise positional language (*above, below*); dimensional language (*wide, long*); nominal language (*square, triangle*); and transformative language (*slide, rotate, reflect*). This is essential for learning early ideas in geometry, but we argue that visualization can be used to explore concepts across mathematics and science.

For example, we used the following visualization exercise with teachers in our study:

> Try this: Imagine you are in an empty room. You notice a ladder leaning against one of the walls. Imagine climbing the ladder to the middle rung. While you stand on the ladder, it begins to slide down the wall toward the floor. Now, can you imagine the path your body takes through the air as the ladder slides to the floor?

The absence of mathematical or scientific terminology in this visualization prompt is by design. For everyone in the classroom to be able to imagine the scenario, it must use concrete, active and plain language grounded in students' experiences. When students begin to share their visualizations (through class discussion or sketching, for example), teachers can begin to use the more precise spatial language discussed above.

Make Short Visualization Exercises a Routine in the Classroom

Many teachers already use mental mathematics routines (such as number talks). Short visualization routines can be included just as easily.

For example, while the sliding ladder exercise could be the focus of a secondary mathematics or physics lesson, teachers could also use it in the early years classroom as an opening routine in a lesson about transformations. In fact, the exercise need not be linked to mathematical or scientific content at all.

Exercises like this capitalize on the rich experiences and intuitions students bring with them into the classroom. This means that the process of visualizing can be just as important as the content to be visualized.

One way teachers can simply and effectively incorporate visualization exercises as regular routines in the classroom is to make small tweaks to the exercises. Consider the following prompt:

> Imagine a cube. Slice the cube in half vertically at a right angle. Pull one half of the cube away and look at its cross-section. What shape is the cross-section?

Most of us possess the requisite knowledge such that we do not need to imagine looking at half of a cube. We know that when we slice a cube in half as described, the cross-section will be a square. We might think we are so familiar with cubes that no visualization could yield any surprises. But consider this variation, which we presented to teachers in our study:

> Try this: Imagine a cube. Now imagine setting the cube on the desk in front of you such that it sits on a vertex and the line formed between that vertex and its diagonal opposite is at a right angle to your desk. Using a plane parallel to the desk and starting at the very top of the cube, start making thin slices. Each time you slice a piece of the cube off, look at the shape of the resulting cross-section. What do you see?

As we found, a small change in the orientation of the cube yields some surprising results.

Create a Safe Shared Space for Students to Discuss What They Imagine

We have led visualization exercises with young children, high-achieving secondary students, undergraduate students and experienced teachers. We have found that everyone visualizes differently, which is an important implication for classroom practice. Sometimes those differences can cause unnecessary anxiety and stress.

A common misconception we encounter is that visualization is about seeing crisp, static images in the mind's eye. Although some people can, in fact, see eidetically, they are the exception, not the norm. Far more often, our mental imagery is imperfect and ephemeral, but it is still useful in developing mathematical and scientific understanding—and it can be improved with use.

To foster good visualization practices, we must give students time and space to discuss what they see and allow them to sketch, make gestures and engage with the material world.

For example, when we used the cube-slicing exercise with teachers, we had them mould cubes out of clay before engaging in discussion about their visualizations. The physical cubes served as a shared point of departure for teachers to talk about what they saw, to consider each other's ideas and to subsequently refine their own.

Closing Remarks

Research suggests that strong spatial skills benefit students' overall mathematics achievement. Visualization is one way to support the development of strong spatial skills in the classroom.

Although we have highlighted several gaps and ambiguities in Alberta's outgoing and incoming curricula for mathematics and science, we see great potential for teachers at all levels to incorporate visualization practices in their classrooms.

Note

1. It is worth emphasizing how prominent the process of visualization is in the outgoing mathematics curriculum: 60 of 71 (85 per cent) outcomes for Grades 1–3 are linked explicitly to this process.

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Using Variation Theory to Meet the Needs of Postpandemic Math Learners

Roxanne Sekura, Alisa Cooper and Carlene Duke

The shift to remote learning and other school disruptions during the COVID-19 global pandemic resulted in education access gaps and a subsequent push to move on without reflecting, modifying curriculum or taking time for deliberate review. This has taken a toll on teachers and students—particularly in the math classroom.

A recent study (Kuhfeld, Soland and Lewis 2022, 5) found that, in the United States, "math achievement dropped across the first two years of the pandemic," and the authors suggest that "the effects of the early phase of the pandemic were mainly on math" (p 7).

Classroom teachers see and feel this acutely.

Based on our review of the literature and our classroom experience as teachers, we contend that the variation theory of learning can be a valuable design to bridge and deepen understanding in the postpandemic math classroom.

In this article, we offer an entry point to using variation theory to meet the needs of all learners, including postpandemic math learners. We also address how variation theory can enhance teachers' ability to bring about both student engagement and desired learning outcomes (Armor et al 1976; Bandura 1977; Tschannen-Moran and Hoy 2001).

How Can Teachers Support Students in the Math Classroom?

Some math teachers struggle to find a balance between procedural fluency and problem-solving skills.

Schoenfeld and Pearson (2009) contend that too much focus on the basics binds students to set procedures while failing to give them the skills to think independently for applications of mathematics. A problem is supposed to be exactly that—a problem, a unique application.

On the other hand, a focus on strategic thinking without foundational knowledge deprives students of the skills they need for fluid, competent performance in mathematics.

The question becomes, How do we as teachers support students in building their confidence in the math classroom?

The answer may lie in variation theory. Marton and Booth (1997, 1) discuss variation as an opportunity to "empower people to gain knowledge." Learning mathematics through variation can empower students to gain the knowledge and build the computational skills they need to become problem solvers.

What Is Variation Theory?

Variation theory is not new. It is a theory of learning based on cognitive science that informs teaching and the design of learning by tapping into students' prior knowledge while adhering to student self-efficacy. This is done by attending to the limitations of working memory.

By design, lessons developed using variation theory offer deliberate, tightly woven sequential tasks that allow students to experience success by bringing their attention to an object of learning through varying its critical features.

When using variation theory in the math classroom, teachers focus on designing a sequence of tasks that allow learners to reflect on the effects of their actions. By presenting concepts as a series of critical and necessary discernments, teachers can help students recognize key relationships between progressive tasks.

As Watson and Mason (2006) explain, in variation theory, learners are exposed to mathematical structure in the traditional sense while making their own sense of a carefully designed collection of questions.

What Is the Cognitive Science Behind Variation Theory?

Variation theory is based on a key finding of cognitive science—that humans are attuned to difference and change. Variation theory also implicitly supports findings about working memory and its limitations.

For teachers, the intention behind variation is to capture students' attention, so that information can be shifted from their temporary memory to their working memory in order to reach full conscious processing.

Sousa (2015, 47) describes working memory as "a place of limited capacity where we can build, take apart, or rework ideas for eventual disposal or storage somewhere else. When something is in working memory, it generally captures our focus and demands our attention."

Cowan (2010) notes that the average capacity of working memory is three or four chunks of information in children aged 5–13 and three to five chunks of information in adolescents 14 and older.

When effectively implemented, variation sequences designed and differentiated appropriately for the learner can help address the limitations of working memory (Preciado Babb, Metz and Davis 2019).

Metz et al (2017b, 170) advise that providing "clear patterns of variation among ideas that may be held together in working memory makes it more likely that connections between those ideas may be discerned." This is known as systematic variation. Systematic variation provides opportunities for students to acknowledge similarities and differences and attends to the working memory limitations of all learners in the classroom.

What Are the Object of Learning and Critical Features?

To teach using variation theory, teachers must carefully design opportunities that draw students' attention to the object of learning by offering variation of its critical features.

The object of learning is the key element the teacher wants students to notice.

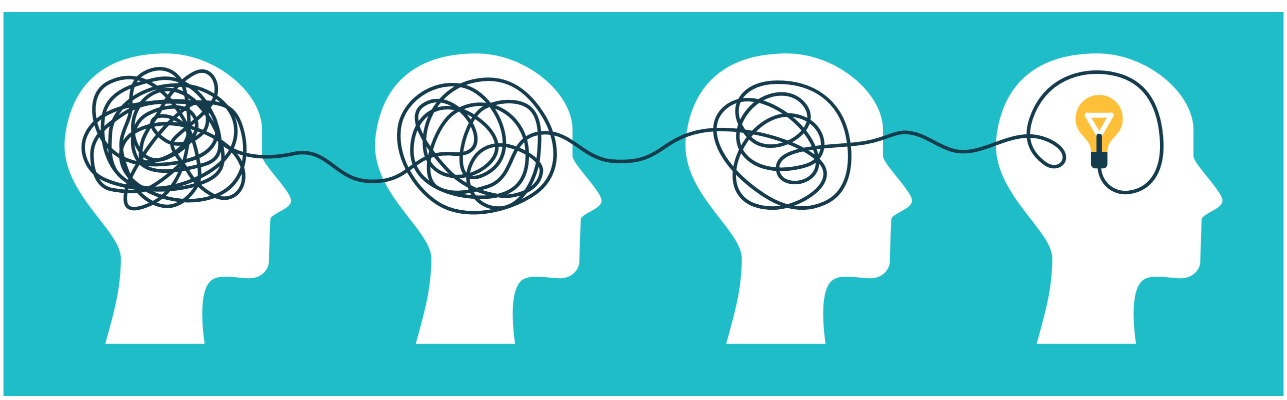
For example, in math, a transition when solving $2x - 2 = 5$ to $2(x - 1) = 5$ would draw students' attention to the parentheses and the need to either distribute or divide by 2 when solving for the variable x. This is a valuable discussion to be had. Integer solutions versus rational solutions could also be discussed.

The object of learning provides clarity around conceptual understanding and, therefore, can differ from student to student, depending on their areas of growth and strengths.

To ensure that students master an object of learning, teachers must consider all its necessary aspects—the critical features that are necessary for learning to take place. Critical features are essential to helping students identify similarities and differences within a concept.

For example, attending to the sign of the coefficient of x is a critical feature of factoring. A task could involve juxtaposing $x^2 + 3x + 2$ with $x^2 - 3x + 2$ when factoring.

When only one aspect of the object of learning is varied at a time, and the variations follow a systematic pattern until the target form is reached, the result is a teacher-initiated exploration for students. All incremental variations required for each discernment attend to working memory limitations and lead up to the final target (Park and Leung 2006).



It is important to pay attention to what varies and what is invariant in the situation, so that learners can establish a connection between their new knowledge and their previous knowledge. As Bragg (2017, 122) writes, "Seeing phenomena in new ways is at the core of learning."

Similarly, Marton (2015, 48–49) explains, "The experience of difference against a background of sameness enables learners to discern critical aspects of objects of learning and critical features of tasks." Variation theory offers insight into how teachers can "handle the object of learning in powerful ways" (p 255).

What Is Ribboning?

We were inspired to attempt to use variation theory in a high school math classroom by Park and Leung's (2006) article about teaching math in Korea, as well as through our exposure to the strategy of ribboning when we worked as research assistants for the Math Minds initiative (a partnership between the University of Calgary's Werklund School of Education, school districts in Alberta and British Columbia, and JUMP Math).

Metz et al (2017a, 181) describe ribboning as follows:

> We have recently begun to use the term "ribboning" to describe the way key lesson elements are separated for attention. . . . We have found that effective lessons typically resulted in narrow bands of colour that resemble ribbons. In other words, these lessons alternated frequently between drawing attention to important discernments and checking whether students made the intended discernments.

Unlike the "I do, we do, you do" model, ribboning allows students to think for themselves as they move through the lesson.

Ribboning is also unlike the "you try" examples often found in textbooks, which are best described as "questions [that are] slightly different in a seemingly arbitrary way so that learners tend to proceed in a stop–start fashion" (Watson and Mason 2006). When presented with a systematic progression to more-complex expressions in variation, students are quick to engage.

Consider Figure 3 (discussed later in this article). The offering of critical features is presented in the first column, and the second column gives students an opportunity to engage with those features. Universal Design for Learning (UDL) would describe this as mastery-oriented feedback, as it guides learners toward mastery.[1] The emphasis is on effort and practice while providing feedback that encourages perseverance.

How Do Both Student Engagement and Teacher Efficacy Increase When Teaching with Variation?

As teachers, we realized that using variation theory in the math classroom could increase student engagement by offering opportunities for learners to see the critical features before they "learn to do" by applying them (Marton 2015).

In this age of technology, when pencil-and-paper mathematics is still a curricular requirement, math teachers are concerned about student engagement. As Prensky (2005) emphasizes, old ways of teaching will no longer captivate most students.

This speaks to the importance of teachers being lifelong learners who continually strive to evolve their practice. Martin, Sass and Schmitt (2012) posit that teacher efficacy in terms of student engagement is a driving force behind how teachers approach instruction.

According to UDL, teachers can enhance student engagement by providing tasks that allow for active exploration and participation.[2] Variation theory is the key to creating such tasks, optimizing relevance and value for the learner and, therefore, enhancing both student engagement and teacher efficacy.

In discussing students with learning challenges or a history of failure, Scott, Hirn and Alter (2014, 193) state that "student attention to task and achievement are affected by the quality of instruction" and "success during instruction is the most basic reinforcer for learning." They write,

> Students will only maintain engagement with instruction when they are experiencing success. The consistency of student success should be the criterion by which the effectiveness of instruction is judged. Thus, effective instruction should be based on the question, "How can I teach this concept, skill, rule, or behavior in a manner such that I know the student will succeed?" (p 193)

Thus, “Simply providing opportunities to learn is insufficient. Instruction for these students must be calculated for success” (p 193). That means that every lesson must be designed and framed for success.

We believe that variation theory offers opportunities for instruction “calculated for success” (Scott, Hirn and Alter 2014, 193), reaching learners of all levels while adhering to the limitations of working memory. Concepts presented in this way offer an opportunity to bridge gaps between high- and low-level learners and provide levelled instruction to meet each student at their academic understanding by simplifying concepts to their basic components and progressively increasing the difficulty in a systematic fashion.

Tasks designed with systematic variation, in an environment that alternates offerings of critical features with opportunities to engage with those features, present students with numerous opportunities for success throughout a lesson. This could be enough to maintain student engagement.

When teachers use variation theory in the math classroom to scaffold conceptually related tasks (Watson and Mason 2006; Zawojewski and Silver 1998), students become active participants in their own learning and successes (Huang and Leung 2005). In fact, students are encouraged to work ahead of the teacher as they start to notice and attend to the critical features.

Moreover, as Watson and Mason (2006) note, when using variation theory, teachers’ confidence in their efficacy increases when they understand all possible trajectories a learner can take with a particular lesson’s object of learning.

This is where frequent formative assessment comes into play. In assessing student responses, teachers can reflect on the range of student thinking in order to design a lesson in variation that narrows that range to the target object of learning.

What Does Variation Theory Look Like in the Math Classroom?

Watson and Mason (2006) emphasize the key components of constructing and planning the stages of lessons in variation to support the range of student thinking. They note that many North American textbooks have not been designed using variation theory.

When using variation theory in the math classroom, teachers must first identify the object of learning—the key mathematical underpinning students need to internalize—and then design examples that draw students’ attention to that specific concept.

In choosing the object of learning, teachers should identify concepts that students struggle with, through marking student work, conferencing during group activities or reflecting on class discussion. Then, they should design lessons with variation accounting for common errors. They should consider all possible learning trajectories, with the recognition that “you have to go slow to go fast” (Liljedahl 2021, 30).

Take, for example, the introduction of operations with fractions. Typically, students will learn how to add and subtract fractions at one grade level and then how to multiply and divide fractions at the subsequent grade level. This structure does not support mixed-practice comparison.

Figure 1 shows procedural variation with tasks in which the operation (add, subtract, multiply, divide) is the only element of change. Juxtaposing operation gives students an opportunity for sense-making, with procedural fluency as the goal.

$$\frac{1}{2}+\frac{1}{4}=\frac{1(2)}{2(2)}+\frac{1}{4}=\frac{2}{4}+\frac{1}{4}=\frac{3}{4}$$

$$\frac{1}{2}-\frac{1}{4}=\frac{1(2)}{2(2)}-\frac{1}{4}=\frac{2}{4}-\frac{1}{4}=\frac{1}{4}$$

Need a common denominator.

$$\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{2}\cdot\frac{1}{4}=\frac{1\cdot 1}{2\cdot 4}=\frac{1}{8}$$

$$\frac{1}{2}\div\frac{1}{4}=\frac{1}{2}\cdot\frac{4}{1}=\frac{4}{2}=2$$

Do not need a common denominator. Why?

FIGURE 1. Procedural variation with tasks in which the operation is the only element of change.

Notice how the fractions and their order remain the same in each task. The only difference is the mathematical operation used. This helps students directly compare the tasks and see how the operation influences the procedure used in the computation. The object of learning—when a common denominator is and is not required—becomes more apparent when students can distinguish the critical feature (the operator in the equation).

A goal of conceptual variation is to connect what students know with what they need to know.

For example, when asking students to simplify the rational expressions

$$\frac{x^2-25}{x-5},$$

$$\frac{x^2-25}{5-x}$$

and

$$\frac{x^2-25}{5+x},$$

the teacher can provide a series of tasks that draw attention to order of terms, as found in Figure 2.

Task 1
Does order of terms matter when adding and subtracting?

5 + 6 = __ 6 + 5 = __ 5 + 2 = __ 2 + 5 = __
5 – 6 = __ 6 – 5 = __ 5 – 2 = __ 2 – 5 = __

Task 2
Now take it to variable expressions.
Does order matter?

Is $x + 5$ the same as $5 + x$?
Is $x - 5$ the same as $5 - x$?

Task 3
Simplify with integers.

$$\frac{(6-5)}{(5-6)}=\frac{1}{-1}=-1$$

$$\frac{(5-3)}{(3-5)}=\frac{2}{-2}=-1$$

$$\frac{(5+3)}{(3+5)}=\frac{8}{8}=1$$

Task 4
Simplify each rational expression.
(Discussion about greatest common factor, or GCF, can occur here.)

$$\frac{(x-5)}{(5-x)}=-1$$

$$\frac{(3-x)}{(x-3)}=-1$$

$$\frac{(x+5)}{(5+x)}=1$$

Task 5
Factor and simplify.

$$\frac{x^2-25}{x-5}=\frac{(x-5)(x+5)}{(x-5)}=\frac{(x-5)}{(x-5)}\cdot(x+5)=1\cdot(x+5)=x+5$$

$$\frac{x^2-25}{5-x}=\frac{(x-5)(x+5)}{(5-x)}=\frac{(x-5)}{(5-x)}\cdot(x+5)=-1\cdot(x+5)=-x-5$$

$$\frac{x^2-25}{5+x}=\frac{(x-5)(x+5)}{(5+x)}=(x-5)\cdot\frac{(x+5)}{(5+x)}=(x-5)\cdot 1=x-5$$

FIGURE 2. Conceptual variation tasks for high school mathematics with order of terms as the object of learning.

Solve the following degree 1 linear equations.

$3(2x-1)=4$	$2(x+3)=8$
$3(2x-1)=4x$	$2(x+3)=8x$
$3(2x-1)=4(x+5)$	$2(x+3)=8(x+1)$
$4(x-1)+3x=3(x+2)$	$2(x+3)+2x=8(x+1)$
$4(x-1)+3(x+1)=3(x+2)$	$2(x+3)+2(x-1)=8(x+1)$

FIGURE 3. A sample of procedural variation found in teacher-developed workbook lessons, demonstrating progressive variation leading up to the target linear equation.

Figure 3, a sample of procedural variation found in teacher-developed workbook lessons, demonstrates progressive variation leading up to the target linear equation to be solved. Sense-making, procedural fluency and attention to detail are presented with increasing complexity. This task also offers opportunities for student engagement. The first column is an offering from the teacher, and the second column is an opportunity for students to engage in those offerings. Each sample results in rational solutions and integers that do not require a calculator to compute.

Bragg (2017) discusses how teacher prompts highlighting the variation can help bridge perceptual jumps that are too big and support engaging learners.

Conclusion

Variation theory is old research put to good use in the postpandemic learning environment, or in any classroom with a range of learners with varied background preparation. By considering the limitations of working memory and students' ability to pay attention to detail, variation theory allows teachers to create opportunities for success in the math classroom.

Variation does not guarantee that learning will take place, but by providing opportunities for students of all levels, it makes learning possible (Kullberg, Kempe and Marton 2017).

Providing students with the tools and skills necessary to succeed in math, regardless of circumstances, is a worthy goal for teachers. Therefore, exploring variation theory would offer a promising professional development journey for both individual and collective growth in the mathematics education community.

Notes

1. The UDL Guidelines, CAST, https://udlguidelines.cast.org (accessed October 25, 2024).

2. See note 1.

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Roxanne Sekura embodies dedication and expertise in mathematics, supported by an academic background that includes a master's degree and over two decades of teaching experience, spanning from middle school to calculus. Her commitment to fostering a culture of grit among her students is evident through her emphasis on deliberate practice and the recognition of effort as a crucial component of success. By instilling the value of perseverance, she empowers her students to tackle complex mathematics with resilience.

Alisa Cooper taught in Canada for 10 years before moving abroad to teach at an international school in Thailand. She is the International Baccalaureate (IB) Diploma Programme coordinator at her school. Her professional interests include increasing student engagement and helping students find meaning in their math education.

Carlene Duke is deeply committed to building teacher capacity and fostering the belief that everyone is a math person through her role as K–12 numeracy coordinator in her school division. She holds a BEd and an MEd and has over a decade of experience in the elementary classroom. She works toward creating a supportive environment where both teachers and students can flourish in their mathematical pursuits, fostering skills and attitudes that extend beyond the classroom and into their everyday lives.

Inquiry Report: Analyzing Math Talk Practices in the Classroom

Stephanie Topic, Victoria Lorenzo and Sophie Brisbois

As three master's students from differing perspectives (kindergarten, elementary French immersion and high school mathematics), we realized that our students, regardless of age, were demonstrating a negative mindset toward mathematics. So began a two-year inquiry into transforming our teaching practices and the culture in our math classrooms to improve our students' attitudes toward and success in math.

Traditionally, there has been a push toward structure and rigidity in the approach to teaching mathematics in K–12 classrooms. The disconnection between maintaining the prevailing normative structures in the math classroom and establishing successful thinkers and problem solvers led us to begin our inquiry into how we could create a culture of thinking in our classrooms through brain-enriched learning opportunities, as well as determine what pedagogical methods would engage learners to take an active and strategic role in their own learning.

What we discovered and implemented shifted our classroom cultures toward viewing math as a multidimensional subject in which collaboration, discussion and the exploration of multiple solutions are at the forefront.

Background

A common theme that emerged in the literature, as well as in our own observations, is that math is often taught as a performance subject in which the goal is to get the correct answer (Boaler 2016).

For example, asking students to select the correct answers in a multiple-choice exam is a common practice, especially in the higher grades.

When success in math is defined as selecting the right answer, students become frustrated—especially when they have completed the problem and tried different strategies but have not reached the correct answer.

When assessing our students, we noticed that the conventional method is to assess the speed and accuracy of their routine and procedural calculation.

We began to wonder, Is this a result of how we have been taught and evaluated? Has it become the default for us as teachers?

Furthermore, teaching math through telling and direct modelling can become commonplace in the classroom and can result in students not thinking (Liljedahl 2021). When we expect students to follow prescribed methods and procedures to complete a

math problem, we are not setting them up to engage in higher-level thinking or problem solving. Ultimately, the skills we want our students to develop in math go beyond performing rote calculations and following procedural steps.

Peter Liljedahl's (2021) *Building Thinking Classrooms in Mathematics* was the catalyst to our inquiry, which we conducted using the self-study methodology.

When we implemented the strategies outlined in our self-study, students in all three of our settings became more engaged and successful in math as they became increasingly active learners who were encouraged to share their strategies and representations. Broadening the criteria for success allowed math to become more accessible to all. Multiple approaches were encouraged, and students were able to communicate their thinking, develop a deeper understanding and conceptualize that understanding.

Methods

We conducted a self-study in the form of narrative inquiry, documenting our reflections on our own practice to facilitate our professional growth and improved student learning.

A self-study focuses on the process rather than the product, and we sought to understand the meaning of a particular phenomenon while developing local knowledge that would also be useful to the larger educational community (LaBoskey 2004).

We gathered data in the form of recorded conversations, and we documented the changes we saw in our learners and the subsequent shifts in classroom culture.

Reciprocally, our self-study informed our teaching, and our experiences informed the research. We sought to connect theory to our own practices and experiences in the mathematics classroom in order to inform how we could collectively improve mathematical thinking skills from kindergarten to Grade 12.

Findings

Common themes that emerged from our self-study were mathematical discourse, scaffolding, facilitation of conversation and learning, brain-enriched tasks, motivation to mastery, collaboration, and challenges faced.

We determined that the three most impactful classroom practices across all grades were

- collaboration,
- mathematical discourse and
- brain-enriched tasks.

These three practices were the most critical in transitioning a math classroom into a thinking classroom.

We argue that when teachers engage with these practices, they can begin to establish the culture shift necessary to encourage students to think, problem solve and enjoy math rather than merely follow prescribed steps.

Collaboration

We would argue that collaboration led to the biggest shift in classroom culture in all our settings, specifically in how employing this teaching strategy emphasized collaboration between our students.

For students, collaboration (working in groups on their math) led to significant changes in behaviour. They were developing group work skills that would translate to other subjects, and being assigned to random collaborative groupings eliminated social structures that could have developed into barriers to thinking in the classroom (Liljedahl 2021).

Rabel and Wooldridge (2013) found that 83 per cent of students preferred to work in groups and that students were able to scaffold each other's responses. We noticed a similar effect in our classrooms: students were keen to share their strategies and thinking within their groups and with the whole class.

Many other researchers have observed the tremendous positive effects of collaboration in the classroom. Vygotsky (1978) discusses the great benefits of collaborative learning, especially when a student is interacting with a more advanced partner. When working in pairs, students are challenged to participate in more-complicated problem solving than they would be capable of independently. Wells (1996) suggests that by working on problems together, the group can construct a solution that none could have achieved alone; individuals are forced to rise above themselves.

Our practice had involved engaging students in math done primarily silently and independently and done for the practice and reinforcement of concepts. We transitioned to sharing students' strategies through math discussions and offering multiple representations. As we did so, the way we planned and taught our math lessons and led our classes shifted.

We started to plan math with the goal of getting students thinking and also participating in small and large groups to work on tasks that would both challenge and engage them. This collaborative work encouraged students to become engaged in the process, and we noticed that this led them to attempt to solve additional concept questions. High school students wanted additional thinking tasks and would be highly engaged in finding multiple solutions. The same was true of elementary students.

Additionally, we focused on teaching students skills such as how to constructively work in a group, how to actively listen and how to share their thinking. With the younger students, this required more teacher support, modelling and scaffolding. However, we noted another benefit: the skills that students gained transferred into other subjects. We were teaching them to be better communicators, listeners and, ultimately, problem solvers.

An example from each classroom context is provided below.

Kindergarten

The kindergarten classroom provides the perfect opportunity for students to develop their sharing and listening skills in a safe and caring environment.

Teachers can build daily number talks into the routine of the kindergarten classroom. During large-group discussions, teachers can mirror the elements of a number talk at an appropriate developmental level through focusing on explaining how students understand and see numbers, rather than on getting the right answer (Parrish 2014).

Students in kindergarten typically can sustain focused attention for only 10–15 minutes. Thus, teachers should use a variety of manipulatives and approaches, such as cards, buttons, pencils, whiteboards and jewels (as shown in Figure 1).

FIGURE 1

After a number talk, teachers can move into counting collections to allow students to organize, strategize, document their thinking and count with a partner. Students should be given many ways to represent their thinking, such as taking pictures of collections, writing, building and creating a pictorial representation (Figure 2). This process allows teachers to understand students' mathematical processes and to collaboratively scaffold student learning.

FIGURE 2

In kindergarten, the routine of counting collections is best done in partners. This is based on a social constructivist framework, and the counting should be a social endeavour. In pairs, students are provided with the opportunity to co-construct their understanding, making the learning more accessible for all students. They can take pride in their work and share their findings with classmates.

For early learners, counting collections makes numbers tangible and gives numbers a purpose.

Teachers can increase the difficulty of counting collections as the school year progresses to meet the varied needs of students.

High School

Math stations allow students to work in groups and collaborate with their peers. I (Stephanie) use this structure to encourage my high school students to explore new concepts on their own and gain a deeper understanding of why particular skills are required.

Using a random-number generator, I divide students into groups. Then, I give each group a specific mathematical skill to learn.

I usually present the skill to students through a raw example. They identify the key elements and structure. Then, they attempt to answer similar questions on their own.

Once the groups have mastered their skills, students form new groups. Each new group includes members from all the original groups, and students present and peer teach the skill they learned in their original group. The result is a collaborative approach to peer teaching and learning.

I have used this method to build skills in trigonometry, polynomials and factoring. Students build on each other's knowledge, which facilitates learning. They gain confidence in their own practice as they develop ownership over their learning and become cocreators of knowledge. Finally, they become more likely to offer their ideas to the whole class and to see the value in the practice questions.

Elementary French Immersion

Boaler (2016) cites a study (Schwartz and Bransford 1998) that found that new methods for solving math problems are best introduced when students need a method to solve a particular problem and the brain is primed and ready to explore.

I (Sophie) put this idea into practice when I introduced my elementary French immersion students to division.

I gave students *jetons* (chips) and asked them to work in pairs to find as many ways as possible to group 16 chips. Important French vocabulary emerged, such as *groupes égaux* (equal groups), *diviser* (divide), *partager* (share) and *matrice* (matrix). The students figured out on their own that the activity involved division. They even discussed what would happen if not all the chips fit in a group. See Figure 3.

FIGURE 3

Introducing division conceptually and visually—something I had never done before—made division much easier for students. As Boaler (2016, 66) writes, "Students did not think they were finding a standard answer . . . ; they thought they were exploring methods."

Students can also do this activity while discovering fractions, as shown in Figures 4 and 5.

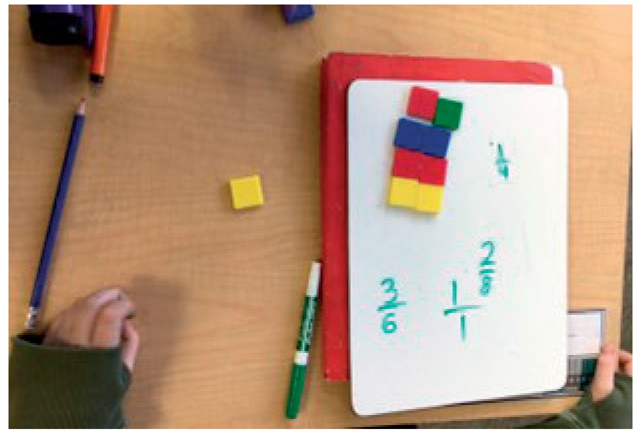


FIGURE 4

FIGURE 5

A real-life example for fractions could involve showing students a photo of the components of a lunch box and asking them to identify the fractions they see or various ways to share the lunch.[1] The teacher could even have students present the fractions they see in their own lunch boxes.

Mathematical Discourse

We identified the need for ongoing and intentional development of mathematical discourse in the classroom as another common theme.

Rabel and Wooldridge (2013) state that both learning a language and developing mathematical understanding are socially constructed activities. We feel that students at all grade levels can benefit from classroom conversation, dialogue and connection with their peers.

We all found success in improving our students' mathematical competencies by extending their understanding of problem-based questioning as an intentional strategy to make sense of each other's ideas, provide constructive criticism and reflect on their own understanding. Through peer discussion, students demonstrated higher-order thinking skills and enhanced math reasoning. They further developed the ability to effectively communicate their ideas by formulating responses as to how they would approach the problems presented.

Research shows that developing an environment in which students are expected to share their thinking and listen to the thinking of others also serves to deepen their mathematical understanding (Carpenter, Franke and Levi 2003). Mathematical discourse serves to make students' thinking visible, allowing math ideas to surface, contradictions to be exposed and understanding to be developed (Stein et al 2008).

We gave our students opportunities to communicate their own approaches, strategies and critiques of each other's work through group conversation, whiteboard demonstration and written accounts.

Building students' vocabulary is critical, especially in kindergarten and elementary French immersion classrooms, where language development is an essential component. Barwell (2016) suggests that teachers play a significant role in encouraging the transition from informal, everyday language to specific, formal mathematical language.

With early learners and elementary French immersion students, we supported their language development by using sentence starters and prompts; mathematical vocabulary, symbols and terminology; and word walls.

We viewed the need to effectively communicate one's thinking as also being important in the high school math classroom, where students worked collaboratively to explain their thinking and solve complex problems with multiple unique solutions. In this setting, mathematical discourse was encouraged through integrating math-specific vocabulary, symbolism and collaborative discussions with peers.

We found that students' self-confidence and ability to engage in subject-specific discourse translated into all subject areas, not just mathematics, as students worked better in groups and were more likely to share their ideas. To put it succinctly, we discovered that integrating language and literacy into daily activities is essential to nurturing critical-thinking skills, creativity, collaboration and communication.

An example from each classroom context is provided below.

Kindergarten

Teachers can use number talks in the kindergarten classroom to develop students' mathematical discourse.

During a number talk, students mentally solve a problem, while the teacher records their thought processes. The teacher should start with a line of questioning that is accessible for all students.

For example, the teacher can show students 10 frames (Figure 6) to facilitate various discussions—for example, "Tell me how many different ways you can count the dots. How many more dots do you need to make 10? How did you come to that conclusion?"

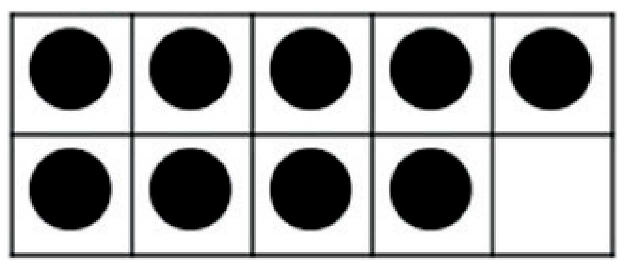


FIGURE 6. A 10 frame.

Teachers should emphasize student explanations, as opposed to correct answers, and maintain a safe and positive environment in which student contributions to the discussion are valued. As the school year progresses, students will be better equipped to verbalize their understanding. Mathematical language takes time to develop in a kindergarten classroom, through ample teacher and peer modelling.

Elementary French Immersion

An example of a math talk teachers can use in the elementary French immersion classroom is a warm-up task called Quel est l'intrus? In this task, students determine which item in a set of four does not belong.

This task can be adapted for any curricular outcome, including number patterns, numbers, two- and three-dimensional shapes, operations, fractions, and graphs. Students are required to justify their choice using mathematical language, and the question is designed in such a way that any of the four items could be the one that does not belong, which leaves the problem open to student interpretation and skill level. In kindergarten, students can compare pictures, shapes, animals or colours.

The Which One Doesn't Belong? website (http://wodb.ca/numbers.html) shares many puzzles of this nature. As students engage in mathematical discussion about the first puzzle, which includes the numbers 9, 16, 25 and 43, they develop vocabulary in French associated with place value, digits, odd and even numbers, and the sum of or difference between the digits.

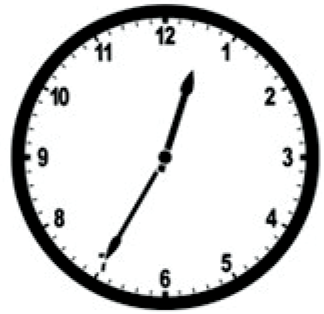
Mystery numbers with clues or a Hedbanz-style game can be used to build French vocabulary, such as *les chiffres* (the digits), *pair* (even), *impair* (odd), *position de valeur* (place value), *la somme des chiffres* (the sum of the digits), *plus grand que/supérieure* (greater than) and *plus petit que/inférieure* (less than). It also encourages students to ask and answer questions in full sentences in French.

Using real-life imagery (such as egg cartons, animals or trays of chocolates) allows students to explain their observations using mathematical language or equations.

High School

In high school math, teachers can give students opportunities to collaboratively formulate responses using higher-order thinking skills and mathematical language by creating mathematical representations of problems. Students complete prompting questions and problem-solving questions in groups, whiteboard demonstrations or think–pair–share activities in which discourse is framed around communicating ideas and strategizing for the best possible solution.

Figure 7 is an example of a question I (Stephanie) used in the classroom. I put students into groups of three, using a random-number generator. I gave each group a whiteboard marker and a vertical whiteboard and asked them to solve the question.



In degrees, what is the size of the smaller angle between the hands of a clock at 12:35pm?

FIGURE 7

Students communicated their formulations to their peers and to me, in both oral and written formats. As a result, their use of mathematical language to formulate responses and communicate ideas improved. Their proposed solutions were extraordinary, as they developed varied and unique strategies for finding a solution.

I continue to use similar questions in my classroom.

Brain-Enriched Tasks

The types of math tasks we gave our students challenged the notion that math is a performance subject and that the students who answer correctly the quickest are good at math.

Challenging and open math tasks that allow for multiple strategies, representations or answers promote a thinking classroom. Teachers can adjust and modify traditional tasks to allow students to think, explain, justify, reason and apply prior knowledge, with the goal of fostering idea generation rather than procedural application (English and Gainsburg 2016). Such tasks allow students to collaborate, as well as to actively engage, discuss and, most important, think about possible solutions.

These brain-enriched tasks focus on whiteboard use, learning by doing and by exploring different methods, verbal communication, real-life application problems, mathematical language, and the use of math vocabulary to effectively explain one's thinking. We implemented all these elements to successfully build thinking classrooms.

Warm-up questions were important in kindergarten and elementary, to check for understanding. The use of games reinforced skills and emphasized math-specific vocabulary.

We noticed that the challenging tasks allowed students at all levels to make deeper connections between their prior knowledge and the application of their skills in new scenarios. At first, especially in elementary, students were frustrated when they did not find the answer during their first attempt. They had a quick-reward mentality, believing that they should be able to solve the task immediately.

By redefining problem solving as an experience or a journey, not just a destination or an answer (Boaler 2016), we discovered that students became comfortable with being uncomfortable. They learned to take risks to challenge their understanding in order to seek improvement, which ultimately led to content mastery.

Challenging math tasks also helped students develop a resilience mindset and engage in active learning. They applied math concepts rather than following passive procedural processes and standard algorithms (Boaler 2016). The goal in our classrooms became for students to collaborate in groups, share their work and personal strategies, explain their thinking verbally and through diagrams, and persevere with tasks that may take various attempts and different approaches.

The types of questions we offered our students played a large role in promoting active learning and engagement. The questions were vital in helping students move from rote routine and trivial math to complex and nonroutine thinking, as well as challenging the notion that math is a performance subject.

An example from each classroom context is provided below. These examples demonstrate that math is a creative subject (Wolfram 2010) and show how to focus on idea generation (English and Gainsburg 2016).

Kindergarten

After reading about the grocery store (Figure 8) and engaging in a grocery store play centre (Figure 9), students looked at photos of a local grocery store (Figure 10) and explained what they thought the sorting rule would be for specific shelves.

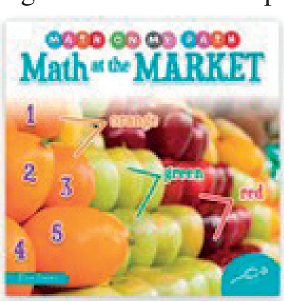


Figure 8

Figure 9

Figure 10

They were then given baskets of play food and tasked with being grocery store owners who needed to sort the food for customers. Afterward, their peers would guess the sorting rule.

Subject integration occurred throughout the sorting task, as it presented a unique learning opportunity for students to engage simultaneously in science, language, health and mathematics

Providing play opportunities with pretend food allowed students to verbalize and demonstrate their understanding of shape, size, mass, colour and so on in a safe and nurturing environment. Incorporating play and real-life problem solving allowed early learners to immerse themselves in the task, explain their thinking and take risks.

Elementary French Immersion

I (Sophie) had students work through problems together on a large whiteboard surface.

A "Would you rather . . ." problem (Figure 11) introduced multiple elements for students to work through and verbalize.

FIGURE 11

For an added challenge, I gave groups (usually three students per group) only one whiteboard marker, further emphasizing the importance of effective communication.

"Would you rather . . ." math problems are relevant to the real world and provide students with authentic situations in which to use their math skills.

High School

With high school students, teachers can make real-world connections that students understand.

I (Stephanie) gave my students a brain-enriched task that connected their in-class learning to real life by having them model COVID-19 trends in a selected country.

In pairs, students chose a country from a list provided. Using regression techniques and graphing calculators, they modelled the country's COVID-19 data over a six-month period (Figure 12). They were asked to comment on trends, predictions and observations.

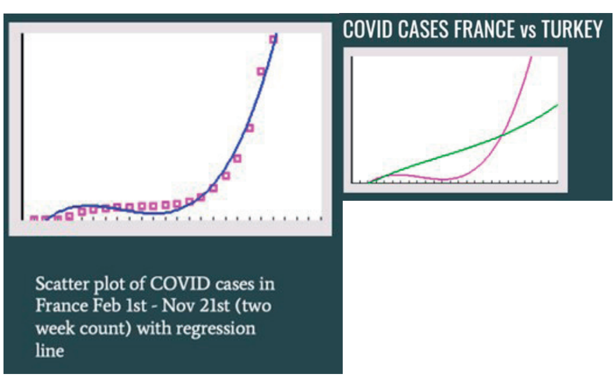


FIGURE 12

It was evident that students made a deeper connection with the classroom content and were better able to conceptualize curricular outcomes. This type of question gave them a chance to formulate and communicate their own unique approaches and ideas, to peer teach, to justify and reason, and to conceptualize the why behind developing specific skill sets in math.

Conclusion

We believe that mathematics education requires a culture shift from trivial and routine assignments to multifaceted and spontaneous engagement.

We argue that success in the math classroom is defined the wrong way and that our aims for learning should focus on developing non-academic skills that will transform students from passive participants to active learners. This involves providing opportunities for learners to think and to engage with the math curriculum (Liljedahl 2021).

As educators, we have shifted our perspective toward viewing mathematics as a multidimensional subject, taking a conceptual approach to teaching mathematics that connects theoretical math processes to real-world problem solving (Boaler 2016). This shift has allowed us to broaden the criteria of success for our students, with a focus on developing thinking skills and striving for individual excellence.

By implementing the three critical elements of collaboration, math discourse and brain-enriched tasks, math teachers can take the first steps to building thinking classrooms in their schools and establishing the culture shift necessary for true change in math education.

Note

1. A photo of a lunch box can be found at https://web.archive.org/web/20230609213255/https://myeverydaytable.com/aldi-chicken-taquitos-lunch-box-recipe/ (accessed November 1, 2024).

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Sophie Brisbois is an elementary French immersion teacher working for a Catholic school board in Alberta. She holds a master's degree from the University of Calgary, with a focus on encouraging thinking in mathematics. She is passionate about second language learning and seeks ways for students to interact and practise their second language skills as much as possible across all subjects.

Expanding Possibilities for Spatial Reasoning and Mathematical Thinking with Board Games

Krista Francis, Bogusia Gierus, Heather Lai and Pam Mah

During her first time playing Qwirkle in math class, Bradley, a Grade 6 student, places a tile that completes an intersecting row and column.[1] *She gives herself a score of four points for the row.*

Motioning along the intersecting column, which comprises three tiles, Krista asks, "What about this?"

Bradley questions, "Two more?"

Krista replies, "You count the entire column, including the tile that you counted in the row. So your total score is . . . ?"

Bradley accurately states, "Four plus three is seven."

As the game progresses, Bradley continues to have difficulty scoring intersecting rows and columns. She continues to count only the tiles in the row. Each time, Krista reminds her to include the column.

By her last play, Bradley is able to correctly score the combined rows and columns.

Implementing board games in the math classroom can enhance students' spatial reasoning and mathematical thinking, connect to curriculum, build a cohesive classroom environment, and increase student engagement.

In this article, we share four board games we found valuable for observing Grades 5 and 6 students' engagement in spatial reasoning and mathematical thinking, as well as links to videos of students playing the games in the classroom.

The purpose of this article is twofold. First, we hope the videos will help readers identify students' rich spatial actions in the context of mathematical thinking while playing board games. Second, we hope teachers will be inspired to incorporate board games in their classrooms.

Spatial Reasoning and Board Games

Spatial reasoning is important for success in math (Mix et al 2016), and students' spatial reasoning can improve through deliberate training (Hawes, Gilligan-Lee and Mix 2022).

We assert that playing board games in the math classroom provides opportunities for developing students' spatial reasoning. In fact, playing a move in a game can be similar to spatial-training activities, such as building shapes from illustrations, as described by Hawes, Tepylo and Moss (2015).

Figure 1 draws upon Davis, Okamoto and Whiteley's (2015) elements of spatial reasoning to illustrate the opportunities to reason spatially while playing a board game. More than simply visualizing moving objects in space, students playing a game engage in bodily experiences of spatial elements, such as locating, orienting, decomposing/recomposing, shifting dimensions, balancing, diagramming, symmetrizing, navigating, transforming, comparing, scaling, feeling and gesturing. For example, to place a tile or game piece, a student may need to locate a position, orient and rotate the tile or game piece to compose or extend a feature, compare a picture of a game feature, and so on.

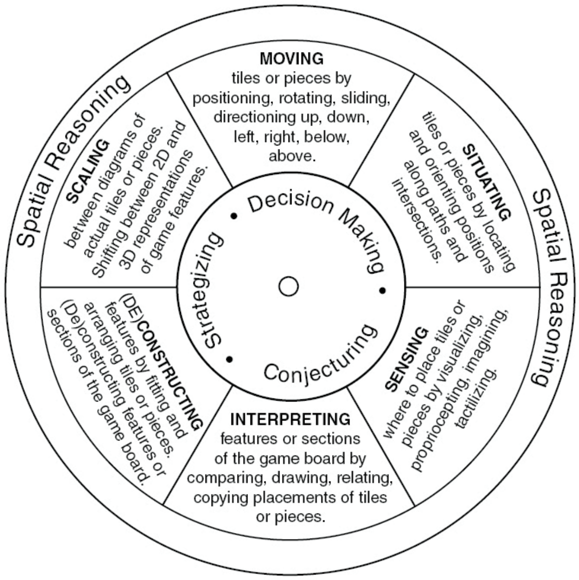


FIGURE 1. Reasoning spatially while playing a board game. Based on Davis, Okamoto and Whiteley's (2015) elements of spatial reasoning.

The Project

This article is informed by a two-year design research project that included collaborative professional learning before, during and after cycles of board game play.

At an initial professional learning workshop, we played and then discussed various board games. When considering bringing a board game into the classroom, teachers should first play the game with a like-minded colleague or two. Experience playing a game is useful for helping students in class.

By the end of the workshop, we had selected four games that were spatially rich and suitable for Grades 5 and 6 students:

- Qwirkle
- Shifting Stones
- NMBR 9
- Carcassonne

Two teachers (Heather and Pam) would then implement board game play in their classrooms.

We considered each board game a cycle. For each cycle, teachers attended two half-day professional learning sessions.

In the first year of the project, the professional learning focused on playing the games, labelling them, developing scoresheets, creating quick-reference guides and designing student reflection sheets. (All resources are available at https://stem-education.ca.)

In the second year, the professional learning focused on analyzing student reflection sheets, identifying student misconceptions and refining the resources.

The teachers chose to have their classes play the same games at the same time, allowing them to share implementation strategies and have meaningful conversations.

Each class set of eight games (seven for the students and one for the teacher) cost about $400.

Implementing Board Game Play in the Classroom

In the classroom, students were immersed in each board game for four consecutive weeks (70 minutes per week).

During the first week, the teachers focused on introducing the game. They then focused on reinforcing the rules during the second week, targeting strategies during the third week and continuing game play during the fourth week.

Generally, the teachers placed students into groups of three to five and assigned each group a specific board game set. This motivated students to become stewards of the board games and ensure their longevity.

Students could choose to play on desks or on the floor. They needed to ensure that the area could serve as a playable surface with no gaps and was clear and free of food.

To introduce the game, the teachers showed students a video explaining the rules. (All these videos are available at https://stem-education.ca.) They also provided a short demonstration of how to play, allowing students to ask questions and identify starting strategies.

Students then dispersed into their groups to play the game. Beginning by playing cooperatively (either as a team or in pairs) allowed inexperienced students to learn and experienced students to provide leadership.

Students then filled out a reflection sheet to help the teachers assess their understanding of rules, scoring and strategies.

In subsequent weeks, the teachers began with a quick review of what happened in the previous week to clarify any misconceptions identified in the student reflection sheets.

The teachers then directed students to incorporate a specified intention into their game play. For example, by allowing all the players to see each other's cards or tiles, they could support each other in learning the rules. Another day, the focus might be sportsmanship.

During game play, students used scoresheets to score their games so that the teachers could gauge each student's understanding with respect to rules and scoring. As illustrated in the opening anecdote, game play is self-directed and requires minimal teacher intervention, allowing teachers to observe and clarify misunderstandings with small groups or individual students.

At the end of each class, students completed reflection sheets.

Finally, students took inventory of the game pieces during cleanup.

Details of the Game Play

We played the games in the following order, according to increasing complexity of rules: Qwirkle, Shifting Stones, NMBR 9 and Carcassonne.

The games are different, but they all encourage mathematical and spatial reasoning, use grids, and are aesthetically appealing.

In this section, we describe each game and report our findings from viewing short videos from the third week of each game's play. The videos have been extracted from videos of complete games in order to show one player's move (our unit of analysis). The data was rich, and any move would illustrate students' engagement in mathematical and spatial reasoning.

For each video, we report instances of spatial engagement (italicized) and mathematical thinking (bolded).

Qwirkle

Qwirkle is an easy-to-learn four-player attribute game in which players match tiles by shape or colour to form adjoining lines. (Think Scrabble but with shapes and colours.)

Drawing on Davis, Okamoto and Whiteley's (2015) elements of spatial reasoning, mathematical strategizing in Qwirkle primarily engages the spatial elements of *situating* and *constructing* as players *locate* where they can *compose* and *fit* one continuous line (1-line), two intersecting lines (2-line) or a grid.

After just a few weeks of game play, students were able to mostly follow the rules and choose a strategy (1-line, 2-line or grid) with their available tiles. More details about the strategies can be found at https://stem-education.ca.

Students' Spatial Reasoning and Mathematical Thinking

Before reading on, watch the one-minute video extract at https://vimeo.com/818846761. As you watch, notice the spatial elements (both verbal and gesturing) each child is engaging in (as per Figure 1).

Maya is on the lower left, and Chelsea is on the right. Maya places a tile (*composing*) in the *intersection* of two lines. She then questions the scoring of each line. Following is an analysis of the students' engagement in spatial reasoning and mathematical thinking.

Making a Conjecture

As Maya draws (*diagramming*) a line with her finger above the vertical row of two clubs, she asks, "Would I get just the clubs points?"

Trying a Conjecture

As Maya draws (*diagramming*) a line with her finger above the horizontal row (*pathfinding*) of five purple shapes, she counts, "One, two, three, four, five." Then, she asks, "Would I get any of these points?" as she redraws (*diagramming*) the horizontal and vertical lines with her index finger.

Testing the Conjecture

Pointing (*locating*) to the vertical row of clubs, Maya asks, "Or would I just get this one?"

Adopting the Conjecture

Chelsea redraws (*diagramming*) the horizontal and vertical lines (*pathfinding*) with her finger as she confirms, "You got this and that." Maya responds, "So, seven," as she gestures along the row.

This short mathematical conversation provides insight into Maya's questioning of the scoring rules. Unlike the opening anecdote, in which a teacher clarifies the rules, here a peer clarifies the rules. Maya questions how to score her *composition* of two *intersecting* lines (2-line). Chelsea explains that she should score both intersecting lines.

Chelsea's explanation can be considered a specific instance of what Mason, Burton and Stacey (2010) call specialized mathematical thinking, in which students make, try, test and adopt specific conjectures for the game play. In a board game, a conjecture could be about strategies, applying rules or scoring. As these instances build toward general game strategies, they become what Mason, Burton and Stacey call generalized mathematical thinking.

Figure 2 illustrates Maya's conjecture about scoring intersecting lines in Qwirkle. Her specialized conjecture applies to her specific move of forming intersecting lines with two clubs (horizontal) and five purple shapes (vertical). With experience, conjecturing about applying rules and scoring can lead to broader game strategies (generalized conjecture). Notice the spatial reasoning (italicized) embedded in the mathematical thinking.

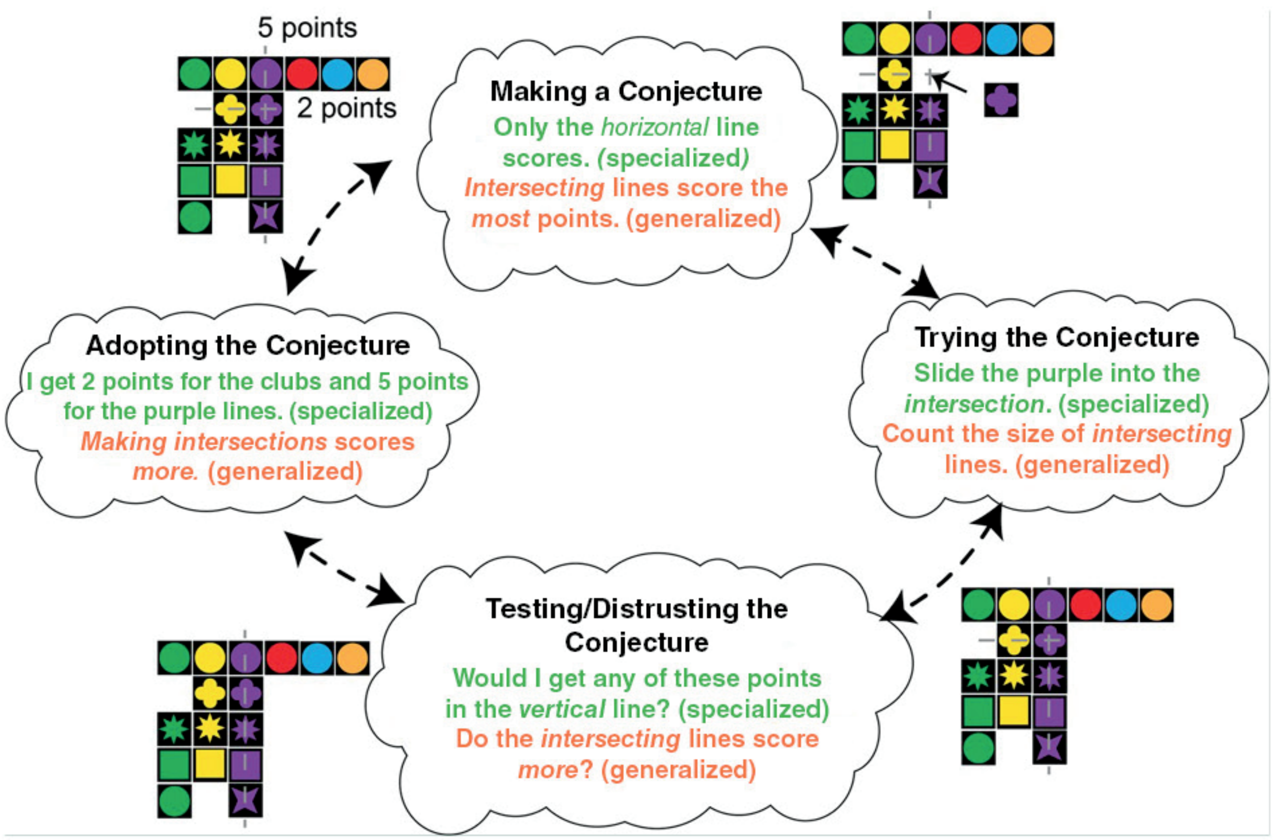


FIGURE 2. An example of specialized and generalized mathematical thinking in Qwirkle, with spatial reasoning in italics.

As students gain more experience playing Qwirkle, they will **evaluate** which tile placement will **maximize** their score. This process repeats several times during the game. In other words, Qwirkle provides multiple opportunities for engaging in mathematical thinking.

Shifting Stones

Shifting Stones is a tile-manipulation game suitable for one to five players and takes 15–20 minutes to play. Rather than an emergent grid (like Qwirkle), this game is played on a fixed 3 × 3 grid. Players try to move or flip tiles to match the patterns in their hand of cards. With each move, they must discard a card, requiring an economy of cards scored versus cards expended to move tiles.

Drawing on Davis, Okamoto and Whiteley's (2015) elements of spatial reasoning, the game's mathematical strategizing primarily engages *situating* as players *orient* the patterns of the cards to the game's north (*directioning*) and investigate the possible paths (*pathfinding*) to change the *location* of the tiles to match their hand. The game also requires *visualizing* what is on the *bottom* of an *upward*-facing tile to *flip* a tile.

Students' Spatial Reasoning and Mathematical Thinking

The video (https://vimeo.com/819670016) shows students playing Shifting Stones.

Jordan (in pink) is on the lower left, facing the game's north. North is indicated by where the pickup cards are placed. The video begins with Anjuli (upper left) making her first move. She faces the northeast position of the game. (As an aside, the two girls on the right hold their cards upside down to compensate for the game's north being at the bottom of their view of the tile grid.)

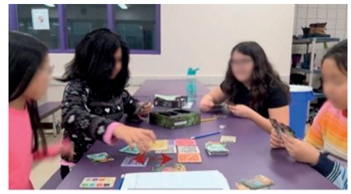
The following describes many elements of spatial reasoning within a specific specialized instance of mathematical thinking, as observed in one move.

Making a Conjecture

Anjuli has a card with the sun on top. Her conjecture is that north faces her and that her card with the sun at the top of a row will score two points.

Trying the Conjecture

To make one move, Anjuli discards a card by *moving* and *placing* one card on the pile of discards. Anjuli switches the *positons* of the sun tile and the horse tile (*moving, directioning, orienting*).



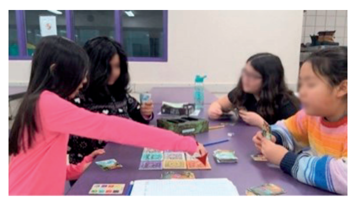
Anjuli scores two points. Jordan looks back and forth (*visualizing, comparing*) between the grid of tiles and one of the cards (*shifting between 2-D and 3-D*).

Testing the Conjecture

Jordan points (*locating, positioning*) to the sun on the card, which is *located* at the *top* (*directioning*) of a column. Motioning to the game's north, she says, "No, that has to be at the *top*."

Adopting the Conjecture

Then, *comparing* the *position* of the sun on the card to the tiles, Jordan *points* (*locating, positioning*) to the sun tile and *moves* her hand up and *points* (*locating, positioning*) to the *top* of the *column*. Anjuli replies, "Oh, I didn't see that." She takes back her scored card and *positions* it in her hand.



Jordan notices that the sun on Anjuli's card is at the top of the column and the sun on the tile arrangement is in the middle of the column. We suspect that Anjuli's strategic error came from not *orienting* the card to the game's north. This card would have worked if the arrangement of tiles had been *rotated* 90° clockwise. Jordan **explains** that Anjuli's card does not score because it "has to be at the top." *The top* could be interpreted as **generalizing,** because it applies to all cards.

Figure 3 shows how the group provides feedback and how the game requires *orienting positions* and *directions* in rows and columns. Depending on where a player sits, orienting to the game's north can be a challenging spatial task.

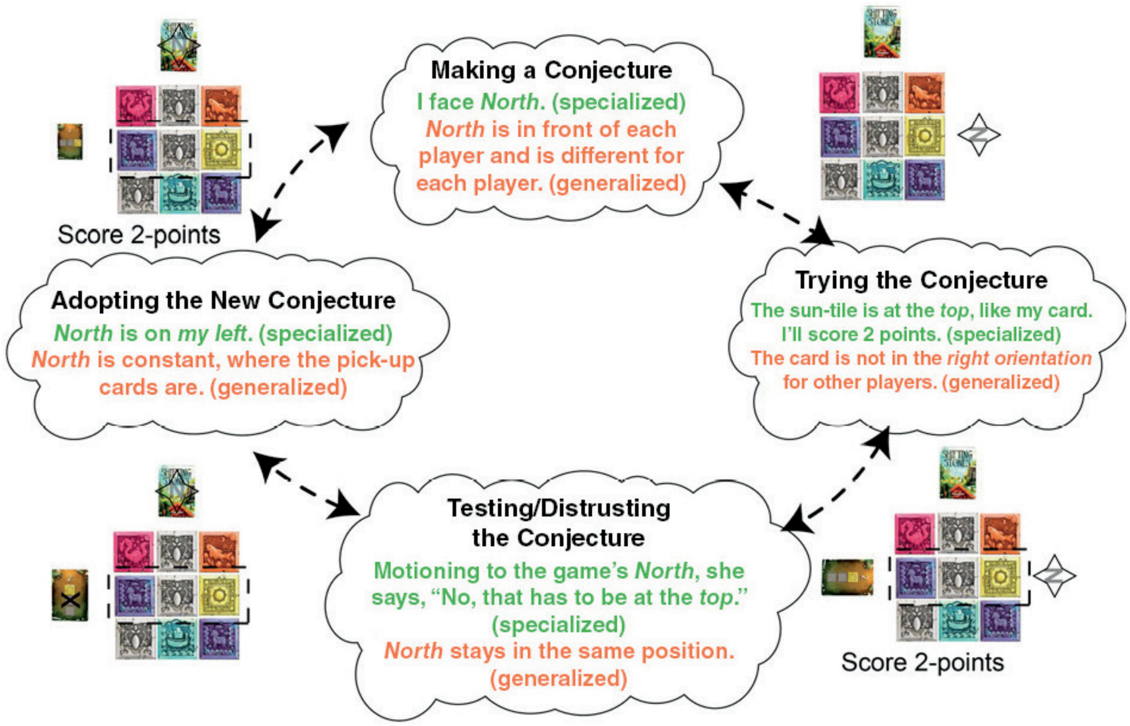


FIGURE 3. An example of specialized and generalized mathematical thinking in Shifting Stones, with spatial reasoning in italics.

Shifting Stones is useful for developing students' experiences with mathematical ideas of space and number (such as number line, Cartesian space, distance, area, perimeter and graphing).

NMBR 9

NMBR 9 is a puzzle game in which players strategically place numbered tiles in foundations and layers. Higher layers score more points.

Since each player plays the same tile on their own stack, the whole class can play simultaneously.

Fitting the numbered tiles is similar to Tetris in that the tighter the player can fit tiles together, the better. Leaving holes makes it difficult to build layers up.

This game strongly engages spatial skills such as *fitting, rotating, locating* and *shifting between 2-D and 3-D.*

Students' Spatial Reasoning and Mathematical Thinking

In the video (https://vimeo.com/819997639), the players are all building strong bases with few gaps. Spatially, they are *fitting* the tiles and *packing* them closely to minimize spaces in the base.

As you watch the video, pay attention to Jonathon (in pink, top right) as he tries to place an 8-tile. As described below, in one move, Jonathon practises *rotating* and *fitting* four times.

Making a Conjecture

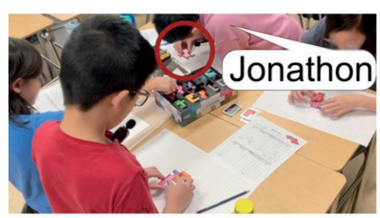
Jonathon's conjecture is that the 8-tile will fit on the next level up (first level) and score points.

Trying a Conjecture

Jonathon tries to *position* the 8-tile on the first level, on top of the 7-tile.

Testing a Conjecture

Because there is a space under the 8-tile, Jonathon lifts the tile *up* (*moving, directioning*) and *rotates* and *positions* (*moving*) it to the *side* of the emergent tile grid. He *refits* the 2-tile and the 7-tile *together* again.



Jonathon's Arrangement

Space under 8

Trying Another Conjecture

Jonathon *rotates* the 8-tile 90° and *positions/fits* it on the *base* level, *above* the 7-tile.

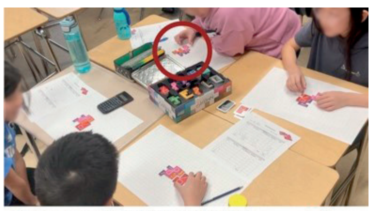
Testing the Conjecture

Jonathon uses his thumb to uncouple (*decompose*) the 8-tile from the 7-tile.

Jonathon's Arrangement

Trying Again, Testing Again, Adopting the Conjecture

Jonathon *rotates* the 8-tile 90° and *locates* and *positions* it to *fit* it *below* the 7-tile on the *base* level.



Jonathon's Arrangement

Jonathon's repeated attempts to **prove** his conjecture that the 8-tile will *fit* on the *first level* shows how spatial reasoning and mathematical thinking are intertwined in NMBR 9. His **testing of the conjectures** coincided with *moving, [de]constructing, locating* and *comparing.* As students *place* the tiles, they *rotate* and hover the tiles over different *positions.* NMBR 9 provides tremendous opportunities for complex, simultaneous spatial experiences as students investigate and evaluate each possibility for tile placement and scoring.

Figure 4 illustrates Jonathon's mathematical thinking, drawn from his articulations and actions, about his specialized and generalized strategies.

Carcassonne

Carcassonne is a tile-placement game in which players create a map of a region in southern France, including the landscape, roads, cities, cloisters and farms.

Placing tiles requires *orienting* and *assembling* tiles in a *grid* to *connect* features. Strategizing requires planning the growth of features, predicting which tiles will be needed and analyzing game plays for offensive or defensive moves.

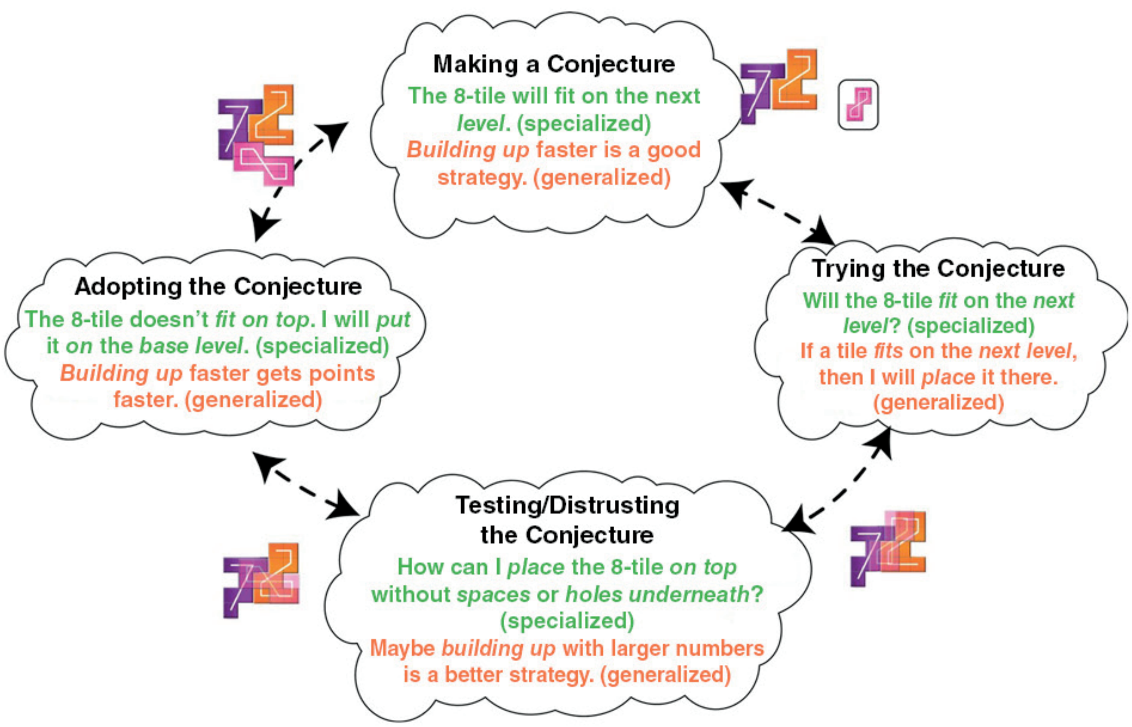


FIGURE 4. An example of specialized and generalized mathematical thinking in NMBR 9, with spatial reasoning in italics.

Students' Spatial Reasoning and Mathematical Thinking

In the video (https://vimeo.com/892308517), it is Bob's turn to place his tile. The other players weigh in with their ideas.

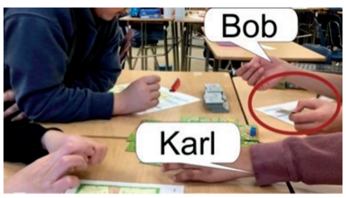
As you watch the video, pay attention to the exchanges between Bob (top right), Karl (bottom right) and, later, Mack (top left), as they deliberate where Bob should place his tile. Adam is in the lower left.

Spatially, Bob tries to follow the rules by *fitting* the tiles to the existing map, making sure that the roads, farms and cities match up, but he also tries to make the best strategic choice for his tile placement.

To make meaning of the map and communicate with each other, the boys *trace* the road and *gesture* to the spatial orientations.

Making a Conjecture

Holding his tile, Bob says, "I got another city tile." He investigates where he might be able to *place* the tile on the map. Finishing the city will optimize his score.



Trying a Conjecture

Bob finishes his already-claimed city.

Testing the Conjecture

Gesturing at the *lower left* part of the board, to show Bob where he wants him to *place* his tile, Karl chants, "Finish it, finish it, finish it, finish the city."

Bob says, "Look, you can see there's a point difference."

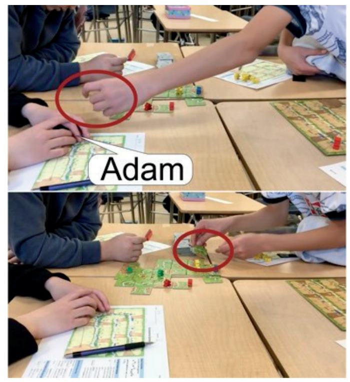
Karl responds, "Save that for later, trust me," as he gestures with his hand at the place Bob wants to *place* the tile.

Bob asks, "Why should I?" as he *rotates* his tile in the place where he wants to *place* the tile (top right) to see how it *fits* and *builds* onto the *constructed* grid.

Karl counters, "Because. . ."

Still Testing the Conjecture

Bob gives in and *moves* the tile to *fit* it in the *bottom left* (*locate*). He *rotates* and *hovers* the tile *above* the *position*. But then he exclaims, "No, but this also helps with Adam's road!"



Adopting the Conjecture

Bob goes back to his original thinking and brings the tile back to the *top right* (*locate*). After *rotating* it in *place* one more time, he *places* the tile to finish (*construct*) his city. He reclaims his yellow meeple (marker).

Justifying the Conjecture

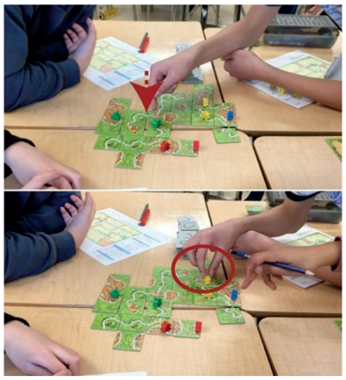
Karl questions, "How does that help Adam's road? It just makes it *longer*."

Mack jumps in, "It gives him one more point at the end," as he *traces the path* of the road with his finger.

Agreeing, Karl responds, "Yeah!"

Still Justifying the Conjecture

Tracing the path with his finger, Bob adds, "Yeah, and his road is already *connected* one way, so if . . . yeah." As he points, he shows how the road is *connected* on the other *side*. He picks *up* his meeple and counts the tiles by *pointing* with the meeple. In this way, he finalizes his decision about his *placement*, and his turn is over.



In this short interaction, the three boys use spatial reasoning and mathematical thinking concurrently and inseparably to try to persuade Bob to make a specific move. As Bob is deciding on the *placement* of his tile (**specializing**), he *rotates* and *hovers* his tile over at least two positions, based on arguments from the two other boys. Karl tries to persuade Bob to *place* the tile on the *lower left.* Bob, however, **questions the idea** of a different placement because it will help with Adam's road. Mack steps in and explains, while *tracing* Adam's road with his finger, that the placement will give him one more point at the end. Bob then quickly realizes that this *placement* is unfavourable to him and *places* his tile back in his originally proposed position. But even after his *placement,* he *traces* Adam's road and tries to **justify** and **explain** why the other position is strategically wrong for him.

Interactions like this abounded as students played Carcassonne. They were highly engaged in not only their own moves but also each other's moves. This is an example of collective mathematical thinking. Their seamless interaction between spatial orienting of the tiles, construction of the map, and mathematical thinking about strategies was evident every moment of the game.

Figure 5 illustrates a similar example of mathematical thinking.

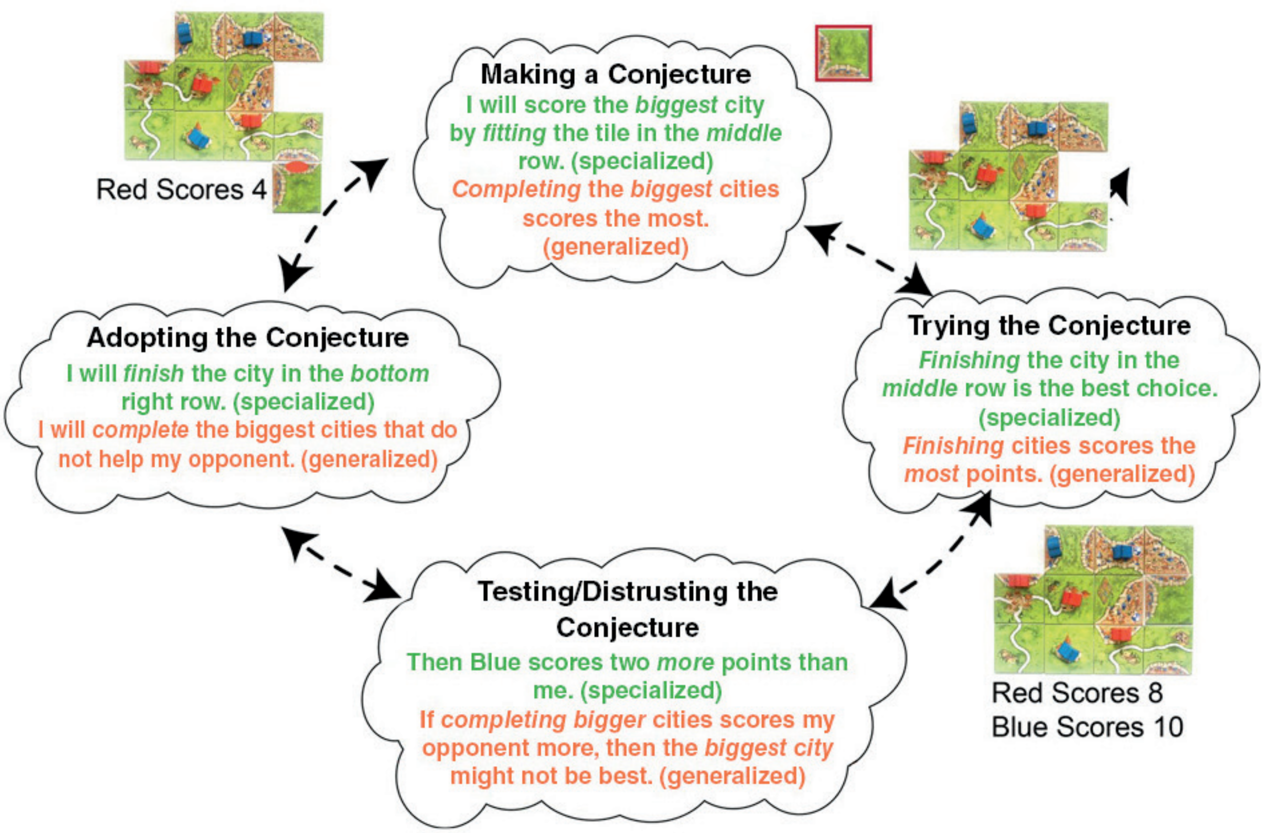


FIGURE 5. An example of specialized and generalized mathematical thinking in Carcassonne, with spatial reasoning in italics.

Discussion

Board games elicit rich opportunities for engaging in spatial reasoning and mathematical thinking. Every move poses a unique problem and offers possibilities for **making, trying, testing** and **adopting conjectures.** As well, opportunities for spatial training emerge organically as students *move, construct, situate, sense, interpret* and *scale* game pieces.

By the end of each four-week cycle, students had developed generalized game strategies, such as making grids in Qwirkle or making spaces for 9-tiles and 8-tiles in upper levels in NMBR 9. Our findings make identifying spatial actions in the context of mathematical thinking easier in both playing board games and engaging in other mathematical endeavours.

We selected board games that are easy to implement in the classroom and that also afford tremendous benefits beyond math class, and we hope teachers will be inspired to try them.

We found that game play helped develop a cohesive classroom environment that spilled over into group work and friendships.

Additionally, the students' enthusiasm for playing games established a positive outlook toward math class. Students also asked to use manipulatives when engaging in other math activities or problems, more so than in years when we did not play games.

Further, as outlined in the appendix, playing board games addressed many learning outcomes in the Grades 5 and 6 mathematics curricula (Alberta Education 2022). Combined learning outcomes are indicative of the networked nature of mathematics.[2] We focused on Grades 5 and 6, but these games could address learning outcomes for most grades.

Playing games also addressed mathematical ideas beyond the curriculum. The games provided opportunities for experiencing multiple instantiations of number: counting tiles, measuring lines or covered areas, moving tiles to specific positions, and scoring along a number line. The game rules offered possibilities for algebraic reasoning and Boolean logic. For example, in Carcassonne, if a city is completed, the city's tiles score double.

In the opening anecdote, when Bradley placed a tile that formed an intersecting row and column in Qwirkle, she had difficulty scoring both. She repeatedly counted tiles in the newly formed rows and columns rather than recognizing the size of each. The game gave her multiple opportunities to notice how a tile formed intersecting horizontal rows and vertical columns. Noticing these intersections is a key skill when locating the position of cardinal points on a Cartesian plane.

Qwirkle and the other games have an underlying grid. While the grid is not explicitly the focus of the games, students had multiple spatial experiences with *building, moving* and *positioning* game pieces in gridified space. While this topic is beyond the scope of this article, we speculate that these spatial experiences would be valuable in learning mathematics concepts such as algebra, geometry and graphing.

To conclude, playing board games in math class can enliven the classroom, immerse students in rich spatial reasoning and problem-solving experiences, and expand students' mathematical possibilities.

Appendix: Curricular Connections

The following learning outcomes are from Alberta's K–6 math curriculum (Alberta Education 2022).

Numbers, Algebra and Patterns

- Students multiply and divide natural numbers within 100 000, including with standard algorithms. (Grade 5)
- Students interpret numerical and algebraic expressions. (Grade 5)
- Students analyze expressions and solve algebraic equations. (Grade 6)
- Students investigate functions to enhance understanding of change. (Grade 6)

Qwirkle

A completed row or column of six scores twice the points, which can be expressed as an algebraic expression. As a function, the score increases with tiles placed.

Shifting Stones

The number on the card dictates the multiplier for scoring points. For example, every five-point card scored is multiplied by five. The number on each card can be considered a variable in an algebraic expression. As a function, the score changes with accumulated cards.

NMBR 9

The level in which a tile is placed indicates the multiplier for the value of tile placed. For example, a 9-tile placed on the third level scores 27. The level can be considered a variable in an algebraic expression. Depending on the level, the function for scoring changes.

Carcassonne

City tiles score double in completed cities. Shields also score double in a completed city. Completing cities changes the variable from 1 to 2 in the algebraic expression. As a function, the score increases with increasing size of the feature of the tile placed.

Geometry

- Students analyze shapes through symmetry and congruence. (Grade 6)

NMBR 9

As tiles are fitted, the shapes are rotated, slid and turned and are still congruent.

Carcassonne

As tiles are matched to identical features, they are rotated, slid and turned and are still congruent.

Coordinate Geometry

- Students relate location to position on a grid. (Grade 5)

Qwirkle

The position and location of tiles on an emergent grid depend on the pattern of colour or shape.

Shifting Stones

The rearrangement of tiles on the fixed grid depends on the pattern on the playing card specifying cardinality.

NMBR 9

The position and location of tiles on an emergent grid depend on the arrangement of the already-placed tiles according to spaces available on the current or underlying layers.

Carcassonne

The position and location of tiles depend on the features of the tile in play and the already-created features on the emergent grid.

- Students explain location and movement in relation to position in the Cartesian plane. (Grade 6)

Shifting Stones

The fixed grid and specific orientation of north dictate the movement and position for rearranging tiles according to the card in play.

Measurement

- Students estimate and calculate area using standard units. (Grade 5)

Qwirkle

A score is counted by the area of the rows and columns formed when a tile is placed.

Carcassonne

A score is counted by the number of squares that complete an area.

Statistics

- Students analyze frequency in categorical data. (Grade 5)
- Students investigate relative frequency using experimental data. (Grade 6)

Qwirkle

The scoresheet requires students to collect and interpret data according to the score of the tiles placed.[3]

Shifting Stones

The scoresheet requires students to collect and interpret data according to the score of the tiles placed.

NMBR 9

The level dictates the category the data is placed in. The scoresheet requires students to collect and interpret data according to the score of the tiles placed.

Notes

This project was supported in part by funding from the Social Sciences and Humanities Research Council (SSHRC) of Canada.

The authors acknowledge the broader project contributions of Stefan Rothschuh, Mischa Simpson, Munesah Rahman and Janelle McFeetors.

1. All students' names have been changed to protect their privacy.

2. This approach to teaching math is discussed in Unit 1 of the Math Minds course, www.structuringinquiry.com (accessed November 12, 2024).

3. All scoresheets are available at https://stem-education.ca, under Board Games (accessed November 12, 2024).

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Two Pythagorean Proofs That Move Students from Knowledge to Understanding

Kevin Hubick

In *The Pythagorean Proposition,* Elisha S Loomis (1940) provides 109 algebraic proofs and 256 geometric proofs of the Pythagorean theorem. He states that the number of algebraic and geometric proofs is limitless (p viii).

In Alberta, students are introduced to the famous theorem in Grade 8 and revisit it in high school. While most students memorize the formula ($a^2 + b^2 = c^2$), the goal of this article is to help teachers guide students from memorization to an understanding of why the theorem works.

Here are two of Loomis's more accessible proofs, one geometric and one algebraic, for students to work through. Ideally, students would work through this process as a guided activity within a vertical surface (Liljedahl 2021) or similar group setting.

A Geometric Proof

The following describes Loomis's (1940, 153–54) geometric proof 90.

Begin with a square of any size. (Grid paper greatly facilitates the process.)

Split the side of the square at any point, leaving the line as two lengths, *a* and *b,* such that $a + b$ equals the full length of the side of the square.

From each corner, split the remaining sides of the square into the same ratio of lengths of *a* and *b,* ensuring that the letters alternate as you proceed along the perimeter of the square. See Figure 1.

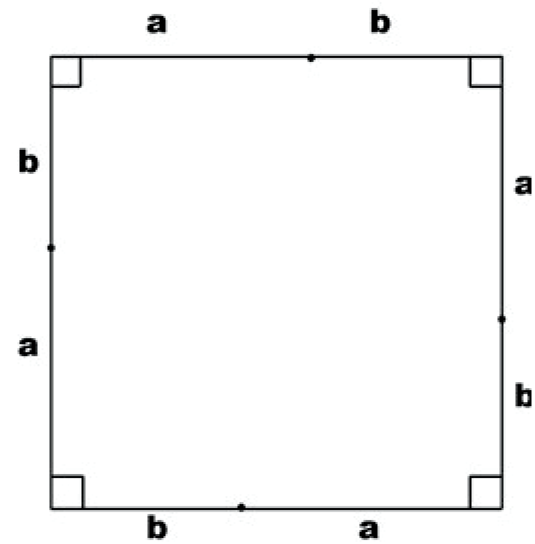

Figure 1

Students should create large squares, but should a student choose to create a square of length 1, have them search the Internet for "YBC 7289" (Yale Babylonian Collection 7289). They will see an ancient cuneiform tablet, dating between 1800 and 1600 BC, that is an example of a square whose sides equal 1. Of note, transcribed along the diagonal and expressed in sexagesimal is an estimate of $\sqrt{2}$ accurate to six decimal places. This hand-sized tablet undeniably indicates knowledge of the Pythagorean theorem but predates Pythagoras by more than a millennium. Inform the student that this is what their square of length 1 would look like and then have them create a larger square in order to complete the proof as intended.

Guiding Question
What are some properties of a square?

Construct a line between adjacent points and continue this process for the other points proceeding along the perimeter. Doing so creates four congruent right triangles by side–angle–side (SAS), as each triangle contains one side of length a and one side of length b, and the contained angle between the sides is 90°. See Figure 2.

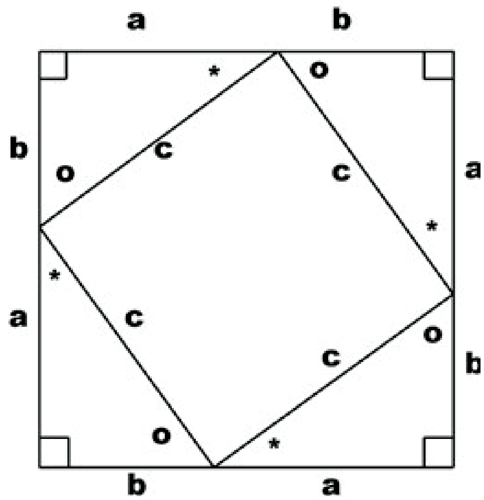

FIGURE 2

Guiding Questions
Do the triangles share common traits?
What does congruency mean for the other angles and the third sides?

Since the triangles are congruent, it follows that their third sides are congruent. As the hypotenuse of a right triangle, this side can be labelled c. It also follows that the two acute angles are congruent. Since the sum of all three angles in each triangle is supplementary (that is, the angles add up to 180°) and each triangle contains a 90° angle, the sum of the two acute angles must be complementary and, hence, add up to 90°.

Guiding Question
What is the corner angle inside the rhombus?
How do we know?

Along each side of the original square, at the points where the hypotenuses end, are three angles: two acute angles (from the adjacent triangles) and a third angle (which forms the corner of the central rhombus). Together, these three angles form a straight line.

Since the two acute angles represent the two acute angles in any of the triangles, which we know add up to 90°, it follows that the rhombus has corner angles of 90°. Therefore, it is a square, since all its corners are right angles and all its sides are equal to the hypotenuse (c) of the congruent triangles. See Figure 3.

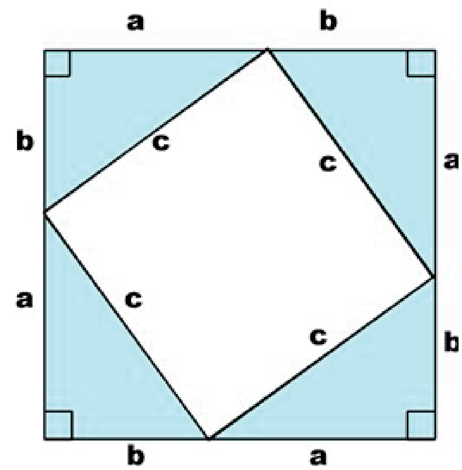

FIGURE 3

This inner square, whose side length is c, has an area of c^2.

Guiding Question
What is the area of the inner square?

If we rearrange and redraw the original square, one pair of congruent triangles placed together forms a rectangle of dimensions $a \times b$. Repeat this process to create a second rectangle, also of dimensions $a \times b$. Placing these rectangles diagonally, in opposite corners of the larger square, creates two smaller squares of side length a (or area a^2) and side length b (or area b^2). See Figure 4.

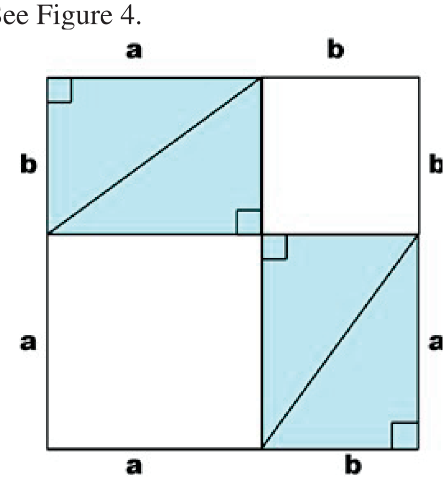

FIGURE 4

Guiding Questions
How do the white areas in Figures 3 and 4 compare with each other?
What does it mean if the white areas are equal?

Since the amount of white space in Figures 3 and 4 is the same, it follows that the sum of the areas a^2 and b^2 must be equal to the area c^2. Using the areas of these three squares, we have a simple geometric proof for the Pythagorean theorem:

$$a^2 + b^2 = c^2.$$

Note for Instructors

When using grid paper in this exercise, have students choose a square whose side lengths are based on common Pythagorean triples (three integers that work in the Pythagorean theorem), such as the following:

- 6–8–10 (side length is 14)
- 5–12–13 (side length is 17)
- 9–12–15 (side length is 21)
- 8–15–17 (side length is 23)

Have students confirm the triple in $a^2 + b^2 = c^2$. Then, have them create a square with the dimensions of the hypotenuse (using the grid paper), cut it out and physically align the square on their original diagram. This is an easy way to concretely verify the theorem.

An Algebraic Proof

The following describes Loomis's (1940, 52) algebraic proof 38.

Start with a right triangle of any size and label it as shown in Figure 5.

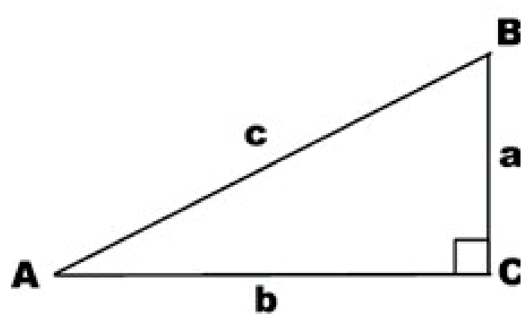

FIGURE 5

Now, construct a line CD such that CD is perpendicular to AB (as shown in Figure 6).

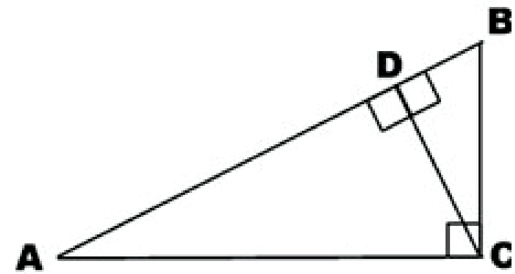

FIGURE 6

Guiding Questions
How many triangles do you see?
What do the triangles share?
What do the triangles have in common?
What can we say if a pair of triangles share two angles?
What is triangle similarity?

Students should recognize three similar triangles: ΔABC, ΔACD and ΔCBD. The triangles are similar because they each contain a right angle and share at least one angle with another triangle. If two pairs of angles are congruent, then the third pair of angles will also be congruent. Hence, all three triangles are similar (or scale factors of each other) by angle–angle–angle (AAA).

Similarity also tells us that the ratios of corresponding sides will be equal for similar triangles (as shown in Figure 7).

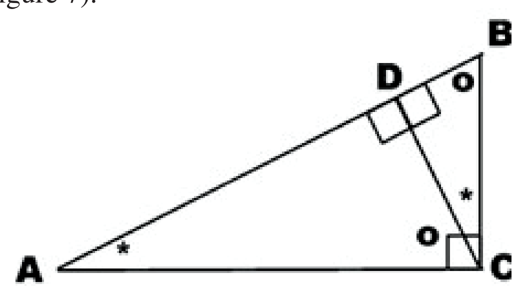

FIGURE 7

Guiding Questions
In the original triangle (ΔABC), what ratio is equivalent to the hypotenuse over the bottom side in the smaller bottom triangle (ΔACD)?
What ratio in the original triangle (ΔABC) would be equivalent to the hypotenuse over the vertical side in the triangle on the right (ΔCBD)?

Have students note the order of sides and that side AB in the large triangle (ΔABC) corresponds to side AC in the bottom triangle (ΔACD).

Comparing ΔABC to ΔACD, we get the ratio of sides:

$$\frac{\mathrm{AB}}{\mathrm{AC}} = \frac{\mathrm{AC}}{\mathrm{AD}}.$$

Cross-multiplying the terms of the equivalent ratios provides the following equality:

$$\frac{\mathrm{AB}}{\mathrm{AC}} = \frac{\mathrm{AC}}{\mathrm{AD}} \longrightarrow (\mathrm{AC})^2 = (\mathrm{AB})(\mathrm{AD}).$$

Repeating the process for ΔABC and ΔCBD, we get

$$\frac{\mathrm{AB}}{\mathrm{CB}} = \frac{\mathrm{CB}}{\mathrm{DB}} \longrightarrow (\mathrm{CB})^2 = (\mathrm{AB})(\mathrm{DB}).$$

Adding the equations, we get

$$(\mathrm{CB})^2 + (\mathrm{AC})^2 = (\mathrm{AB})(\mathrm{AD}) + (\mathrm{AB})(\mathrm{DB}).$$

Factoring the common (AB) from the right side of the equation yields

$$(\mathrm{CB})^2 + (\mathrm{AC})^2 = (\mathrm{AB})[(\mathrm{AD}) + (\mathrm{DB})].$$

Guiding Question
What do the line segments (AD) + (DB) represent in the triangle?

Since (AD) + (DB) represents the hypotenuse of the original triangle (see Figure 6), it equals (AB), which can replace (AD) + (DB) in the equation:

$$(\mathrm{CB})^2 + (\mathrm{AC})^2 = (\mathrm{AB})(\mathrm{AB})$$

or

$$(\mathrm{CB})^2 + (\mathrm{AC})^2 = (\mathrm{AB})^2.$$

Guiding Question
Express this equation using the sides from the original triangle.

Using the notation of the sides from the original triangle (Figure 5), we directly get the Pythagorean theorem:

$$a^2 + b^2 = c^2.$$

Curricular Connections

The Pythagorean theorem is part of the Shape and Space strand in Alberta's Grade 8 math curriculum (Alberta Education 2007) and prerequisite knowledge for high school mathematics.

Using proofs to demonstrate the context for the theorem promotes deeper thinking skills in the following high school math outcomes (Alberta Education 2008):

Mathematics 10C

- Develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles. (Measurement, specific outcome 4)

Mathematics 10-3

- Solve problems that require the manipulation and application of formulas related to:
  - perimeter
  - area
  - the Pythagorean theorem
  - primary trigonometric ratios
  - income. (Algebra, specific outcome 1)
- Demonstrate an understanding of the Pythagorean theorem by:
  - identifying situations that involve right triangles
  - verifying the formula
  - applying the formula
  - solving problems. (Geometry, specific outcome 2)

Mathematics 20-1

- Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. (Trigonometry, specific outcome 2)

Mathematics 30-1

- Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians. (Trigonometry, specific outcome 5)
- Prove trigonometric identities, using:
  - reciprocal identities
  - quotient identities

- Pythagorean identities
- sum or difference identities (restricted to sine, cosine and tangent)
- double-angle identities (restricted to sine, cosine and tangent). (Trigonometry, specific outcome 6)

Mathematics 20-2

- Derive proofs that involve the properties of angles and triangles. (Geometry, specific outcome 1)

The availability of accessible proofs can help teachers present this enriching material and solidify student understanding of the content and context of the Pythagorean theorem. The goal is no longer to have students simply see and memorize an equation but, rather, to have them develop an understanding of why the Pythagorean theorem works.

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Kevin Hubick has a BEd from the University of Alberta, with a major in physical sciences and a minor in mathematics. He teaches at Sturgeon Composite High School and has also served as mathematics department head and as a grade coordinator. He takes great joy in helping students and colleagues understand and appreciate both the application and the beauty of mathematics. As an electronics hobbyist and amateur coder, he has spent the last decade developing the school's computing science program and robotics club. For mentoring students working on an autonomous robot, he received an Association of Professional Engineers and Geoscientists of Alberta (APEGA) Innovation in Education Award. The robot was designed to use machine learning and computer vision to help clear parks and other urban areas of biohazardous waste (such as used needles, medical masks and gloves).

Using Math Routines to Increase Student Engagement

Kendra Boyd

Mathematical ideas are like fingerprints—unique to each person.

Every classroom is filled with students waiting to explore and leave their fingerprints all over what they are learning. This can feel messy, especially in the math classroom, so teachers often offer students rules and procedures to memorize. After all, that is how most of us were taught math.

The problem with this approach is that students need to know more than just formulas, steps and how to solve equations.

One way we can help students grapple with math content is by mathematizing their world.

Enter daily instructional math routines. These routines are "specific and repeatable designs for learning that support both the teacher and students . . . enabling all students to engage more fully in learning opportunities while building crucial mathematical thinking habits" (Kelemanik, Lucenta and Creighton 2016).

Mathematical understanding develops over time and is unique to each person. Having clearly defined routines gets students thinking about math and communicating in more meaningful ways. Math class becomes less about mimicking and memorization and more about connecting concepts conceptually, which deepens students' mathematical understanding.

Have you considered using instructional math routines that engage students in meaningful conversations? These routines usually do not happen spontaneously, so teachers must find ways to embed them. Routines are a great way to enhance the content you are teaching, to build students' mathematical curiosity and to spiral back in meaningful ways.

I have curated a collection of free math routines you can use in your classroom right away, including number talks for various grade levels that you can explore with your students.

To access the document, scan the QR code below or go to https://drive.google.com/file/d/1yPZIKETIxifwXAys05UsofltojmMYkN4/view. Save the document to your own Google Drive.

Try starting or ending your math class with one of these routines.

You can also create your own routines, or challenge your students to create routines to share with their classmates.

Reference

Kelemanik, G, A Lucenta and S J Creighton. 2016. *Routines for Reasoning: Fostering the Mathematical Practices in All Students.* Portsmouth, NH: Heinemann.

Kendra Boyd has been a learning coach with the Medicine Hat Public School Division since 2017. She began her teaching career in 2001 and has taught in Canada and the United States, including nine years in Omaha, Nebraska, at a Title 1 school (a school that receives supplemental funding due to a high population of students from low-income families). That experience highlighted the need to teach the whole child, as well as the types of learning experiences necessary to support and engage all learners. Her passion is sharing best instructional practices and collaborating with other teachers. She holds a BEd from the University of Providence; an MA in teacher education, learning and leadership from the University of Nebraska; and a master's certificate as a math specialist. She sits on the MCATA executive as vice-president (publications).

Open-Ended Questions to Naturally Differentiate for Students in Math

Kendra Boyd and Lorelei Boschman

We have curated open-ended questions for you to use right away with your math students.

These questions will initiate deeper thinking and help you determine students' levels of understanding.

The conversations these questions will provoke will help students clarify and extend their thinking and give you insight into their progress and understanding.

Because the questions are open ended, they will naturally differentiate for students. They give students choice, and you can tailor the numbers to fit your students' needs.

These questions are wonderful for math talks, small-group centres and partner sharing, and students could even solve them on a vertical surface.

Feel free to adjust the numbers based on grade level and students' needs.

Division and topic	Check what they know	Understanding in practice	Extending student learning
Division I Topic: Comparing whole numbers	Choose three numbers to make this statement true: ____ is a lot closer to ____ than to ____. Note for the teacher: • Clarify how many place values you want for these numbers.	Choose two numbers. Then, create as many representations as you can to prove that one number is larger (or smaller) than the other number. Notes for the teacher: • Clarify how many place values you want for these numbers. • Representations can take the form of a number line, a manipulative, a drawing, coloured dots and so on. • Encourage students to share and explain their representations in pairs.	Create a number line, skip counting by ____. Choose four other numbers. Label them on your number line. Note for the teacher: • Choose the number based on where your students are.

Division II Topic: Fractions	Draw a picture to show what 1/2 looks like. Name real-world examples of where you would find or use 1/2.	Choose a fraction other than 1/2. Then, create as many representations as you can for that fraction. Notes for the teacher: • Representations can take the form of a number line, a picture, a drawing, coloured dots, a real-world connection and so on. • Encourage students to share and explain their representations in pairs.	Draw two pictures to show how the same amount represents 1/2 in the first picture and 1/3 in the second picture.
Division III Topic: Fractions	Create three fractions with different denominators. Order them from least to greatest. Explain how you know. You can use visuals to help you. Note for the teacher: • You can put more restrictions on this, as needed. For example, tell students that they cannot use unit fractions (such as 1/3 or 1/4), the denominator cannot be 2, and so on.	Create four fractions, using any number only once. Order them from least to greatest. Explain how you know.	What do you know for sure about *a*/4 and *b*/6? Think about what *a* and *b* could be. Talk about and compare the fractions they make.

Division IV Topic: Conics (parabolas)	The amusement park Parabolic Plunge features a thrilling ride that lets riders experience the heights and depths of a parabolic trajectory. The height (h) in metres of a rider above the ground after a certain amount of time (t) in seconds is modelled by the equation $h = -5t^2 + 10t + 15$. Investigate how changing the coefficients of the equation $h = at^2 + bt + c$ affects the shape of the parabola and the behaviour of the ride. Provide at least three sets of coefficients and describe the corresponding ride experiences.	Suppose the amusement park wants to modify the ride to make it reach greater heights while maintaining a similar overall duration. Propose a new equation for the height (h) in terms of time (t) that achieves this goal. Explain your reasoning and discuss the potential impact of this modification on the ride experience.	Consider the scenario where the ride undergoes maintenance, and the coefficient of the t^2 term in the equation is changed to a positive value. Discuss how this alteration affects the ride experience in terms of height, duration and overall excitement.

Kendra Boyd has been a learning coach with the Medicine Hat Public School Division since 2017. She began her teaching career in 2001 and has taught in Canada and the United States, including nine years in Omaha, Nebraska, at a Title 1 school (a school that receives supplemental funding due to a high population of students from low-income families). That experience highlighted the need to teach the whole child, as well as the types of learning experiences necessary to support and engage all learners. Her passion is sharing best instructional practices and collaborating with other teachers. She holds a BEd from the University of Providence; an MA in teacher education, learning and leadership from the University of Nebraska; and a master's certificate as a math specialist. She sits on the MCATA executive as vice-president (publications).

Lorelei Boschman, BEd, MEd, has been an educator since 1988 as an elementary teacher, a high school math teacher and an instructor at Medicine Hat College (MHC), where she is also chair of education. Educating preservice teachers and creating meaningful and thorough learning experiences to prepare them for teaching is paramount to her. She has various publications and has presented at numerous conferences on the topics of mathematics teaching and preservice teacher training. She serves MCATA as the editor of delta-K.

Reflections from the 2023 Winner of the Dr Arthur Jorgensen Chair Award

Evelyn Matthews

In fall 2023, I received MCATA's Dr Arthur Jorgensen Chair Award as a mathematics education student at the University of Alberta. This was an incredible honour and a testament to my dedication to mathematics education.

As part of the award, I sat on the MCATA executive for one year. This was an excellent opportunity to learn from inservice math educators.

This award has been a significant milestone in my journey, reinforcing my passion for teaching and my commitment to fostering a love of mathematics in my students.

I officially started my journey toward becoming an educator in September 2022. However, this journey has been ongoing since I was 10 years old, even if I didn't realize it back then.

For as long as I can remember, I have loved math. Manipulating numbers, solving puzzles and reasoning through any problem have always drawn me in. Growing up, I knew I wanted to pursue a career in mathematics, but so many fields (dare I say all fields?) depend on mathematics and reasoning, so narrowing it down took time.

In 2016, I graduated from high school, ready to start my education in actuarial mathematics at the University of Manitoba. However, after a year in the program, I was still searching for something that numbers alone couldn't provide. This led me to explore other fields, including accounting, economics and human resources.

My work in human resources at Transport Canada as a co-op student helped me realize what had been missing. I found that I thrived in an environment where I could work with and help others, and I knew that this was a must if I was to be happy in my future career.

Throughout university, I worked in various positions with children, including as a nanny, a daycare worker and a tutor. My experience tutoring a Grade 8 student with autism spectrum disorder (ASD) significantly contributed to the educator I am today. This student reminded me of my younger sister, who was diagnosed with dyslexia in Grade 6. They both had stories of struggling in school with little support, particularly in mathematics. I realized that there was a strong need for mathematics teachers in elementary and middle school.

In 2021, I graduated from the University of Manitoba with a BA in geography and a renewed passion for mathematics.

Since then, I have taken on many roles, including a student, a researcher, a teacher and a mentor.

In September 2022, I started working toward a BEd after degree, with a specialization in mathematics, at the University of Alberta. This gave me many opportunities, including my research position with Josh Markle. In this role, I worked with him, along with Janelle McFeetors and Mijung Kim, to explore the role of visualization in mathematics and science. Having these professors as mentors has been a privilege and has shaped the educator I am today. Their guidance and expertise have deepened my understanding of mathematics and instilled a commitment to research-based teaching methods. I am grateful for their mentorship and the opportunities they have provided me with.

Today, I stand as a recent graduate and a new educator, excited to start sharing my passion for mathematics with my students and helping the next generation see themselves as "math people." My goal is to create an environment where students aren't afraid to take risks, embrace challenges and view mistakes as learning opportunities. MCATA's goal of developing creative, curious and courageous mathematicians aligns with my teaching philosophy and the type of classroom culture I aim to generate.

Reflecting on my journey, I am reminded that education is about not just imparting knowledge but also inspiring and supporting students to reach their full potential. Moving forward, I am excited to continue working with MCATA and to move into my teaching career.

Evelyn Matthews is a dedicated mathematics educator and a recent graduate of the University of Alberta's education program. As an emerging educator, she is passionate about exploring students' mathematical reasoning and enhancing their understanding and engagement in mathematics.

Mathigon

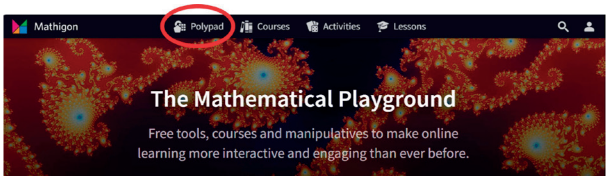
https://mathigon.org

Mathigon is an award-winning website that offers excellent digital manipulatives, activities and lesson plans for Grades 3–12 students.

Manipulatives are powerful because they make learning more visual for students. Hands-on manipulatives are preferable, but Mathigon gives you access to many manipulatives that you may not have at your school.

Explore the many ready-to-go activities and lessons, as well as tutorials on how to make learning mathematics fun by engaging students. Consider using these resources as part of your lessons, as well as for intervention or enrichment.

Let's explore the tabs at the top of the website.



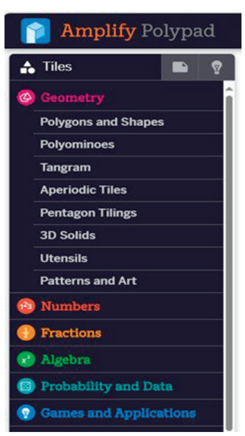
Polypad

The Polypad tab leads to a website that allows teachers and students to easily create and manipulate number lines, polygons, fraction bars, algebra tiles, polygons and more. Click on Launch Polypad to see what's available.

Note the coloured words in the Polypad menu. These can be expanded to show the manipulatives available for each topic.

Teachers can use Polypad to create visualizations or to copy to slide decks.

Students can use Polypad to create and to show their thinking for math work and formative or summative assessments.

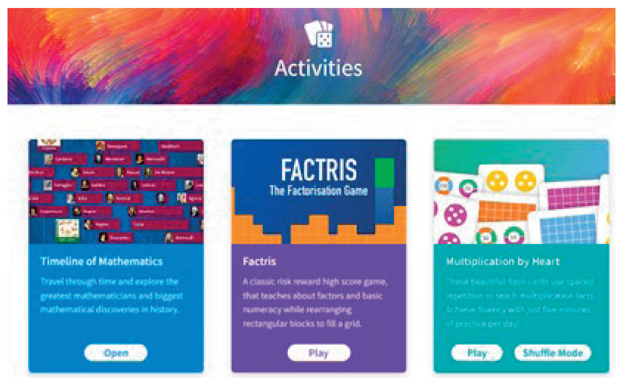


Activities

The Activities tab allows teachers to find ready-to-go curated activities, games and lesson ideas to use with students.

All are easily shareable and easy to navigate.

Consider using one of these activities for an if-time activity for the week. (I played with the Tangram Builder for quite some time!)



Lessons

The Lessons tab leads to lessons that make learning engaging and visible to students.

The search bar on the left allows you to search for specific topics, such as fractions, clocks, balance scales, completing the square or exploring the Pythagorean theorem. To further target your search, you can select student explorations, lesson plans, puzzles and games, or teaching ideas.

It is worth taking a few minutes to see what lessons are available for your math topics. Many junior and senior high math lessons and activities are also available.

The websites highlighted in delta-K *are shared solely for informational purposes. MCATA does not endorse, guarantee or assume responsibility for the content, products or services provided on any third-party website. When engaging with these websites, users are encouraged to evaluate the content independently and use their discretion. Any reliance on the content of or interactions with these websites is at the user's own risk.*

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